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BOUNDING VOLUMES OF SINGULAR FANO THREEFOLDS

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Abstract. Let (X,Δ) be an n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pair. We give an upper bound for the volume $\mathrm{Vol}(X,\Delta)=(-(K_X+\Delta))^n$ when $n=2$, or $n=3$ and X is $\mathbb{Q}$-factorial of $\rho(X)=1$. This bound is essentially sharp for $n=2$. The main idea is to analyze the covering families of tigers constructed in J. M^{c}Kernan (*Boundedness of log terminal fano pairs of bounded index*, preprint, 2002, arXiv:0205214). Existence of an upper bound for volumes is related to the Borisov–Alexeev–Borisov Conjecture, which asserts boundedness of the set of ϵ-klt log $\mathbb{Q}$-Fano varieties of a given dimension n.

According to the minimal model program, Fano varieties are the building blocks for varieties of negative Kodaira dimension. The set of Fano varieties of a given dimension is expected to satisfy certain boundedness properties. For example, the set of all the n-dimensional smooth Fano manifolds forms a bounded family by [KMM92]. Since the need of singularities arises naturally in the minimal model program, the set of mildly singular Fano varieties is also expected to be bounded. This is known for terminal $\mathbb{Q}$-Fano $\mathbb{Q}$-factorial threefolds of Picard number one by [Kaw92] and for canonical $\mathbb{Q}$-Fano threefolds by [KMMT00]. However, if one considers the set of all klt $\mathbb{Q}$-Fano varieties with Picard number one of a given dimension, [Lin03] and [Oka09] have shown that (birational) boundedness fails. The problem is that the category of klt singularities is too big to be bounded as it contains finite quotients of arbitrarily large order. Instead, one restricts to a smaller class of singularities, known as ϵ-klt singularities where $0<\epsilon<1$. Since the notion of log pairs naturally arises in the context of minimal model program, we also consider log Fano varieties.

Definition. A pair (X,Δ) consists of a normal projective variety X and an effective Weil $\mathbb{Q}$-divisor Δ such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let $\pi: Y\to X$ be a log resolution of a pair (X,Δ) and write $K_Y+\Delta_Y=\pi^*(K_X+\Delta)$.

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For $0 < \epsilon < 1$, we say that the pair (X, Δ) is ϵ-klt if all the coefficients of Δ_Y are less than $1 - \epsilon$. In particular, all the coefficients of Δ lie in $[\,0, 1 - \epsilon\,)$. Note that smaller ϵ corresponds to worse singularities.

An ϵ-klt (weak) log $\mathbb{Q}$-Fano variety is an ϵ-klt pair (X, Δ) such that the $\mathbb{Q}$-Cartier divisor $-(K_X + \Delta)$ is ample (resp. nef and big).

Definition. We say that a collection of varieties $\{X_\lambda\}_{\lambda\in\Lambda}$ is bounded if there exists a morphism of finite type of Noetherian schemes $h : \mathcal{X} \to S$ such that for each X_λ, $X_\lambda \cong \mathcal{X}_s$ for some closed point $s \in S$.

In this paper, we are interested in the following conjecture which is still open in dimension three and higher.

Borisov–Alexeev–Borisov Conjecture. *Fix $0 < \epsilon < 1$, an integer $n > 0$, and consider the set of all the n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pairs (X, Δ). Then the set of underlying varieties $\{X\}$ is bounded.*

A. Borisov and L. Borisov establish the B–A–B Conjecture for toric varieties in [BB92]. Alexeev establishes the two-dimensional B–A–B Conjecture in [Ale94] with a simplified argument given in [AM04]. The author's original motivation of studying the B–A–B Conjecture is that it is related to the conjectural termination of flips in the minimal model program: According to [BS10], log minimal model program, ACC[1] for minimal log discrepancies, and B–A–B Conjecture in dimension $\leqslant d$ implies termination of log flips in dimension $\leqslant d + 1$ for effective pairs.

The following statements concerning log $\mathbb{Q}$-Fano pairs (X, Δ) are relevant to the B–A–B Conjecture:

(i) The Cartier index of $K_X + \Delta$ of an n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pair (X, Δ) is bounded from above by a fixed integer $r(n, \epsilon)$ depending only on $n = \dim X$ and ϵ.

(ii) The anticanonical volume $\mathrm{Vol}(X, \Delta) = (-(K_X + \Delta))^n$ of an n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pair (X, Δ) is bounded from above by a fixed integer $M(n, \epsilon)$ depending only on $n = \dim X$ and ϵ.

(iii) (*Batyrev Conjecture*) For given integers $n, r > 0$, consider the set of all the n-dimensional klt log $\mathbb{Q}$-Fano pairs (X, Δ) such that $r(K_X + \Delta)$ is Cartier. Then the set of underlying varieties $\{X\}$ is bounded.

[1] An ACC (resp. DCC) set is a set of real numbers satisfying the ascending (resp. descending) chain condition, that is, it contains no infinite strictly increasing (decreasing) sequences.

It is clear that the B–A–B Conjecture follows from (i) and (iii) (cf., Proposition below). The Batyrev Conjecture (iii) in dimension three is established by Borisov, [Bor96, Bor01]. Recently Hacon, M[c]Kernan, and Xu have announced a proof of the Batyrev Conjecture (iii) in any dimension. In general, it is very hard to establish (i). Ambro in [Amb09] has proved (i) for toric singularities assuming that only the standard coefficients $\{1-\frac{1}{l}|l\in\mathbb{Z}_{\geqslant 1}\}\cup\{1\}$ are allowed to be the coefficients of Δ. A necessary condition for (i) is that the coefficients of Δ must lie in a fixed DCC set: A counterexample is given by the set of pairs $(\mathbb{P}^1, \frac{1}{N}\{\mathrm{pt}\})$ for $N\geqslant 1$.

For convenience of the reader, we include a well-known (to the experts) argument establishing the B–A–B Conjecture under conditions (i) and (ii) in the case when $\Delta=0$, or X is $\mathbb{Q}$-factional of $\rho(X)=1$.

PROPOSITION. *Suppose that $\Delta=0$ or X is $\mathbb{Q}$-factional of $\rho(X)=1$, then the B–A–B Conjecture follows from the statements (i) and (ii) above.*

Proof. Let X be an ϵ-klt $\mathbb{Q}$-Fano variety of dimension n and let r_X be the Cartier index of K_X. The following statements together imply boundedness:

(1) There is an upper bound $r(n,\epsilon)$ of the Cartier index r_X of K_X depending only on n and ϵ;
(2) The divisor $M(-K_X)$ is a very ample line bundle for a fixed M depending only on n and ϵ;
(3) The set of Hilbert polynomials $\mathfrak{F}=\{P(t)=\chi(\mathcal{O}_X(-r_XK_X)^{\otimes t})\}$ associated to all the n-dimensional ϵ-klt $\mathbb{Q}$-Fano variety is finite.

In fact, (2) and (3) imply that the set of n-dimensional ϵ-klt $\mathbb{Q}$-Fano varieties is contained in the finite union of Hilbert schemes $\coprod_{P(t)\in\mathfrak{F}}\mathcal{H}_{P(t)}$, where each $\mathcal{H}_{P(t)}$ is Noetherian. Note that (1) is essential for getting (2).

The statement (1) is simply (i).

Let $r=r(n,\epsilon)$ as in (i). Since rK_X is a line bundle, by [Kol93] $|-mrK_X|$ is base point free for any $m>0$ divisible by a constant $N_1(n)>0$ depending only on $n=\dim X$. Since $|-mrK_X|$ is ample and base point free for $m>0$ sufficiently divisible, it defines a finite morphism. By [Kol97, Theorem 5.9], the map induced by $|-lrK_X|$ for $l>0$ divisible by $N_2(n)$ is birational where $N_2(n)$ also depends only on $n=\dim X$. Since a finite birational morphism of normal varieties is an isomorphism, it follows that there exists an effective embedding by $|M(-rK_X)|$ for some fixed $M>0$ depending only on $n=\dim X$ and ϵ. This is (2).

By [KM83], the coefficients of the Hilbert polynomial $P(t) = \chi(\mathcal{O}_X(tH))$ of a polarized variety (X, H) with H an ample line bundle can be bounded by H^n and $|H^{n-1} \cdot K_X|$. Since by (1) there exists an integer $r(n, \epsilon) > 0$ depending only on $n = \dim X$ and ϵ such that $r(n, \epsilon)(-K_X)$ is an ample line bundle, set $H = r(n, \epsilon)(-K_X)$ and apply (ii), it follows that there are only finitely many Hilbert polynomials for the set of anticanonically polarized ϵ-klt Fano varieties $\{(X, r(n, \epsilon)(-K_X))\}$. This proves (3).

If X is $\mathbb{Q}$-factorial of $\rho(X) = 1$, then $-(K_X + \Delta)$ being ample implies that $-K_X$ is also ample. It is clear that X is also ϵ-klt and hence boundedness follows from the same proof as above. $\square$

An effective upper bound in (ii) is obtained for smooth Fano n-folds in [KMM92] and for canonical $\mathbb{Q}$-Fano threefolds in [KMMT00]. In this work, we obtain an effective answer to question (ii) in dimension two, that is, for log del Pezzo surfaces (cf., Theorem 4.3).

THEOREM A. *Let (X, Δ) be an ϵ-klt weak log del Pezzo surface, then the anticanonical volume* $\mathrm{Vol}(X, \Delta) = (K_X + \Delta)^2$ *satisfies*

$$(K_X + \Delta)^2 \leqslant \max\left\{64, \frac{8}{\epsilon}\right\}.$$

Moreover, this upper bound is in a sharp form: There exists a sequence of ϵ-klt del Pezzo surfaces whose volume grows linearly with respect to $1/\epsilon$.

Let (X, Δ) be an ϵ-klt weak log del Pezzo surface and $X_{\min}$ be the minimal resolution of (X, Δ). Alexeev and Mori have shown in [AM04, Theorem 1.8] that $\rho(X_{\min}) \leqslant 128/\epsilon^5$. Also from [AM04, Lemma 1.2] (or see the proof of Theorem 4.3), an exceptional curve E on $X_{\min}$ over X satisfies $1 \leqslant -E^2 \leqslant 2/\epsilon$. In case $\Delta = 0$, since the Cartier index of K_X is bounded from above by the determinant of the intersection matrix $(E_i.E_j)$ of the exceptional curves E_i's on $X_{\min}$ over X, it follows that the Cartier index bound $r(2, \epsilon)$ in statement (i) satisfies

$$(\lozenge) \qquad r(2, \epsilon) \leqslant 2(2/\epsilon)^{128/\epsilon^5}.$$

REMARK 0.1. An upper bound of $(K_X + \Delta)^2$ is implicitly mentioned in [Ale94] but not clearly written down. It is also not clear if the upper bound $(\lozenge)$ is optimal. In view of Theorem A, this seems unlikely.

We also obtain an upper bound of the anticanonical volumes for ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefolds of Picard number one (cf., Theorem 5.17).

THEOREM B. *Let (X,Δ) be an ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefold of $\rho(X)=1$. Then the degree $-K_X^3$ satisfies*

$$-K_X^3 \leqslant \left(\frac{24M(2,\epsilon)R(2,\epsilon)}{\epsilon}+12\right)^3,$$

where $R(2,\epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S)=1$ and $M(2,\epsilon)$ is an upper bound of the anticanonical volume $\operatorname{Vol}(S)=K_S^2$ for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S)=1$. Note that $M(2,\epsilon)\leqslant \max\{64,16/\epsilon\}$ from Theorem A and $R(2,\epsilon)\leqslant 2(4/\epsilon)^{128/\epsilon^5}$ from ($\diamondsuit$).

For a $\mathbb{Q}$-factorial ϵ-klt log $\mathbb{Q}$-Fano pair (X,Δ) of $\rho(X)=1$, since $-(K_X+\Delta)^3\leqslant -K_X^3$ and X is also ϵ-klt, by Theorem B we get an upper bound of the anticanonical volume $\operatorname{Vol}(X,\Delta)=-(K_X+\Delta)^3$. However, it is not expected that the bound in Theorem B is sharp or in a sharp form. As a corollary of Theorem B, this gives a proof of the Batyrev conjecture in the case of log $\mathbb{Q}$-Fano threefolds of Picard number one.

Note that $\mathbb{Q}$-factoriality is a technical assumption. However, this condition is natural in the sense that starting from a smooth variety, each variety constructed by a step of the minimal model program remains $\mathbb{Q}$-factorial. In dimension two, surfaces with rational singularities, for example, klt singularities, are always $\mathbb{Q}$-factorial.

Instead of using deformation theory of rational curves as in [KMMT00], the Riemann–Roch formula as in [Kaw92], or the sandwich argument of [Ale94], we aim to create isolated non-klt centers by the method developed in [M^{c}K02]. The point is that deformation theory for rational curves on klt varieties is much harder and so far no effective Riemann–Roch formula is known for klt threefolds.

This paper is organized as follows: In Section 1, we study non-klt centers. In Section 2, we illustrate the general method in [M^{c}K02] for obtaining an upper bound of anticanonical volumes in Theorems A and B. In Section 3, we review the theory of families of non-klt centers in [M^{c}K02]. In Section 4, we study weak log del Pezzo surfaces and prove Theorem A. In Section 5, we prove Theorem B.

Through this article we work over field of complex numbers $\mathbb{C}$.

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§1. Non-klt centers

For the definition and properties of the singularities in the minimal model program, we refer to [KM98].

Definition 1.1. Let (X, Δ) be a log pair. A subvariety $V \subseteq X$ is called a **non-klt center** if it is the image of a divisor of discrepancy at most -1. A **non-klt place** is a valuation corresponding to a divisor of discrepancy at most -1. The **non-klt locus** $\mathrm{Nklt}(X, \Delta) \subseteq X$ is the union of the non-klt centers. If there is a unique non-klt place lying over the generic point of a non-klt center V, then we say that V is **exceptional**. If (X, Δ) is log canonical along the generic point of a non-klt center V, then we say that V is **pure**.

The non-klt places/centers here are the log canonical (lc) places/centers in [M^cK02].

A standard way of creating a non-klt center on an n-dimensional variety X is to find a very singular divisor: Fix $p \in X$ a smooth point. If Δ is a $\mathbb{Q}$-Cartier divisor on X with $\mathrm{mult}_p \Delta \geqslant n$, then $p \in \mathrm{Nklt}(X, \Delta)$. Indeed, consider the blow up $\pi : Y = \mathrm{Bl}_p X \to X$ and let E be the unique exceptional divisor with $\pi(E) = p$, then the discrepancy

$$a(E, X, \Delta) = \mathrm{mult}_E(K_Y - \pi^*(K_X + \Delta)) = (n-1) - \mathrm{mult}_E(\pi^*(\Delta)) \leqslant -1,$$

as $n - 1 = \mathrm{mult}_E(K_Y - \pi^* K_X)$ and $\mathrm{mult}_E(\pi^*(\Delta)) = \mathrm{mult}_p \Delta \geqslant n$.

Existence of singular divisors can be shown by the following standard estimation.

Lemma 1.2. *Let X be an n-dimensional complete complex variety and D be a divisor with $h^i(X, \mathcal{O}(mD)) = O(m^{n-1})$ for all $i > 0$, for example, when D is big and nef. Fix a positive rational number α with $0 < \alpha^n < D^n$. Then for $m \gg 0$ and any $x \in X_{\mathrm{sm}}$, there exists a divisor $E_x \in |mD|$ with $\mathrm{mult}_x(E_x) \geqslant m \cdot \alpha$.*

Proof. This is [Laz04, Proposition 1.1.31]. □

We will apply Lemma 1.2 to the case where (X, Δ) is an n-dimensional log $\mathbb{Q}$-Fano variety: Write $(-(K_X + \Delta))^n > (\omega n)^n$ for some rational number $\omega > 0$, then as the cohomology groups $h^i(X, \mathcal{O}_X(m(-K_X + \Delta))) = 0$ for $i > 0$ and $m > 0$ sufficiently divisible by Kawamata–Viehweg vanishing theorem, we can find for each $p \in X_{\mathrm{sm}}$ an effective $\mathbb{Q}$-divisor Δ_p such that $\Delta_p \sim_{\mathbb{Q}} -(K_X+\Delta)/\omega$ with $\mathrm{mult}_p(\Delta_p) \geqslant n$. In particular, $p \in \mathrm{Nklt}(X, \Delta+\Delta_p)$.

The non-klt centers satisfy the following Connectedness Lemma of Kollár and Shokurov, whose proof is simply a formal consequence of the Kawamata–Viehweg vanishing theorem. This is the most important ingredient in this work.

LEMMA 1.3. *Let (X, Δ) be a log pair. Let $f: X \to Z$ be a projective morphism with connected fibers such that the image of every component of Δ with negative coefficient is of codimension at least two in Z. Assume that $-(K_X+\Delta)$ is big and nef over Z, then the intersection of* $\mathrm{Nklt}(X, \Delta)$ *with each fiber $X_z = f^{-1}(z)$ is connected.*

Proof. For simplicity, we assume that $Z = \mathrm{Spec}(\mathbb{C})$ is a point and (X, Δ) is log smooth, that is, X is smooth and Δ has simple normal crossing support. Then the identity map $\mathrm{id}_X : X \to X$ is a log resolution of (X, Δ) and $\mathrm{Nklt}(X, \Delta) = \llcorner \Delta \lrcorner$. Consider the exact sequence

$$\cdots \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X, \mathcal{O}_{\llcorner \Delta \lrcorner}) \to \mathrm{H}^1(X, \mathcal{O}_X(-\llcorner \Delta \lrcorner)) \to \cdots .$$

Since $-\llcorner \Delta \lrcorner = K_X + \{\Delta\} - (K_X+\Delta)$ and $(X, \{\Delta\})$ is klt, we have $\mathrm{H}^1(X, \mathcal{O}_X(-\llcorner \Delta \lrcorner)) = 0$ by Kawamata–Viehweg vanishing theorem as $-(K_X+\Delta)$ is nef and big. Since $\mathrm{H}^0(X, \mathcal{O}_X) \cong \mathbb{C}$, we see that $\mathrm{Nklt}(X, \Delta) = \llcorner \Delta \lrcorner$ is connected.

For general case, see [Cor07, Theorem 17.4]. ∎

Here is a nonexample showing that $-(K_X+\Delta)$ being nef and big is necessary in the Connectedness Lemma 1.3.

EXAMPLE 1.4. Let X be $\mathbb{P}^1 \times \mathbb{P}^1$ and denote F (resp. G) to be the fiber of the first (resp. second) projection to $\mathbb{P}^1$. Consider $\Delta_1 = F_1 + F_2$ the sum of two distinct fibers of the first projection to $\mathbb{P}^1$ and $\Delta_2 = F + G$ the sum of two fibers with respect to two different projections to $\mathbb{P}^1$'s. Then $\mathrm{Nklt}(X, \Delta_1) = F_1 + F_2$ is not connected while $\mathrm{Nklt}(X, \Delta_2) = F + G$ is connected. Note that $-(K_X+\Delta_1)$ is nef but not big while $-(K_X+\Delta_2)$ is nef and big.

Later on, we will try to produce not only non-klt centers but *isolated* non-klt centers. The following theorem is the main technique which allows us to cut down the dimension of non-klt centers.

THEOREM 1.5. [Kol97, Theorem 6.8.1] *Let (X, Δ) be klt, projective and $x \in X$ a closed point. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $(X, \Delta + D)$ is log canonical in a neighborhood of x. Assume that $\mathrm{Nklt}(X, \Delta + D) = Z \cup Z'$ where Z is irreducible, $x \in Z$, and $x \notin Z'$. Set $k = \dim Z$. Let H be an ample $\mathbb{Q}$-divisor on X such that $(H^k.Z) > k^k$. Then there is an effective $\mathbb{Q}$-divisor $B \equiv H$ and rational numbers $0 < \delta \ll 1$ and $0 < c < 1$ such that*

(1) *$(X, \Delta + (1-\delta)D + cB)$ is non-klt in a neighborhood of x, and*
(2) *$\mathrm{Nklt}(X, \Delta + (1-\delta)D + cB) = W \cup W'$ where W is irreducible, $x \in W$, $x \notin W'$ and $\dim W < \dim Z$.*

§2. Guiding example

The idea in [M^cK02] for obtaining an upper bound of the anticanonical volumes is to create isolated non-klt centers and then use the Connectedness Lemma 1.3: For simplicity, we assume that $\Delta = 0$. Write $(-K_X)^n > (\omega n)^n$ for a positive rational number ω. For each $p \in X_{\mathrm{sm}}$, we can find an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -K_X/\omega$ such that $\mathrm{mult}_p \Delta_p \geqslant n$ and hence $p \in \mathrm{Nklt}(X, \Delta_p)$. The observation is that if $\omega \gg 0$, then for general $p \in X$, $p \in \mathrm{Nklt}(X, \Delta_p)$ cannot be an isolated point. Indeed, if this is not true, then for two general points $p, q \in X$, the set $\mathrm{Nklt}(X, \Delta_p + \Delta_q)$ would contain $\{p, q\}$ as isolated non-klt centers. But the pair $-(K_X + \Delta_p + \Delta_q) \sim_{\mathbb{Q}} \left(1 - \frac{2}{\omega}\right)(-K_X)$ is nef and big for $\omega > 2$. By the Connectedness Lemma 1.3, $\mathrm{Nklt}(X, \Delta_p + \Delta_q)$ must be connected, a contradiction.

Therefore, for general $p \in X$ the minimal non-klt center $V_p \subseteq \mathrm{Nklt}(X, \Delta_p)$ passing through p is typically positive dimensional. We would like to show that the restricted volume $\mathrm{Vol}(-K_X|_{V_p})$ on the minimal non-klt center V_p is large if $\omega \gg 0$. Hence, we can cut down the dimension of non-klt centers by Theorem 1.5. After doing this finitely many times, we get isolated non-klt centers and we are done.

In general, it is hard to find a lower bound of the restricted volume $\mathrm{Vol}(-K_X|_{V_p})$ on the minimal non-klt center V_p. We illustrate M^cKernan's method of obtaining a lower bound of the restricted volumes on the non-klt center of an ϵ-klt log $\mathbb{Q}$-Fano variety by the following guiding example (cf., [M^cK02]).

EXAMPLE 2.1. Let X be the projective cone over a rational normal curve of degree $d \geqslant 2$ with the unique singular point $O \in X$. The blow up

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$\pi : Y = \mathrm{Bl}_O X \to X$ is a resolution of X where Y is a $\mathbb{P}^1$-bundle $f : Y \to \mathbb{P}^1$ over $\mathbb{P}^1$:

$$\begin{tikzcd} X & Y \arrow[l, "\pi"'] \arrow[d, "f"'] & \supseteq & F_t \cong \mathbb{P}^1 \arrow[d] \\ & \mathbb{P}^1 & \ni & t. \end{tikzcd}$$

It is easy to show that

(a) $K_Y = \pi^* K_X + (-1 + 2/d)E$, where E is the unique π-exceptional divisor and hence X is ϵ-klt for $\epsilon = 1/d$;
(b) X is $\mathbb{Q}$-factorial of Picard number one and $-K_X \sim_{\mathbb{Q}} (d+2)l$ is an ample $\mathbb{Q}$-Cartier divisor, where l is the class of a ruling of X. Hence X is an ϵ-klt del Pezzo surface;
(c) $\mathrm{Vol}(-K_X) = d + 4 + 4/d$ is a linear function of $d = 1/\epsilon$ and provides the required example in Theorem A.

Let $p \in X$ be a general point. Then p is not the vertex O and the unique ruling l_p passing through p is the non-klt center of the log pair (X, l_p), that is, $l_p = \mathrm{Nklt}(X, l_p)$. Moreover, the proper transform F_p of l_p on Y is a fiber of the $\mathbb{P}^1$-bundle $f : Y \to \mathbb{P}^1$. In this case, one interprets the $\mathbb{P}^1$-bundle structure of Y as a covering family of non-klt centers of X since the map $\pi : Y \to X$ is dominant.

For $p, q \in X$ two general points, let l_p and l_q be the rulings passing through p and q respectively. Consider the pair $K_Y + (1 - 2/d)E = \pi^* K_X$. By the Connectedness Lemma 1.3, the non-klt locus $\mathrm{Nklt}(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q))$ containing $F_p \cup F_q$ is connected as

$$-\,(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q)) = -\pi^*(K_X + l_p + l_q) \equiv d\pi^* l$$

is nef and big. In fact, the fibers F_p and F_q are connected by E in $\mathrm{Nklt}(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q))$ as

$$\begin{aligned} F_p \cup F_q &\subseteq \mathrm{Nklt}(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q)) \\ &\subseteq \pi^{-1}(\mathrm{Nklt}(K_X + l_p + l_q)) = F_p \cup F_q \cup E, \end{aligned}$$

where the second inclusion follows from the definition of non-klt centers. In particular,

$$\mathrm{mult}_E(\pi^*(l_p + l_q)) \geqslant \frac{2}{d} = 2\epsilon.$$

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By symmetry, $\pi^* l_p$ must contribute multiplicity at least $1/d = \epsilon$ to the component E (and in fact is exactly $1/d$ in this case), that is,

$$\pi^* l_p \geqslant \epsilon E. \tag{2.1}$$

Note that

$$l_p \sim_{\mathbb{Q}} \frac{-K_X}{\sqrt{d} \cdot \sqrt{\mathrm{Vol}(X)}}. \tag{2.2}$$

By intersecting both sides of (2.1) with a general fiber F of $f : Y \to \mathbb{P}^1$, we get for the ruling $l = \pi_*(F)$,

$$\frac{1}{\sqrt{d} \cdot \sqrt{\mathrm{Vol}(X)}} \deg_l(-K_X) = \pi^* l_p.F \geqslant \epsilon E.F. \tag{2.3}$$

Since F is a general fiber meeting the horizontal divisor E at a smooth point, $E.F \geqslant 1$. (In this case $E.F = 1$.) Combining all of these, we obtain a lower bound of the restricted volume $\deg_l(-K_X)$,

$$\deg_l(-K_X) \geqslant \epsilon \sqrt{d} \cdot \sqrt{\mathrm{Vol}(X)}.$$

Note that since in this case $\deg_l(-K_X) = -K_X.l = -K_Y.\pi^* l \leqslant 2$, it follows that the anticanonical volume $\mathrm{Vol}(X) = K_X^2 \leqslant 4d = 4/\epsilon$.

In summary, the method of getting an upper bound of the anti-canonical volumes is to obtain a lower bound of the restricted volume $\mathrm{Vol}(-(K_X + \Delta)|_{V_p})$ on the non-klt center V_p. This is outlined in the following steps:

- Suppose that $\mathrm{Vol}(X, \Delta) = (-(K_X + \Delta))^n > (\omega n)^n$ for a positive rational number ω. We will show that $\omega > 0$ cannot be arbitrarily large.
- For general $p \in X$, choose

$$\Delta_p \sim_{\mathbb{Q}} \frac{-(K_X + \Delta)}{\omega}$$

  so that $p \in \mathrm{Nklt}(X, \Delta + \Delta_p)$. Let $V_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$ be the minimal non-klt center containing p.
- Construct covering families of non-klt centers by "lining up" (part of the) non-klt centers $\{V_p\}$; see Section 3. This is the generalization of the $\mathbb{P}^1$-bundle structure in the Example 2.1 and is called the *covering families of tigers* in [M^cK02].

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- Use the Connectedness Lemma 1.3 to obtain a lower bound of the restricted volume

$$\mathrm{Vol}(-(K_X+\Delta)|_{V_p}) = (-(K_X+\Delta)|_{V_p})^{\dim V_p}$$

  on the non-klt center V_p in terms of ω and ϵ. This is the most technical part.
- If $\omega \gg 0$, then we cut down the dimension of non-klt centers by Theorem 1.5. After finitely many steps, we get isolated non-klt centers and hence a contradiction to the Connectedness Lemma 1.3.

The difficulty of this argument arises in dimensions three in many places. First of all, the non-klt centers can be of dimension one or two and we have to deal them separately. When we have one-dimensional covering families of tigers, it is subtle to detect the contribution of the ϵ-klt condition from some horizontal subvariety, which is the analogue to the exceptional curve E in the Example 2.1. This is done by applying a differentiation argument to construct a better behaved covering family of tigers; see 5.3. In case we have two-dimensional non-klt centers, complications arise for computing intersection numbers as the total space Y of a covering family of tigers is in general not $\mathbb{Q}$-factorial. This can be fixed by replacing Y with a suitable birational model. To finish the proof, we also need to run a relative minimal model on the covering families of tigers and study the geometry of all possible outcomes.

§3. Covering families of tigers

The main reference for this section is [M^cK02].

Definition 3.1. [M^cK02, Definition 3.1] Let (X,Δ) be a log pair with X projective and D a $\mathbb{Q}$-Cartier divisor. We say that pairs of the form (Δ_t, V_t) form a **covering family of tigers** of dimension k and weight ω if

(1) there is a projective morphism $f: Y \to B$ of normal projective varieties such that the general fiber of f over $t \in B$ is V_t;
(2) there is a morphism of B to the Hilbert scheme of X such that B is the normalization of its image and f is obtained by taking the normalization of the universal family;
(3) If $\pi: Y \to X$ is the natural morphism, then $\pi(V_t)$ is a pure non-klt center of $K_X + \Delta + \Delta_t$;
(4) π is generically finite and dominant;

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(5) $\Delta_t \sim_{\mathbb{Q}} D/\omega$, where Δ_t is effective;
(6) the dimension of V_t is k.

Note that by definition $k \leqslant \dim X - 1$ and $\pi|_{V_t} : V_t \to \pi(V_t)$ is *finite* and *birational*. A covering family of tigers is illustrated in the following diagram:

$$\begin{tikzcd} X & Y \arrow[l, "\pi"'] \arrow[d, "f"'] & \supseteq & V_t \arrow[d] \\ & B & \ni & t. \end{tikzcd}$$

We will sometimes also refer V_p as a pure non-klt center of $(X, \Delta + \Delta_p)$ containing p.

For (X, Δ) a log $\mathbb{Q}$-Fano variety, we will always assume that $D = -\lambda(K_X + \Delta)$ for some $\lambda > 0$. In particular, D is big and semiample.

The existence of a covering family of tigers is achieved by constructing non-klt centers at general points of X and then fitting a subcollection of them into a family. In order to fit the non-klt centers into a family, we use exceptional non-klt centers where we patch up the unique non-klt place associated to each of them; see Lemma 3.3. The following lemma allows us to create exceptional non-klt centers.

LEMMA 3.2. *Let (X, Δ) be a log pair and let D be a big and semiample $\mathbb{Q}$-Cartier divisor. Write $D^n > (\omega n)^n$ for some positive rational number ω. For every $p \in X_{\mathrm{sm}}$, we can find an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ such that the unique minimal non-klt center $V_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$ containing p is exceptional.*

Proof. Fix a rational number $0 < \lambda < 1$. By Lemma 1.2, for any $p \in X_{\mathrm{sm}}$ we can find an effective divisor $\Delta'_p \sim_{\mathbb{Q}} \frac{D}{\lambda\omega}$ such that $\mathrm{mult}_p \Delta'_p \geqslant n$ and hence $p \in \mathrm{Nklt}(X, \Delta + \Delta'_p)$.

Fix $p \in X_{\mathrm{sm}}$, pick $0 < \delta_p \leqslant 1$ the unique rational number such that $(X, \Delta + \delta_p \Delta'_p)$ is log canonical but not klt at p. By [Amb98, Proposition 3.2], we can find an effective divisor $M_p \sim_{\mathbb{Q}} D$ and some rational number $a > 0$ such that for any rational number $0 < \mu < 1$, the pair $(X, (1-\mu)(\Delta + \delta_p \Delta'_p) + \mu\Delta + \mu a M_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. If we write

$$\Delta_p := (1-\mu)\delta_p \Delta'_p + \mu a M_p \sim_{\mathbb{Q}} \frac{1}{\omega'_p} D,$$

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then

$$\omega'_p = \frac{\omega}{(1-\mu)\delta_p/\lambda + \mu a\omega}$$

satisfies

$$(1-\mu)\delta_p/\lambda + \mu a\omega < 1$$

for $\delta_p \approx \lambda$ and $0 < \mu \ll 1$. Hence $\omega'_p > \omega$. Since D is semiample, we can add a small multiple of D to Δ_p so that $\Delta_p \sim_{\mathbb{Q}} D/\omega$ is the required divisor. □

For a topological space X, we say that a subset P is *countably dense* if P is not contained in the union of countably many proper closed subsets of X. The following proposition is the construction of the covering family of tigers; see also [McK02, Lemma 3.2] or [Tod07, Lemma 3.2].

PROPOSITION 3.3. *Let (X, Δ) and Δ_p be the same as in Lemma 3.2. Then there exists a covering family of tigers $\pi : Y \to X$ of weight ω with $p \in V_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$.*

Proof. For each $p \in X_{\mathrm{sm}}$, there is an integer m_p such that $m_p\Delta_p$ and m_pD/ω are linearly equivalent Cartier divisors. For $m > 0$ such that mD/ω is Cartier, set $C_m := \{p \in X_{\mathrm{sm}} | m\Delta_p \in |mD/\omega|\}$. Then by construction $X = \cup_m \overline{C_m}$, where the union is taken over all $m > 0$ such that mD/ω is Cartier. Since X is countably dense, there is an integer $m > 0$ such that $\overline{C_m} = X$.

Choose $m > 0$ an integer as above so that mD/ω is integral Cartier and $\overline{C_m} = X$. Let B be the Zariski closure of the set of points $\{m\Delta_p | p \in X_{\mathrm{sm}}\} \subseteq |mD/\omega|$ in the projective space $|mD/\omega|$. Replace B by an irreducible component which contains an uncountable subset Q of B such that the set $\{p \in X | \Delta_p \in Q\}$ is dense in X. This is possible since $\overline{C_m} = X$ and X is countably dense. Let $H \subseteq X \times |mD/\omega|$ be the universal family of divisors defined by the incidence relation and $H_B \to B$ the restriction to B. Take a log resolution of $H_B \subseteq X \times B$ over the generic point of B and extend it over an open subset U of B. By assumption the log resolution over the generic point of B has a unique exceptional divisor of log discrepancy zero since it is true over $Q \subseteq B$. Let Y be the image of this unique exceptional divisor in $X \times B$ with the natural projection map $\pi : Y \to X$. By construction $\pi : Y \to X$ dominates X.

Possibly taking a finite cover of B and passing to an open subset of B, we may assume that any fiber V_t of $f : Y \to B$ over $t \in B$ is a non-klt center of $K_X + \Delta + \Delta_t$. Possibly passing to an open subset of B, we may assume that $f: Y \to B$ is flat and B maps into the Hilbert scheme. Replace B by the normalization of the closure of its image in the Hilbert scheme and Y by the normalization of the pullback of the universal family. After possibly cutting by hyperplanes in B, we may assume that π is generically finite and dominant. The resulted family is a required covering family of tigers. $\square$

In fact, the original construction of covering families of tigers is carried out in a more general setting.

COROLLARY 3.4. *Let (X, Δ) be an n-dimensional log pair and let D be a big $\mathbb{Q}$-Cartier divisor. Let ω be a positive rational number such that $\operatorname{Vol}(D) > (\omega n)^n$. Let P be a countably dense subset of X. Suppose that for every point $p \in P$ we may find a pair (Δ_p, V_p) such that V_p is a pure non-klt center of $K_X + \Delta + \Delta_p$ where $\Delta_p \sim_{\mathbb{Q}} D/\omega_p$ for some $\omega_p > \omega$. Then we may find a covering family of tigers of weight ω together with a countably dense subset Q of P such that for all $q \in Q$, V_q is a fiber of π.*

Proof. See [McK02, Lemma 3.2] or [Tod07, Lemma 3.2]. $\square$

The following lemma shows that we can assume the covering families of tigers under our consideration are always positive dimensional as suggested in the guiding example 2.1.

LEMMA 3.5. *Let (X, Δ) be a projective klt pair and $D = -(K_X + \Delta)$ be a big and nef $\mathbb{Q}$-Cartier divisor. Then a covering family of tigers (Δ_t, V_t) of weight $\omega > 2$ is positive dimensional, that is, $k = \dim V_t > 0$.*

Proof. This is [McK02, Lemma 3.4] and we include the proof for convenience of the reader. Suppose that there exists a zero-dimensional covering family of tigers of weight $\omega > 2$. For p_1 and p_2 general, there are divisors Δ_1 and Δ_2 with $\Delta_i \sim_{\mathbb{Q}} D/\omega$ such that p_i is an isolated non-klt center of $K_X + \Delta + \Delta_i$. As p_1 and p_2 are general, it follows that Δ_2 does not contain p_1 and $\operatorname{Nklt}(X, \Delta + \Delta_1 + \Delta_2)$ contains p_1 and p_2 as disconnected non-klt centers. But $-(K_X + \Delta + \Delta_1 + \Delta_2) \sim \left(1 - \frac{2}{\omega}\right) D$ is nef and big if $\omega > 2$. This contradicts to Lemma 1.3. $\square$

Recall that we want to obtain a lower bound of restricted volumes on the non-klt centers by studying the associated covering families of tigers so that we can cut down the dimension via Theorem 1.5 to get isolated non-klt centers. If the new non-klt centers after cutting down the dimension are still positive dimensional, then we have to create new covering families of tigers and repeat the process. The following proposition due to McKernan, [M^cK02, Lemma 5.3], enables us to create covering families of tigers of new non-klt centers after cutting down the dimension. We include the proof here for the convenience of the readers. It starts with two lemmas.

LEMMA 3.6. *Let $\pi : Y \to X$ be a smooth morphism of smooth varieties. Let Δ_1 be a $\mathbb{Q}$-divisor on X and let Γ_1 be the pullback of Δ_1 to Y. Let Γ_2 be a boundary on Y, such that the support B of Γ_2 dominates X and $\pi|_B$ is smooth. Then (X, Δ_1) is log canonical (respectively Kawamata log terminal, etc.) if and only if $(Y, \Gamma = \Gamma_1 + \Gamma_2)$ is log canonical (respectively Kawamata log terminal, etc.).*

Proof. The property of being log canonical is local in the analytic topology. On the other hand, locally in the analytic topology, Y is a product $X_1 \times X_2$, where X_1 is isomorphic to X and Γ_2 is the pullback of a divisor Δ_2 on X_2 whose support is smooth, so that $\Gamma = \Delta_1 \times X_2 + X_1 \times \Delta_2$. The result follows from the same computation as in [Kol97, Proposition 8.21]. □

LEMMA 3.7. *Let (X, Δ) be a log pair where X is projective and Δ is effective. Suppose that V is an exceptional log canonical center of $K_X + \Delta$. Then there is an open subset U of the smooth locus of V with the following property:*

For all divisors Θ on X, which do not contain the generic point of V and subvarieties W of V such that $W|_U$ is a pure log canonical center of $K_U + \Theta|_U$, then W is a pure log canonical center of $K_X + \Delta + \Theta$.

Proof. This result is local about the generic point of V so we are free to replace X by any open set that contains the generic point of V. The idea is to reduce to the case of a divisor, when the result becomes an easy consequence of inversion of adjunction. Pick a log resolution $\pi : Y \to X$ of the pair (X, Δ) and let Γ be the divisor defined by $K_Y + \Gamma = \pi^*(K_X + \Delta)$. By assumption there is a unique divisor E lying over V of log discrepancy zero. Let $f : E \to V$ be the restriction of π to E. Replacing X by an open subset, we may assume that f and V are both smooth, and that K_V and $K_X + \Delta$ are $\mathbb{Q}$-linearly equivalent to zero. Passing to a smaller open set of X, we can assume that $\Gamma \geqslant 0$. By adjunction we may write $(K_Y + \Gamma)|_E = K_E + B$ for some effective divisor B, where both sides are $\mathbb{Q}$-linearly equivalent to zero. Passing to a smaller open set again, we may assume that every component of B dominates V and that $f|_B$ is smooth. Then

$$K_E + B = (K_Y + \Gamma)|_E = \pi^*(K_X + \Delta)|_E = f^*((K_X + \Delta)|_V) \sim_{\mathbb{Q}} f^*(K_V).$$

Suppose that W is a pure log canonical center of $K_V + \Theta|_V$. Set $\Theta' = \pi^*\Theta$. As Θ does not contain the generic point of V, E is not a component of Θ', so that $f^*(\Theta|_V) = \Theta'|_E$. It follows by Lemma 3.6 that $f^{-1}(W)$ is a pure log canonical center of $K_E + \Theta'|_E$.

The result now follows from the inversion of subadjunction (cf., (17.1.1), (17.6) and (17.7) of [Kol92]). ∎

PROPOSITION 3.8. *Let (X, Δ) be a log pair and let D be a $\mathbb{Q}$-Cartier divisor of the form $A + E$ where A is ample and E is effective. Let (Δ_t, V_t) be a covering family of tigers of weight ω and dimension k. Let A_t be $A|_{V_t}$. Suppose that there is an open subset $U \subseteq B$ such that for all $t \in U$ we may find a covering family of tigers $(\Gamma_{t,s}, W_{t,s})$ on V_t of weight ω' with respect to A_t. Then for (X, Δ) we can find a covering family of tigers (Γ_s, W_s) of dimension less than k and weight*

$$\omega'' = \frac{1}{1/\omega + 1/\omega'} = \frac{\omega\omega'}{\omega + \omega'}.$$

Proof. Pick r so that rA is Cartier and let $L = \mathcal{O}_X(rA)$ be the corresponding line bundle. Note that by Serre vanishing $\mathrm{H}^1(X, \mathcal{I}_V \otimes L^{\otimes m}) = 0$ for m large enough. Hence we may lift $\Gamma_{t,s}$ to a $\mathbb{Q}$-divisor $\Theta_{t,s}$ on X $\mathbb{Q}$-linearly equivalent to the same multiple of A. Adding on a multiple of E we may assume that $\Theta_{t,s}$ is in fact a multiple of D. Since adding E only effect a proper closed subset of X, there is a countably dense collection (Δ_t, V_t) satisfying the hypothesis of Lemma 3.7. Thus by Corollary 3.4 applied to $(\Delta_t + \Theta_{t,s}, W_{t,s})$, we may find a covering family of tigers (Γ_s, W_s) of weight ω''. ∎

We will apply Proposition 3.8 with the ample divisor $D = -(K_X + \Delta)$ and only in the case when we have a two-dimensional covering family of tigers $\pi : Y \to X$ with $\deg(\pi) > 1$.

In the process of obtaining lower bound of the restricted volume on the non-klt centers, if we have one-dimensional non-klt centers, then we can control the restricted volume of D (cf., [M^cK02, Lemma 5.3]).

COROLLARY 3.9. *Let (X, Δ) be a log pair and let D be an ample divisor. Let (Δ_t, V_t) be a covering family of tigers of weight $\omega > 2$ and dimension one. Then $\deg(D|_{V_t}) \leqslant 2\omega/(\omega - 2)$.*

Proof. Suppose that $\deg(D|_{V_t}) > 2\omega/(\omega - 2)$, then by Lemma 3.2, Theorem 1.5, and Corollary 3.4 we may find a covering family $(\Gamma_{t,s}, W_{s,t})$ of

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tigers of weight $\omega' > 2\omega/(\omega - 2)$ and dimension zero on V_t. By Proposition 3.8, there exists a covering family of tigers of dimension zero and weight

$$\omega'' = \frac{\omega\omega'}{\omega + \omega'} > 2$$

for X. This contradicts Lemma 3.5. ∎

§4. Log Del Pezzo surfaces

Let (X, Δ) be an ϵ-klt weak log del Pezzo surface. The minimal resolution $\pi : Y \to X$ of (X, Δ) is the unique proper birational morphism such that Y is a smooth projective surface and $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ for some effective $\mathbb{Q}$-divisor Δ_Y on Y. Note that minimal resolutions always exist for two-dimensional log pairs. It is easy to see that (Y, Δ_Y) is also an ϵ-klt weak log del Pezzo surface with volume

$$\mathrm{Vol}(Y, \Delta_Y) = (K_Y + \Delta_Y)^2 = (K_X + \Delta_X)^2 = \mathrm{Vol}(X, \Delta_X).$$

Hence replacing (X, Δ) by its minimal resolution, we can assume that X is smooth.

Write $(K_X + \Delta)^2 > (2\omega)^2$. For a general point $p \in X$, let $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ be an effective $\mathbb{Q}$-divisor constructed from Lemma 1.2 such that $p \in \mathrm{Nklt}(X, \Delta + \Delta_p)$. Assume that $\omega > 2$, then by Lemma 3.5 the unique minimal non-klt center F_p of $(X, \Delta + \Delta_p)$ containing p is one-dimensional. Note that for general $p \in X$, $F_p \leqslant \Delta_p$.

LEMMA 4.1. *For a very general point $p \in X$, the numerical class $F := F_p$ on X is well defined and F is nef.*

Proof. An effective integral one-cycles F_p satisfies $F_p \leqslant \Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ and hence forms a bounded set in the Mori cone of curves. As $\mathbb{C}$ is uncountable, for $p \in X$ a very general point the numerical class $F := F_p$ is well defined. Since $\{F_p\}$ moves, the class F is nef. ∎

The following lemma shows that if we assume the weight ω is large, then the non-klt centers $\{F_p\}$ on X already possess a nearly fiber bundle structure analogous to a covering family of tigers.

LEMMA 4.2. *Assume that $\omega > 4$, then $F^2 = 0$, that is, $F_p \cap F_q = \emptyset$ for $p, q \in X$ two very general points.*

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Proof. Suppose that $\omega > 4$ and assume that $F_p \cap F_q \neq \emptyset$ for $p, q \in X$ two very general points. Since by Lemma 4.1 the curve class $F = F_p$ is nef, then for $H = -(K_X + \Delta)/\omega$ we have

$$1 \leqslant F_p.F_q = F.F_q \leqslant F.\Delta_q = \deg(H|_{F_q}).$$

Since H is big and nef, one can cut down the dimension of the non-klt centers by Theorem 1.5.[2]

To be precise, choose x an intersection point of F_p and F_q. Pick $0 < \delta_1 \leqslant 1$ such that $(X, \Delta + \delta_1\Delta_p)$ is log canonical at x. If $(X, \Delta + \delta_1\Delta_p) = \{x\}$, then this contradicts the Connected Lemma 1.3 as $\mathrm{Nklt}(X, \Delta + \delta_1\Delta_p + \Delta_q)$ containing x and F_q is disconnected but $-(K_X + \Delta + \delta_1\Delta_p + \Delta_q)$ is nef and big. Hence we may assume that $\mathrm{Nklt}(X, \Delta + \delta_1\Delta_p)$ is one-dimensional. By Theorem 1.5, there exists rational numbers $0 < \delta \ll 1$, $0 < c < 1$, and an effective $\mathbb{Q}$-divisor $B_x \equiv H$ such that $\mathrm{Nklt}(X, \Delta + (1-\delta)\delta_1\Delta_p + cB_x) = \{x\}$ in a neighborhood of x. By switching p and q, we can assume that $q \neq x$. Similarly we have $\mathrm{Nklt}(X, \Delta + (1-\delta')\delta_2\Delta_q + c'B_q) = \{q\}$ in a neighborhood of q for some divisor $B_q \equiv H$ and rational numbers $0 < \delta_2 \leqslant 1$, $0 < \delta' \ll 1$, and $0 < c' < 1$. The set $\mathrm{Nklt}(X, \Delta + (1-\delta)\delta_1\Delta_p + cB_x + (1-\delta')\delta_2\Delta_q + c'B_q)$ contains isolated non-klt centers x and q and hence is disconnected. Since the pair is numerically equivalent to $\big(1 - \frac{(1-\delta)\delta_1 + c + (1-\delta')\delta_2 + c'}{\omega}\big)(-(K_X + \Delta))$ which is nef and big if $\omega > 4$, we have a contradiction to the Connectedness Lemma 1.3. ∎

Recall that for any two Weil divisors $A = \sum a_i D_i$ and $B = \sum b_i D_i$, we define $A \wedge B$ to be

$$A \wedge B = \sum_i \min\{a_i, b_i\} D_i.$$

THEOREM 4.3. *Let (X, Δ) be an ϵ-klt weak log del Pezzo surface. Then the anticanonical volume* $\mathrm{Vol}(X, \Delta) = (K_X + \Delta)^2$ *satisfies*

$$(K_X + \Delta)^2 \leqslant \max\left\{64, \frac{8}{\epsilon}\right\}.$$

Proof. Replacing (X, Δ) by its minimal resolution, we may assume that X is smooth. Write $(K_X + \Delta)^2 > (2\omega)^2$, then for each general point

[2]By adding a small multiple of $-(K_X + \Delta)$, we may assume that the inequality $\deg(H|_{F_q}) \geqslant 1$ is strict with a smaller modified ω and hence Theorem 1.5 applies.

$p \in X$ there exists an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -(K_X+\Delta)/\omega$ such that $p \in \mathrm{Nklt}(X, \Delta+\Delta_p)$. From Lemma 3.5, we may assume that $\omega > 2$ and the unique minimal non-klt center $F_p \subseteq \mathrm{Nklt}(X, \Delta+\Delta_p)$ containing p is one-dimensional. Note that $F_p \leqslant \Delta_p$ for general $p \in X$. By Lemmas 4.1 and 4.2, we may assume that $\omega > 4$ and for very general $p \in X$ the numerical class F of F_p is well defined and nef with $F^2 = 0$.

For two very general points $p, q \in X$, $\Delta_p.\Delta_q > 0$ and hence $F_p = \mathrm{Supp}(F_p) \subsetneqq \mathrm{Supp}(\Delta_p)$: Otherwise $\Delta_q \equiv \Delta_p \leqslant NF_p$ for some $N > 0$ and $0 < \Delta_p.\Delta_q \leqslant N^2F_p^2 = N^2F^2 = 0$, a contradiction. Denote $E_p = \mathrm{Supp}(\Delta_p) - F_p \neq 0$. By the Connectedness Lemma 1.3, $\mathrm{Nklt}(X, \Delta+\Delta_p+\Delta_q) \supseteq F_p \cup F_q$ is connected. By Lemma 4.2, $F_p \cap F_q = \emptyset$ and hence E_p must contain a connected curve $E \leqslant E_p$ such that $F_p.E \neq 0$, $F_q.E \neq 0$, and $\mathrm{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q \cup E$. Furthermore, we can assume that E is irreducible since $E.F_q \neq 0$ as $F_q \equiv F_p$ for $q \in X$ a very general point. By symmetry and the ϵ-klt condition, E satisfies $\frac{\epsilon}{2}E \leqslant \Delta_p$ (cf., Example 2.1).

Suppose that $E^2 \geqslant 0$, then as $H = -(K_X+\Delta)/\omega$ is nef

$$H.E = \Delta_p.E = (\Delta'_p.E + a'_E E^2) \geqslant F_p.E \geqslant 1$$

where we write $\Delta_p = \Delta'_p + a'_E E$ with $\Delta'_p \wedge E = 0$, $\Delta'_p \geqslant F_p$, and $a'_E > 0$. In particular, we can use Theorem 1.5 again to cut down the dimension of non-klt centers as in Lemma 4.2. This contradicts the Connectedness Lemma 1.3.

We may assume that $E^2 < 0$, and thus

$$\begin{aligned} -2 \leqslant 2g_a(E) - 2 &= (K_X+E).E \\ &= (K_X+\Delta).E + (1-\epsilon-a_E)E^2 - \Delta'.E + \epsilon E^2 \leqslant \epsilon E^2 \end{aligned}$$

where $\Delta = \Delta' + a_E E$ with $\Delta' \wedge E = 0$ and $a_E \in [\,0, 1-\epsilon\,)$ by the ϵ-klt condition. This implies that

$$1 \leqslant -E^2 \leqslant 2/\epsilon$$

where the first inequality follows from the fact that $E^2 \in \mathbb{Z}$ as X is smooth. Since $F^2 = 0$ for F the numerical class of F_p, where $p \in X$ is very general, by Nakai's criterion the divisor $H_s = F + sE$ with $0 < s \leqslant 1/(-E^2)$ is nef and big. By the Hodge index theorem (see [Har77, V 1.1.9(a)]), we get the inequality

$$(K_X+\Delta)^2 \leqslant \frac{(-(K_X+\Delta).H_s)^2}{H_s^2}. \tag{1}$$

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From $\Delta.F \geqslant 0$ and $F^2 = 0$, we have that

$$-(K_X + \Delta).F \leqslant -(K_X + F).F \leqslant 2. \tag{2}$$

Also for $\Delta = \Delta' + a_E E$ with $\Delta' \wedge E = 0$ and $a_E \in [\,0, 1 - \epsilon\,)$, we have that

$$\begin{aligned} -(K_X + \Delta).E &= -K_X.E - \Delta'.E - a_E E^2 \\ &\leqslant E^2 + 2 - a_E E^2 = (a_E - 1)(-E^2) + 2 \leqslant 2 - \epsilon(-E^2). \end{aligned} \tag{3}$$

Put $s = 1/(-E^2)$, all together we get

$$\begin{aligned} (K_X + \Delta)^2 &\leqslant \frac{(-(K_X + \Delta).(F + sE))^2}{H_s^2} \\ &\leqslant \frac{(2 + s(2 - \epsilon(-E^2)))^2}{2sE.F + s^2E^2} \\ &\leqslant (-E^2)\left(2 - \epsilon + \frac{2}{-E^2}\right)^2 \\ &= (-E^2)(2 - \epsilon)^2 + 4(2 - \epsilon) + \frac{4}{-E^2} \\ &\leqslant \frac{8}{\epsilon} \end{aligned}$$

where the first inequality is (1), the second inequality follows from (2), (3), and $F^2 = 0$, and the third inequality is given by $E.F \geqslant 1$, and thus $2sE \cdot F + s^2E^2 \geqslant 2s + s^2E^2 = 1/(-E^2)$. The last inequality follows from the fact that the function $f(x) = (2 - \epsilon)^2 x + 4/x$, defined on $1 \leqslant x \leqslant 2/\epsilon$, achieves its maximum at $x = 2/\epsilon$. ∎

REMARK 4.4. Note that by applying Corollary 3.9 one can only obtain an upper bound of order $1/\epsilon^2$. Hence Theorem 4.3 is a nontrivial result.

§5. Log Fano threefolds of Picard number one

Let (X, Δ) be an ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefold of Picard number $\rho(X) = 1$. Note that by hypothesis X is ϵ-klt and $-K_X$ is ample with $-K_X^3 \geqslant \mathrm{Vol}(X, \Delta) = -(K_X + \Delta)^3$. Hence it is sufficient to assume that X is an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number $\rho(X) = 1$ and to find an upper bound of $\mathrm{Vol}(X) = -K_X^3$.

We will obtain an upper bound of the volumes by studying covering families of tigers. The weight of any covering families of tigers in our study will always be the weight with respect to $-K_X$.

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Let X be an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number $\rho(X)=1$ and write $\mathrm{Vol}(X)=-K_X^3>(3\omega)^3$ for some positive rational number ω. Denote $D=-2K_X$, we have $D^3>(6\omega)^3$. By Lemma 1.2, we can fix an affine open subset $U\subseteq X$ such that for each $p\in U$ there exists an effective divisor $\Delta_p\sim_{\mathbb{Q}} D/\omega$ with $\mathrm{mult}_p\Delta_p\geqslant 6$. We do not assume that Δ_p creates exceptional non-klt centers as in Lemma 3.2, otherwise we lose control on multiplicity. We pick divisors Δ_p's in the following systematic way so that we can control their multiplicities uniformly.

5.1 Construction

Let $\Delta_U\subseteq U\times U$ be the diagonal and $\mathcal{I}_{\mathcal{Z}}$ be the ideal sheaf of $\mathcal{Z}=\overline{\Delta_U}\subseteq X\times U$. For each $p\in U$, by the existence of $\mathbb{Q}$-divisor $\Delta_p\sim_{\mathbb{Q}} D/\omega$ with $\mathrm{mult}_p\Delta_p\geqslant 6$, there exists $m_p>0$ such that $L_{m_p}=m_pD/\omega$ is Cartier and $H^0(X,L_{m_p}\otimes\mathcal{I}_p^{\otimes 6m_p})\neq 0$, where $\mathcal{I}_p$ is the ideal sheaf of $p\in U$. In particular, we can write $U=\cup U_m$ where $m>0$ runs through all sufficiently divisible integers such that $L_m=mD/\omega$ is Cartier and $U_m=\{p\in U|H^0(X,L_m\otimes\mathcal{I}_p^{\otimes 6m})\neq 0\}$. In particular, each $\overline{U_m}$ is closed in X and $X=\cup\overline{U_m}$. Since the base field $\mathbb{C}$ is uncountable, X cannot be a countable union of proper closed subsets and there exists some $m>0$ such that U_m is dense in X.

Fix an $m>0$ so that $L_m=mD/\omega$ is Cartier and $U_m=\{p\in U|H^0(X,L_m\otimes\mathcal{I}_p^{\otimes 6m})\neq 0\}$ is dense in X. Denote $\mathrm{pr}_X: X\times U\to X$ and $\mathrm{pr}_U: X\times U\to U$ the projection maps. Since $\mathrm{pr}_U: X\times U\to U$ is flat, by [Har77, III 12.11] after restricting to a smaller open affine subset of U, we can assume that the map

$$(\mathrm{pr}_U)_*(\mathrm{pr}_X^*L_m\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m})\otimes\mathbb{C}(p)\to H^0(X,L_m\otimes\mathcal{I}_p^{\otimes 6m})$$

is isomorphic for each $p\in U$. Since U_m is dense in U, the sheaf $(\mathrm{pr}_U)_*(\mathrm{pr}_X^*L_m\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m})\neq 0$ on U and hence $H^0(X\times U,\mathrm{pr}_X^*L_m\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m})\neq 0$ as U is affine. Let $s\in H^0(X\times U,\mathrm{pr}_X^*L_m\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a nonzero section with $F=\mathrm{div}(s)$ the corresponding divisor on $X\times U$. For each $p\in U$, denote $F_p=F\cap(X\times\{p\})$ the associated divisor on $X\cong X\times\{p\}$. Since $\mathrm{mult}_{\mathcal{Z}}(F)\geqslant 6m$, by Lemma 5.1 below the $\mathbb{Q}$-divisor $\Delta_p=F_p/m\sim_{\mathbb{Q}} D/\omega$ on X satisfies $\mathrm{mult}_p\Delta_p\geqslant 6$ for general $p\in U$.

LEMMA 5.1. [Laz04, Lemma 5.2.11] *Let $g: M\to T$ be a morphism of smooth varieties, and suppose that $\mathcal{Z}\subseteq M$ is an irreducible subvariety dominating T:*

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$$\begin{tikzcd} \mathcal{Z} \arrow[dr] & \subseteq & M \arrow[dl, "g"] \\ & T. & \end{tikzcd}$$

Let $F \subseteq M$ be an effective divisor. Then for a general point $t \in T$ and any irreducible component $\mathcal{Z}'_t \subseteq \mathcal{Z}_t$,

$$\operatorname{mult}_{\mathcal{Z}'_t}(M_t, F_t) = \operatorname{mult}_{\mathcal{Z}}(M, F)$$

where $\operatorname{mult}_{\mathcal{Z}}(M, F)$ *is the multiplicity of the divisor F on M along a general point of the irreducible subvariety $\mathcal{Z} \subseteq M$ and similarly for* $\operatorname{mult}_{\mathcal{Z}'_t}(M_t, F_t)$.

For a given collection of $\mathbb{Q}$-divisors $\{\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega | p \in U \text{ general}\}$ associated to a nonzero section in $H^0(X \times U, \mathrm{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ as above, by Lemma 3.2 we can modify Δ_p's such that the unique non-klt centers $V_p \subseteq \mathrm{Nklt}(X, \Delta_p)$ passing through p are exceptional. By Lemma 3.3 (or in general Corollary 3.4), we can construct covering families of tigers from these divisors.

In order to obtain an upper bound of ω, which is enough for bounding the volumes, we will pick up a "well-behaved" nonzero section $s \in H^0(X \times U, pr_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ and study the corresponding covering families of tigers.

5.2 Cases

By 5.1, there exists an open affine subset $U \subseteq X$ and an integer $m > 0$ such that $H^0(X \times U, \mathrm{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. Let $s \in H^0(X \times U, \mathrm{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a nonzero section with divisor $F = \mathrm{div}(s)$ on $X \times U$ and $\{\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega | p \in U \text{ general}\}$ be the associated collection of $\mathbb{Q}$-divisors. We consider two cases:

(1) (*Small multiplicity*) For each irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$, $\mathrm{mult}_{\mathcal{W}}(F) \leqslant 3m$, that is, for general $p \in U$ we have $\mathrm{mult}_W(\Delta_p) \leqslant 3$ for any irreducible component W of $\mathrm{Supp}(\Delta_p)$ passing through p. After differentiating F, we will construct a "well-behaved" covering family of tigers of dimension one. We will derive an upper bound of ω by studying this well-behaved covering family of tigers. See Section 5.3.

(2) (*Big multiplicity*) There exists an irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$ with multiplicity $\mathrm{mult}_{\mathcal{W}}(F) > 3m$, that is, for

general $p \in U$ we have $\operatorname{mult}_W(\Delta_p) > 3$ for an irreducible component W of $\operatorname{Supp}(\Delta_p)$ passing through p. We will construct a covering family of tigers of dimension two and derive an upper bound of ω by studying the geometry of this covering family of tigers. See Section 5.4.

To pick up a "well-behaved" nonzero section in $H^0(X \times U, \operatorname{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$, we will apply the following proposition.

PROPOSITION 5.2. [Laz04, Proposition 5.2.13] *Let X and U be smooth irreducible varieties, with U affine, and suppose that*

$$\mathcal{Z} \subseteq \mathcal{W} \subseteq X \times U$$

are irreducible subvarieties such that $\mathcal{W}$ dominates X. Fix a line bundle L on X, and suppose given on $X \times U$ a divisor $F \in |\operatorname{pr}_X^*(L)|$. Write

$$l = \operatorname{mult}_{\mathcal{Z}}(F), \qquad k = \operatorname{mult}_{\mathcal{W}}(F).$$

Then after differentiating in the parameter directions, there exists a divisor $F' \in |\operatorname{pr}_X^*(L)|$ on $X \times U$ with the property that

$$\operatorname{mult}_{\mathcal{Z}}(F') \geqslant l - k \quad \text{and} \quad \mathcal{W} \nsubseteq \operatorname{Supp}(F').$$

5.3 Small multiplicity

Let X be an ϵ-klt $\mathbb{Q}$-Fano threefold of Picard number one and write $\operatorname{Vol}(X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω. Denote $D = -2K_X$, we have $D^3 > (6\omega)^3$. From 5.1, there is an integer $m > 0$ such that $L = mD/\omega$ is Cartier and an open affine subset $U \subseteq X$ such that $H^0(X \times U, \operatorname{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. We fix a nonzero section $s \in H^0(X \times U, \operatorname{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ with $F = \operatorname{div}(s)$ on $X \times U$.

PROPOSITION 5.3. *With the above set up, assume that $\omega > 4$. Suppose that we are in the case where all the irreducible components $\mathcal{W}$ of* $\operatorname{Supp}(F)$ *passing through $\mathcal{Z}$ satisfy* $\operatorname{mult}_{\mathcal{W}}(F) \leqslant 3m$*, then we have $\omega < 8/\epsilon + 4$. In particular, there is an upper bound of the volume*

$$\operatorname{Vol}(X) = -K_X^3 \leqslant \left(\frac{24}{\epsilon} + 12\right)^3.$$

Proof. Let M be the maximum of $\operatorname{mult}_{\mathcal{W}}(F)$ amongst all the irreducible components $\mathcal{W}$ of $\operatorname{Supp}(F)$ passing through $\mathcal{Z}$, then $M \leqslant 3m$ from the hypothesis. For $\mathcal{W}$ of $\operatorname{Supp}(F)$ a fixed irreducible component passing

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through $\mathcal{Z}$, take M times differentiation of F by Proposition 5.2. Then we obtain a divisor $F' \in |\mathrm{pr}_X^*(L) \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ with the property that

$$\mathrm{mult}_{\mathcal{Z}}(F') \geqslant (6m - M) \geqslant 3m \quad \text{and} \quad \mathcal{W} \not\subseteq \mathrm{Supp}(F').$$

Since there are only finitely many irreducible components of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$ and we are taking a generic differentiation, indeed for a general divisor $F'' \in |\mathrm{pr}_X^*(L) \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ we have $\mathcal{W} \not\subseteq \mathrm{Supp}(F'')$ for any $\mathcal{W}$ an irreducible component of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$. In particular, the base locus $\mathrm{Bs}(|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|)$ contains no codimension one components in a neighborhood of $\mathcal{Z}$.

Let G be a general divisor in $|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ and $\Delta_p = G_p/m$ for $p \in U$ general be the corresponding $\mathbb{Q}$-divisors on X. It follows that $p \in \mathrm{Nklt}(K_X + \Delta_p)$ as $\mathrm{mult}_p \Delta_p \geqslant 3$. The minimal non-klt center $V_p \subseteq \mathrm{Nklt}(K_X + \Delta_p)$ passing through p must be positive dimensional by Lemma 3.5 as the weight of Δ_p is $\omega/2 > 2$. Also V_p can only be one-dimensional, as by construction $m \geqslant 2$, and hence

$$0 \leqslant \mathrm{mult}_W \Delta_p = \mathrm{mult}_w \Delta_p = 1/m < 1,$$

where W is any irreducible component of $\mathrm{Supp}(\Delta_p)$ and $w \in W$ is a general point.

Let $\pi : Y \to X$ with $f : Y \to B$ be a one-dimensional covering family of tigers of weight $\omega' \geqslant \omega/2$ constructed from Δ_p's by Lemma 3.2 and Proposition 3.3. By abusing of the notation, we still denote Δ_p's the divisors associated to this covering family of tigers.

Choose $p, q \in U \subseteq X$ general. By Lemma 1.3, the non-klt locus $\mathrm{Nklt}(\pi^*(K_X + \Delta_p + \Delta_q)) \supseteq V_p \cup V_q$ on Y is connected and there is a one-cycle $C_{p,q}$ connecting V_p and V_q. Since Y is normal, an irreducible component C of $C_{p,q}$ intersecting V_q satisfies $C \cap Y_{\mathrm{sm}} \neq \emptyset$ for $p, q \in X$ general. Since C is in $\mathrm{Nklt}(\pi^*(K_X + \Delta_p + \Delta_q))$, by symmetry we have $\mathrm{mult}_C(\pi^*(\Delta_p)) \geqslant \epsilon/2$. Choose a general point $b \in f(C)$, then $Y_b = f^{-1}(b)$ is a general fiber of $f : Y \to B$ and one has

$$\frac{2}{\frac{\omega}{2} - 2} \geqslant \frac{2}{\omega}(-K_X.V_t) = \pi^*\Delta_p.Y_b \geqslant \frac{\epsilon}{2}$$

where the first inequality follows from Corollary 3.9. For the second inequality, note that we can make $0 \leqslant \mathrm{mult}_W \Delta_p = 1/m \ll 1$ for W any irreducible component of $\mathrm{Supp}(\Delta_p)$. In fact, in 5.2 we can start with the

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section $s^N \in H^0(X \times U, pr_X^* L_{mN} \otimes \mathcal{I}_p^{\otimes 6mN})$ for $N \gg 0$. The dichotomy into small and big multiplicities are the same and the above argument still goes through.

Hence $\omega \leqslant 8/\epsilon + 4$. ∎

REMARK 5.4. The difficulty here when we have covering families of tigers of dimension one is that in general the one-cycle C as in the above proof might be contained in $\mathrm{Supp}(\pi_*^{-1}(\Delta_p))$. In this case, one cannot see the contribution of the ϵ-klt condition from the intersection number $\pi^*\Delta_p.Y_b$ for Y_b a general fiber over $f(C) \subseteq B$, since $Y_b \subseteq \mathrm{Supp}(\pi_*^{-1}(\Delta_p))$ (cf., Example 2.1). The differentiation argument eliminates the contribution of irreducible codimension one components of $\mathrm{Supp}(\pi_*^{-1}(\Delta_p))$ along Y_b and hence we can proceed as in Proposition 5.3.

5.4 Big multiplicity

Again, let X be an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number one. Write $\mathrm{Vol}(X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω and denote $D = -2K_X$. As before, from 5.1 there is an integer $m > 0$ such that $L = mD/\omega$ is Cartier and an open affine subset $U \subseteq X$ such that $H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. We fix a nonzero section $s \in H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ with $F = \mathrm{div}(s)$ on $X \times U$. We now consider the case where there exists an irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$ with multiplicity $\mathrm{mult}_{\mathcal{W}}(F) > 3m$.

LEMMA 5.5. *Suppose that there exists an irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$ with multiplicity $\mathrm{mult}_{\mathcal{W}}(F) > 3m$, then there exists a covering family of tigers of dimension two and weight $\omega' \geqslant \omega/2$.*

Proof. Fix $\mathcal{W}$ to be one of these irreducible components. Cutting down by hyperplanes on U and restrict to a smaller open subset of U, we may assume that $\mathcal{W} \to U$ factors through a Hilbert scheme of X and $\mathcal{W} \to X$ is generically finite. Replace U by the normalization of the closure of its image in the Hilbert scheme and $\mathcal{W}$ by the normalization of universal family, we obtain maps $\pi : Y \to X$ and $f : Y \to B$. Note that a general fiber Y_b is two-dimensional. We claim that the pairs $(\Delta_b = \pi_*(Y_b), V_b = Y_b)$ is a two-dimensional covering of tigers of weight $\omega' \geqslant \omega/2$.

Since X is $\mathbb{Q}$-factorial and $\rho(X) = 1$, the integral divisor $\Delta_b = \pi_*(Y_b)$ for any $b \in B$ on X is $\mathbb{Q}$-linearly equivalent to a multiple of $-K_X$. Also $\pi_*(Y_b) \leqslant F_b$ since by construction this is true for the divisor $\mathcal{W}_b$. In particular, $\pi_*(Y_b) \sim_{\mathbb{Q}} -K_X/\omega'$ for some $\omega' \geqslant \omega/2$. Since any two general divisors $\pi_*(Y_{b_i})$, $i=1,2$, on X are $\mathbb{Q}$-linearly equivalent as the base field is uncountable, and it is clear that $V_b=\pi(Y_b)$ is a pure non-klt center of $\mathrm{Nklt}(X,\Delta_b)$, the claim follows. ∎

Let $\pi:Y\to X$ with $f:Y\to B$ be a covering family of tigers of dimension two and weight $\omega'\geqslant\omega/2$ from Lemma 5.5. We first deal with case where $\pi:Y\to X$ is not birational.

PROPOSITION 5.6. *Suppose that the two-dimensional covering family of tigers $\pi:Y\to X$ with $f:Y\to B$ of weight $\omega'\geqslant\omega/2$ is not birational and assume that $\omega>12$, then $\omega\leqslant 24/\epsilon+12$. In particular, there is an upper bound of volume*

$$\mathrm{Vol}(X)=-K_X^3\leqslant\left(\frac{72}{\epsilon}+36\right)^3.$$

Proof. Let $d\geqslant 2$ be the degree of $\pi:Y\to X$. Fix an open subset $U\subseteq X$ such that for a general point $p\in U$, there are d divisors $\Delta_p^{t_i}$ for some $t_1,\ldots,t_d\in B$ such that $\pi(Y_{t_i})\subseteq\mathrm{Nklt}(X,\Delta_p^{t_i})$ is the unique minimal non-klt center passing through p. Consider the collection of $\mathbb{Q}$-divisors $\{\Delta'_p=\frac{6}{d}\sum_{i=1}^{d}\Delta_p^{t_i}|p\in U\}$, then $\mathrm{mult}_p\Delta'_p\geqslant 6$, $\mathrm{mult}_{W'}\Delta'_p=\frac{6}{d}\leqslant 3$ for $W'\subseteq\mathrm{Supp}(\Delta'_p)$ any irreducible component, and $\Delta'_p\sim_{\mathbb{Q}}\frac{-K_X}{d\omega'/6}$.

By the same construction as in 5.1, possibly after shrinking U to a smaller open affine subset, there exists an integer $m>0$ such that $H^0(X\times U,\mathrm{pr}_X^*L\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m})\neq 0$ where $L=6m(-K_X)/d\omega'$ is Cartier. Let $t\in H^0(X\times U,\mathrm{pr}_X^*L\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a general nonzero section and $G=\mathrm{div}(t)$ be the associated divisor on $X\times U$. Note that $\mathrm{mult}_{\mathcal{Z}}(G)\geqslant 6m$ and $\mathrm{mult}_{\mathcal{W}}(G)\leqslant 6m/d\leqslant 3m$ for $\mathcal{W}$ any irreducible component of $\mathrm{Supp}(G)$ passing through $\mathcal{Z}$. This is true since for general $p\in U$ there is a special divisor Δ'_p with $\mathrm{mult}_p\Delta'_p\geqslant 6$ and $\mathrm{mult}_{W'}\Delta'_p=\frac{6}{d}\leqslant 3$ for $W'\subseteq\mathrm{Supp}(\Delta'_p)$ any irreducible component, but t is a general section and we can use Lemma 5.1 to compute the multiplicity of the general section $t_p=t|_{X\times\{p\}}$.

By a differentiation argument and the same construction as in Proposition 5.3, there is a covering family of tigers (Δ_t,V_t) of dimension one and weight $\omega''\geqslant d\omega'/6\geqslant d\omega/12$ which satisfies the property that the base locus $\mathrm{Bs}(|\mathrm{pr}_X^*L\otimes\mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|)$ contains no codimension one components in a neighborhood of $\mathcal{Z}$, where M is the maximum of $\mathrm{mult}_{\mathcal{W}}(G)$ amongst all the irreducible components $\mathcal{W}$ of $\mathrm{Supp}(G)$ passing through $\mathcal{Z}$. Hence by Corollary 3.9 again, we get

$$\frac{2}{\omega''-2}\geqslant\frac{1}{\omega''}(-K_X.V_t)=\pi^*\Delta_p.Y_b\geqslant\frac{\epsilon}{2}.$$

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In particular,

$$\frac{4}{\epsilon}+2 \geqslant \omega'' \geqslant \frac{d\omega}{12} \geqslant \frac{\omega}{6}$$

and $\omega \leqslant 24/\epsilon + 12$. ∎

Assumption 5.7. From now on, we assume that $\pi : Y \to X$ with $f : Y \to B$ is a *birational* covering family of tigers of dimension two and weight $\omega' \geqslant \omega/2$. Write $K_Y + \Gamma - R = \pi^* K_X$ where Γ and R are effective divisors on Y with no common components.

LEMMA 5.8. *There is a π-exceptional divisor E on Y dominating B. In particular, $\pi : Y \to X$ is not small.*

Proof. Suppose that there is no π-exceptional divisors dominating B. Let A_B be a sufficiently ample divisor on B and $A_Y = f^* A_B$ the pullback. Since $\rho(X) = 1$, the divisor $A_X = \pi_* A_Y$ on X is ample and $\pi^* A_X = A_Y + G$ for some effective π-exceptional divisor G. By assumption $f(G) \subseteq B$ has codimension one and hence $A_Y + G \leqslant f^* H$ for some divisor H on B. This is a contradiction since then $A_Y + G$ is not big but $\pi^* A_X$ is. ∎

The following lemma is crucial for computing the restricted volume. The key point is that it allows us to control the negative part of the subadjunction $-K_X|_{V_t}$. Note that the proof fails in higher dimensions (cf., [McK02, Lemma 6.2]).

LEMMA 5.9. *Let E be a π-exceptional divisor dominating B, then for general points $p, q \in X$ we have that*

$$E \subseteq \mathrm{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_p + \Delta_q)).$$

In particular, if we denote $H = \pi^*(-K_X)$, then for any π-exceptional divisor E dominating B we have

$$\frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geqslant \epsilon E.$$

Proof. Since the construction of covering families of tigers is done by the Hilbert scheme, π is finite on the general fibers V_t of $f : Y \to B$. Recall that $\pi(V_t) \subseteq X$ is a pure non-klt center of $(X, \Delta_{p(t)})$ for some $\Delta_{p(t)}$ passing through a general point $p(t) \in X$. We denote $\Delta_{p(t)}$ by Δ_t for simplicity.

Let E be a π-exceptional divisor dominating B. Take $t_1, t_2 \in B$ general, and consider $\pi(E \cap V_{t_i}) \subseteq \pi(V_{t_i}) \cap \pi(E)$. Since π is finite on the

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general fibers of $f: Y \to B$, $\pi(E)$ is an irreducible curve contained in $\pi(V_{t_1}) \cap \pi(V_{t_2}) \subseteq \mathrm{Nklt}(K_X + \Delta_{t_1} + \Delta_{t_2})$. Pick a general point $x \in \pi(E)$ and consider its preimage on V_{t_i}. Since π is finite on the general fiber V_t, $\pi^{-1}(x) \cap V_{t_i}$ can only be finitely many points. Choose $x_i \in \pi^{-1}(x) \cap V_{t_i}$ over x for $i = 1, 2$. By the Connectedness Lemma 1.3 applied to the pair $(Y, \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$ over X, there is a (possibly reducible) curve contained in $\pi^{-1}(x) \cap \mathrm{Nklt}(Y, \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$ connecting x_1 and x_2. The component of this curve containing x_1 cannot lie on V_{t_1} as the map π is finite on V_{t_1}. As $x \in \pi(E)$ is general, this curve deforms into a dimension two set by moving $x \in \pi(E)$. Denote $\tilde{E}$ the closure of this two-dimensional set, which is another π-exceptional divisor mapping onto $\pi(E)$ and intersects along $E \cap V_{t_1}$. Since there are only finitely many exceptional divisors over $\pi(E)$ and t_1 is general, we can assume $\tilde{E} = E$ as E is irreducible, and hence $E \subseteq \mathrm{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$. In particular, $\mathrm{mult}_E(K_Y + \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2})) \geqslant 1$ and we get $\pi^*(\Delta_p + \Delta_q) \geqslant E$ if $E \not\subseteq \mathrm{Supp}(\Gamma)$ and $\pi^*(\Delta_p + \Delta_q) \geqslant \epsilon E$ if $E \subseteq \mathrm{Supp}(\Gamma)$ from the fact that $\Gamma \in [\,0, 1 - \epsilon\,)$ as X is ϵ-klt. ∎

To study the geometry of the covering family $f: Y \to B$, we would like to run a relative minimal model program of (Y, Γ) over B. However, Y is normal but possibly not $\mathbb{Q}$-factorial. To get a $\mathbb{Q}$-factorial model of (Y, Γ), we adopt Hacon's dlt models (cf., [KK10, Theorem 3.1]). In fact, since the volume bound will be obtained by doing a computation on a general fiber Y_b, it suffices to modify Y over an open subset $U \subseteq B$.

LEMMA 5.10. *After restricting to an open subset $U \subseteq B$ and replacing Y by a suitable birational model, we can assume that Y is $\mathbb{Q}$-factorial and (Y, Γ) is $\epsilon/2$-klt. Moreover, we can assume that for E any π-exceptional divisor dominating U and $p, q \in X$ general, we have that*

$$\frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geqslant \frac{\epsilon}{2} E. \tag{5.1}$$

Proof. Fix $p, q \in X$ general and consider the pair

$$K_Y + \Gamma - R_{\mathrm{d}} + \pi^*(\Delta_p + \Delta_q) - R_{\mathrm{e}} \sim_{\mathbb{Q}} \pi^*(K_X + \Delta_p + \Delta_q) \tag{$\sharp$}$$

where $R = R_{\mathrm{d}} + R_{\mathrm{e}}$ with $(-)_{\mathrm{d}}$ the sum of components dominating B and $(-)_{\mathrm{e}}$ the sum of components mapping to points in B. Restricting Y to $Y_U = f^{-1}(U)$ for a suitable nonempty open set $U \subseteq B$, we may assume that

$R_{\mathrm{e}}=0$ and the pair ($\sharp$) becomes the effective pair

$$K_Y+\Gamma-R_{\mathrm{d}}+\pi^*(\Delta_p+\Delta_q)\sim_{\mathbb{Q}}\pi^*(K_X+\Delta_p+\Delta_q).$$

We abuse the notation: Y is understood to be Y_U if not specified.

Denote $\Gamma_{p,q}=\Gamma-R_{\mathrm{d}}+\pi^*(\Delta_p+\Delta_q)$ and let $\phi:W\to Y$ be a log resolution of $(Y,\Gamma_{p,q})$. Write

$$K_W+\phi_*^{-1}\Gamma_{p,q}+Q\sim_{\mathbb{Q}}\phi^*(K_Y+\Gamma_{p,q})+P$$

where $Q,P\geqslant 0$ are ϕ-exceptional divisors with $Q\wedge P=0$. We aim to modify W by running a relative minimal model program over Y with scaling of an ample divisor so that it contracts $Q^{<1-\epsilon/2}+P$.

Consider $F=\sum_i F_i$ where the sum runs over all the ϕ-exceptional divisors with log discrepancy in $(\epsilon/2,1]$ with respect to $(Y,\Gamma_{p,q})$, then

$$(F+P)\wedge Q^{\geqslant 1-\epsilon/2}=0\quad\text{and}\quad\mathrm{Supp}(F)\supseteq\mathrm{Supp}(Q^{<1-\epsilon/2}).$$

Since $(Y,\Gamma-R)$ is ϵ-klt, the divisor Γ on Y as well as $\phi_*^{-1}\Gamma$ on W has coefficients in $[\,0,1-\epsilon\,)$. For rational numbers $0<\epsilon<\epsilon'<1$ and $0<\delta,\delta'\ll 1$, we have the following $\epsilon/2$-klt pair

$$\begin{aligned}&K_W+\phi_*^{-1}\Gamma+Q^{<1-\epsilon/2}+\delta' Q^{1-\epsilon/2\leqslant\cdot<1}+(1-\epsilon')(Q^{\geqslant 1})_{\mathrm{red}}+\delta F\\&\quad\sim_{\mathbb{Q}}\phi^*(K_Y+\Gamma_{p,q})-(\phi_*^{-1}\Gamma_{p,q}-\phi_*^{-1}\Gamma)-(1-\delta')Q^{1-\epsilon/2\leqslant\cdot<1}\\&\quad-(Q^{\geqslant 1}-(1-\epsilon')(Q^{\geqslant 1})_{\mathrm{red}})+P+\delta F\end{aligned}$$

where $(\sum_i a_iQ_i)^{\alpha\leqslant\cdot<\beta}=\sum_{\alpha\leqslant a_i<\beta}a_iQ_i$ and $(\sum_j b_jG_j)_{\mathrm{red}}=\sum_{b_j\neq 0}G_j$. We denote this pair by (W,Ξ) where $\Xi=\phi_*^{-1}\Gamma+Q^{<1-\epsilon/2}+\delta' Q^{1-\epsilon/2\leqslant\cdot<1}+(1-\epsilon')(Q^{\geqslant 1})_{\mathrm{red}}+\delta F$.

By [BCHM10], a relative minimal model program with scaling of an ample divisor of the pair (W,Ξ) over Y terminates with a birational model $\psi:W\dashrightarrow W'$ over Y with $\phi':W'\to Y$ the induced map. We obtain the following diagram:

$$\begin{tikzcd}
 & & W' \arrow[dll, "\pi'"'] \arrow[d, "\phi'"] & W \arrow[l, dashed, "\psi"'] \arrow[dl, "\phi"] \\
X & Y \arrow[l, "\pi"] \arrow[d, "f"] & Y_U \arrow[l, hook'] \arrow[d] & \\
 & B & U \arrow[l, hook'] &
\end{tikzcd}$$

where $\pi':W'\to X$ is the induced map.

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Write $K_{W'}+\Gamma_{W'}-R_{W'}\sim_{\mathbb{Q}}\pi'^{*}K_X$ where $\pi'=\phi'\circ\pi$. Note that $\Gamma_{W'}\in[\,0,1-\epsilon\,)$ by the ϵ-klt condition and $\Gamma_{W'}-\psi_*\Gamma\geqslant 0$ is ϕ'-exceptional. It follows by the construction that

$$\psi_*^{-1}\Gamma_{W'}\leqslant\phi_*^{-1}\Gamma+\delta' Q^{1-\epsilon/2\leqslant\cdot<1}+(1-\epsilon')(Q^{\geqslant 1})_{\mathrm{red}}.$$

In particular, $(W',\Gamma_{W'})$ is $\epsilon/2$-klt as the pair (W,Ξ) is $\epsilon/2$-klt and the minimal model program does not make singularities worse.

On W', the divisor

$$\begin{aligned}G=\psi_*\Big(&-(\phi_*^{-1}\Gamma_{p,q}-\phi_*^{-1}\Gamma)-(1-\delta')Q^{1-\epsilon/2\leqslant\cdot<1}\\&-(Q^{\geqslant 1}-(1-\epsilon')(Q^{\geqslant 1})_{\mathrm{red}})+P+\delta F\Big)\end{aligned}$$

is ϕ'-nef with $\phi'_*G\leqslant 0$. By [KM98, Negativity Lemma 3.39], we have that $G\leqslant 0$. Since F is ϕ-exceptional and $(F+P)\wedge Q^{\geqslant 1-\epsilon/2}=0$, it follows that $\psi_*(P+\delta F)=0$. In particular, all the ϕ'-exceptional divisors on W' have log discrepancies less than or equal to $\epsilon/2$ with respect to $(Y,\Gamma_{p,q})$.

We now show that for any π'-exceptional divisor E' on W' dominating U, E' satisfies the inequality

$$\frac{2}{\omega'}H'\sim_{\mathbb{Q}}\pi'^{*}(\Delta_p+\Delta_q)\geqslant\frac{\epsilon}{2}E'.$$

where $H'=\psi_*H=\pi'^{*}(-K_X)$. This is easy to see: If $E=\phi'_*(E')\neq 0$ on Y_U, then by Lemma 5.9 $E\subseteq\mathrm{Nklt}(K_Y+\Gamma-R+\pi^*(\Delta_p+\Delta_q))$ and hence $E'\subseteq\mathrm{Nklt}(K_{W'}+\Gamma_{W'}-R_{W'}+\pi'^{*}(\Delta_p+\Delta_q))$. The inequality then follows from the same argument as in Lemma 5.9. If $\phi'_*E'=0$, then by construction $\mathrm{mult}_{E'}(K_{W'}+\Gamma_{W'}-R_{W'}+\pi'^{*}(\Delta_p+\Delta_q))\geqslant 1-\epsilon/2$. Suppose that $E'\subseteq\mathrm{Supp}(R_{W'})$, then

$$\frac{2}{\omega'}H'\sim_{\mathbb{Q}}\pi'^{*}(\Delta_p+\Delta_q)\geqslant E'\geqslant\frac{\epsilon}{2}E'.$$

If $E'\subseteq\mathrm{Supp}(\Gamma_{W'})$, then as $\Gamma_{W'}\in[\,0,1-\epsilon\,)$ we get

$$\frac{2}{\omega'}H'\sim_{\mathbb{Q}}\pi'^{*}(\Delta_p+\Delta_q)\geqslant\left(\left(1-\frac{\epsilon}{2}\right)-(1-\epsilon)\right)E'=\frac{\epsilon}{2}E'.$$

It follows that W' is a required model. ∎

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REMARK 5.11. Write $\Gamma = \pi_*^{-1}\Delta + \Gamma_{\mathrm{d}} + \Gamma_{\mathrm{e}}$ and $R = R_{\mathrm{d}} + R_{\mathrm{e}}$, where $(-)_{\mathrm{d}}$ is the sum of components dominating B and $(-)_{\mathrm{e}}$ is the sum of components mapping to points in B. From the proof of Lemma 5.10, we deduce the following two inequalities:

$$\frac{2}{\omega'}H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geqslant R_{\mathrm{d}} \tag{5.2}$$

and

$$\frac{2}{\omega'}H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geqslant \frac{\epsilon}{2}\Gamma_{\mathrm{d}}. \tag{5.3}$$

Now let $\pi : Y \to X$ with $f : Y \to U$ be the modified birational covering family of tigers of dimension two and weight $\omega' \geqslant \omega/2$ given by Lemma 5.10 where Y is now $\mathbb{Q}$-factorial. Write $K_Y + \Gamma - R \sim_{\mathbb{Q}} \pi^* K_X$, where $\Gamma \geqslant 0$, $R \geqslant 0$ are π-exceptional, and $\Gamma \wedge R = 0$. The pair (Y, Γ) is $\epsilon/2$-klt with $\Gamma \in [\,0, 1-\epsilon\,)$ and note that $H = \pi^*(-K_X)$ is semiample and big on Y.

Recall that for a projective morphism $\phi : Z \to U$, a divisor D on Z is pseudo-effective (PSEF) over U if the restriction of D to the generic fiber is pseudo-effective.

LEMMA 5.12. *Assume that $\omega' > 2$ and consider the pseudo-effective threshold of $K_Y + \Gamma$ over U with respect to H*

$$\tau := \inf\{t > 0 | K_Y + \Gamma + tH \text{ is PSEF over } B\},$$

then

$$1 \geqslant \tau \geqslant 1 - \frac{2}{\omega'} > 0.$$

Proof. Since $K_Y + \Gamma + H \sim_{\mathbb{Q}} R \geqslant 0$, the first inequality is clear. When restrict to a general fiber Y_u of Y over U, we have

$$\begin{aligned}(K_Y + \Gamma + \tau H)|_{Y_u} &= (R - (1-\tau)H)|_{Y_u} \\ &= \left(R_d - \frac{2}{\omega'}H\right)|_{Y_u} - \left(1 - \tau - \frac{2}{\omega'}\right)H|_{Y_u}\end{aligned}$$

which cannot be PSEF if $\omega' > 2$ and $\tau < 1 - \frac{2}{\omega'}$ since the first term is nonpositive by (5.2) and the second term is negative. ∎

Now we run a relative minimal model program with scaling of the covering family of tigers $f : Y \to U$. Since (Y, Γ) is $\epsilon/2$-klt and H is semiample and big, we may assume that $(Y, \Gamma + \tau' H)$ remains $\epsilon/2$-klt for a rational number

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$0 < \tau' < \tau$. Run a relative minimal model program of $(K_Y + \Gamma + \tau' H)$ with scaling of H over U. By [BCHM10], it terminates with a relative Mori fiber space $Y' \to T$ over U with $\dim Y' > \dim T \geqslant \dim U$. Denote the induced maps by $g : Y \dashrightarrow Y'$, $\psi : Y' \to T$, and $\phi : Y' \to U$, we get the following diagram:

$$\begin{tikzcd} X & Y \arrow[l, "\pi"'] \arrow[r, dashed, "g"] \arrow[d, "f"'] & Y' \arrow[dl, "\phi"'] \arrow[d, "\psi"] \\ (B \supseteq) & U & T. \arrow[l] \end{tikzcd}$$

For a general fiber Y'_t of $\psi : Y' \to T$, the Picard number $\rho(Y'_t) = 1$ and $-(K_{Y'} + \Gamma'_d)|_{Y'_t} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_t}$ is ample.

LEMMA 5.13. *There exists a divisor E' on Y' which is exceptional over X and dominates T.*

Proof. Recall that there is a natural map $T \to U \to B$. Hence we can extend $\psi : Y' \to T$ to $\overline{\psi} : \overline{Y'} \to \overline{T}$ over B where $\overline{(-)}$ stands for a projective compactification of $(-)$. Take a common resolution $p : W \to X$ and $q : W \to \overline{Y'}$ and let $A_{\overline{T}}$ be a sufficiently ample divisor on $\overline{T}$. Let $A_{\overline{Y'}} = \overline{\psi}^* A_{\overline{T}}$, $A_W = q^* A_{\overline{Y'}}$, and $A_X = p_* A_W$. Then $p^* A_X = A_W + E = q^* A_{\overline{Y'}} + E = q^* \overline{\psi}^* A_{\overline{T}} + E$ for an effective divisor E on W which is exceptional over X. Since $\rho(X) = 1$, it follows by the same argument as in Lemma 5.8 that one of the irreducible components of E maps to a divisor E' on $\overline{Y'}$. By the same argument as in Lemma 5.8 again, one of the irreducible components of the nonzero divisor $q_*(E)$ dominates $\overline{T}$. ∎

PROPOSITION 5.14. *If $\dim T = 2$, then $\omega' \leqslant 8/\epsilon + 2$.*

Proof. By Lemma 5.13, there exists a divisor E' on Y' which is exceptional over X and dominates T. Note that Y' is normal and hence $\psi(\mathrm{Sing}(Y'))$ is a proper subset of T. In particular, a general fiber Y'_t of $\psi : Y' \to T$ is a smooth projective curve and hence $E'.Y'_t \geqslant 1$. Since $-(K_{Y'} + \Gamma'_d)|_{Y'_t} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_t}$ is ample, a general fiber Y'_t is a smooth rational curve $\mathbb{P}^1$. From (5.1), we know that

$$\frac{2}{\omega'} H' - \frac{\epsilon}{2} E' \sim_{\mathbb{Q}} \text{effective.}$$

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Also from (5.2),

$$-(K_{Y'}+\Gamma').Y'_t=(H'-R').Y'_t=\left(1-\frac{2}{\omega'}\right)H'.Y'_t+\left(\frac{2}{\omega'}H-R'\right).Y'_t$$
$$\geqslant\left(1-\frac{2}{\omega'}\right)H'.Y'_t.$$

It follows that

$$\frac{2}{\omega'}\geqslant\frac{1}{\omega'}(-(K_{Y'}+\Gamma').Y'_t)\geqslant\frac{1}{\omega'}\left(1-\frac{2}{\omega'}\right)H'.Y'_t$$
$$\geqslant\left(1-\frac{2}{\omega'}\right)\frac{\epsilon}{4}E'.Y'_t$$
$$\geqslant\left(1-\frac{2}{\omega'}\right)\frac{\epsilon}{4}$$

where the first inequality is by the adjunction formula on $\mathbb{P}^1$. Hence $\omega'\leqslant\frac{8}{\epsilon}+2$. ∎

PROPOSITION 5.15. *If* $\dim T=1$, *then*

$$\omega'\leqslant\frac{4M(2,\epsilon)R(2,\epsilon)}{\epsilon}+2$$

where $R(2,\epsilon)$ *is an upper bound of the Cartier index of* K_S *for* S *any* $\epsilon/2$-*klt log del Pezzo surface of* $\rho(S)=1$ *and* $M(2,\epsilon)$ *is an upper bound of the volume* $\mathrm{Vol}(S)=K_S^2$ *for* S *any* $\epsilon/2$-*klt log del Pezzo surface of* $\rho(S)=1$.

Proof. Since $f:Y\to U$ has connected fibers, $T\cong U$. Since $-(K_{Y'}+\Gamma'_d)|_{Y'_u}\sim_{\mathbb{Q}}(H'-R_d)|_{Y'_u}$ is ample and $\rho(Y'_u)=1$ for a general point $u\in U$, we see that

$$-K_{Y'_u}\sim_{\mathbb{Q}}(H'+\Gamma'_d-R_d)|_{Y'_u}$$

is ample. By Lemma 5.13, let E' be a divisor on Y' exceptional over X which dominates U, then

$$-K_{Y'_u}\equiv(H'+\Gamma'_d-R_d)|_{Y'_u}\geqslant\left(1-\frac{2}{\omega'}\right)H|_{Y'_u}\geqslant\left(1-\frac{2}{\omega'}\right)\cdot\frac{\omega'\epsilon}{4}E'_u$$

where the second inequality follows by dropping Γ'_d and applying (5.2) while the last one from (5.1). By intersecting with the ample divisor $-K_{Y'_u}$, this

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implies that

$$(-K_{Y'_u})^2 \geqslant (\omega' - 2)\frac{\epsilon}{4}E'_u.(-K_{Y'_u}).$$

Now (Y'_u, Γ'_u) is an $\epsilon/2$-klt log del Pezzo surfaces of Picard number one. Hence Y'_u is an $\epsilon/2$-klt del Pezzo surfaces of Picard number $\rho(Y'_u) = 1$. By Theorem 4.3, $(-K_{Y'_u})^2$ is bounded above by a positive number $M(2, \epsilon)$ satisfying

$$M(2, \epsilon) \leqslant \max\left\{64, \frac{16}{\epsilon} + 4\right\}.$$

Also, by ($\Diamond$) the Cartier index of $K_{Y'_u}$ has an upper bound,

$$R(2, \epsilon) \leqslant r\left(2, \frac{\epsilon}{2}\right) \leqslant 2(4/\epsilon)^{128/\epsilon^5}.$$

It follows that

$$\begin{aligned} M(2, \epsilon) \geqslant (-K_{Y'_u})^2 &\geqslant \frac{1}{R(2, \epsilon)}(\omega' - 2)\frac{\epsilon}{4}E'_u.(\text{Ample Cartier}) \\ &\geqslant \frac{1}{R(2, \epsilon)}(\omega' - 2)\frac{\epsilon}{4} \end{aligned}$$

and hence we get an upper bound,

$$\omega' \leqslant \frac{4M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 2. \qquad \square$$

REMARK 5.16. It has been shown in [Bel08] that a klt log del Pezzo surface has at most four isolated singularities. Also surface klt singularities are classified by Alexeev in [Cor07]. Hence we expect that it is possible to obtain a better upper bound for $R(2, \epsilon)$ and $M(2, \epsilon)$ in Proposition 5.15.

THEOREM 5.17. *Let (X, Δ) be an ϵ-klt log $\mathbb{Q}$-Fano threefold of $\rho(X) = 1$. Then the degree $-K_X^3$ satisfies*

$$-K_X^3 \leqslant \left(\frac{24M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 12\right)^3$$

where where $R(2, \epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S) = 1$ and $M(2, \epsilon)$ is an upper bound of the volume $\mathrm{Vol}(S) = K_S^2$ for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S) = 1$. Note that we have $M(2, \epsilon) \leqslant \max\{64, 16/\epsilon + 4\}$ from Theorem 4.3 and $R(2, \epsilon) \leqslant 2(4/\epsilon)^{128/\epsilon^5}$ from ($\Diamond$).

Proof. Recall that $\omega' \geqslant \omega/2$. The theorem then follows from Propositions 5.3, 5.14 and 5.15. ∎

The following example shows that the cone construction analogous to Example 2.1 only provides ϵ-klt Fano threefolds with volumes of order $1/\epsilon^2$.

EXAMPLE 5.18. (Projective cone of projective spaces) For $n \geqslant 1$ and $d \geqslant 2$, let $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be the embedding by $|\mathcal{O}(d)|$ and X be the associated projective cone. The projective variety X is normal $\mathbb{Q}$-factorial of Picard number one with the unique singularity at the vertex O. X admits a resolution $\pi : Y = Bl_O X \to X$ with exceptional divisor $E \cong \mathbb{P}^n$ of normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$. The variety Y is the projective bundle $\mu : Y \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-d)) \to \mathbb{P}^n$ with tautological bundle $\mathcal{O}_Y(1) \cong \mathcal{O}_Y(E)$. We have:

- $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ and hence $E^{n+1} = (-d)^n$;
- $K_Y = \pi^* K_X + (-1 + \frac{n+1}{d})E$ and hence X is always klt. X is terminal (resp. canonical) if and only if $n + 1 > d \geqslant 2$ (resp. $n + 1 \geqslant d \geqslant 2$);
- $K_Y = \mu^*(K_{\mathbb{P}^n} + \det(\mathcal{E})) \otimes \mathcal{O}_Y(-\mathrm{rk}(\mathcal{E})) \equiv -(n + 1 + d)F - 2E$ where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-d)$ and $F = \mu^* \mathcal{O}^n_{\mathbb{P}}(1)$;
- $F^{n+1} = 0$ and $F^{n+1-k}.E^k = (-d)^{k-1}$ for $1 \leqslant k \leqslant n + 1$;
- $K_Y^{n+1} = K_X^{n+1} + (-1 + \frac{n+1}{d})^{n+1} E^{n+1}$ and

$$K_Y^{n+1} = \frac{-1}{d} \sum_{k=1}^{n+1} \binom{n+1-k}{k} \left(-1 + \frac{n+1}{d}\right)^{n+1-k} (2d)^k$$
$$= \frac{-1}{d}((d - n - 1)^{n+1} - (-(d + n + 1)^{n+1}));$$

- In summary, $-K_X$ is ample with

$$(-K_X)^{n+1} = \frac{(d + n + 1)^{n+1}}{d}.$$

If $n = 2$, then we have an ϵ-klt Fano threefold of Picard number one with $\epsilon = 1/d$. The volume $\mathrm{Vol}(X) = (-K_X)^3$ is of order $1/\epsilon^2$.

In view of Theorem 5.17, it is then interesting to see whether ϵ-klt Fano threefolds with big volumes exist.

QUESTION 5.19. Can one find an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold X of $\rho(X) = 1$ with volume $\mathrm{Vol}(X) = (-K_X)^3 = O\left(\frac{1}{\epsilon^c}\right)$ for $c \geqslant 3$?

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