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# DEFORMATIONS OF DIFFERENTIAL ARCS

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**Abstract**

Let $k$ be field of characteristic zero. Let $f \in k[X, Y]$ be a nonconstant polynomial. We prove that the space of differential (formal) deformations of any formal general solution of the associated ordinary differential equation $f(y', y) = 0$ is isomorphic to the formal disc $\mathrm{Spf}(k[[Z]])$.

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## 1. Introduction

Let $f \in \mathbb{C}[X, Y]$ be a nonconstant polynomial. Let $z(T) \in \mathbb{C}[[T]]$ be a formal solution of the associated ordinary differential equation $f(z'(T), z(T)) = 0$. We call a *differential formal deformation* of $z(T)$ an element $z(U, T) \in \mathbb{C}[[U, T]]$ satisfying $f(\partial_T(z(U, T)), z(U, T)) = 0$ and $z(0, T) = z(T)$. A natural open problem is the study of the space of differential formal deformations of $z(T)$. (See [1] for related, though different, questions.)

The present work addresses a formulation of this question in terms of formal geometry. It may also be seen as a differential analogue of the following theorem of Drinfeld, which generalised a characteristic zero version due to Grinberg and Kazhdan and describes the structure of formal neighbourhoods in arc schemes (see [5, 6], and also [2, 3], or [4]).

Theorem 1.1 [5, Theorem 0.1]. *Let $k$ be a field. Let $V$ be an integral $k$-variety with* $\dim(V) \geq 1$. *Let $\gamma \in \mathscr{L}(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathscr{L}(V_{\mathrm{sing}})$. If $\mathscr{L}(V)_\gamma$ denotes the formal neighbourhood of the $k$-scheme $\mathscr{L}(V)$ at the point $\gamma$, there exist an affine $k$-scheme $S$ of finite type, with $s \in S(k)$, and an isomorphism of formal $k$-schemes:*

$$\mathscr{L}(V)_\gamma \cong S_s \hat{\otimes}_k k[[(T_i)_{i \in \mathbf{N}}]].$$

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In order to give a geometric definition of the space of differential formal deformations, we make use of the formalism of differential algebra proposed by Ritt and Kolchin which is very suitable for our purposes. In this setting, our main result reads as follows.

THEOREM 1.2. *Let $k$ be a field of characteristic zero. Let $f \in k[X, Y]$ be a nonconstant polynomial. Let $C = \mathrm{Spec}(k[X, Y]/\langle f\rangle)$. Let $\gamma \in \mathcal{L}^{\delta}_{\infty}(C)(k)$ be a differential arc. For every test-ring $(A, \mathfrak{m}_A)$ and every differential $A$-deformation $\gamma_A$ of the arc $\gamma$, there exists a unique $a \in \mathfrak{m}_A$ such that $\gamma_A(T) = \gamma(T + a) \in A[[T]]$. In particular, the formal $k$-scheme $\mathcal{L}^{\delta}_{\infty}(C)_{\gamma}$ is isomorphic to $\mathrm{Spf}(k[[X]])$.*

Let us explain the notation in this statement. The $k$-scheme $\mathcal{L}^{\delta}_{\infty}(C)_{\gamma}$, which is the geometric incarnation of the space of differential deformations, is defined as follows. We endow the ring $k[(Y_i)_{i\in\mathbf{N}}]$ with the $k$-derivation $\delta$ defined by $\delta(Y_i) = Y_{i+1}$ for every integer $i \in \mathbf{N}$; the associated differential ring is denoted as usual by $k\{Y\}$ and called the ring of *differential polynomials* (in one variable).

For every subset $S$ of $k\{Y\}$, one denotes by $[S]_{\delta}$ the differential ideal generated by $S$ in the ring $k\{Y\}$. Let $n \geq 1$ be an integer. Then the $k$-scheme $\mathcal{L}^{\delta}_{\infty}(C)$ is defined as $\mathrm{Spec}(k\{Y\}/[f(Y_1, Y_0)]_{\delta})$. If $A$ is a $k$-algebra, a *differential $A$-arc* is an $A$-point of $\mathcal{L}^{\delta}_{\infty}(C)$, that is, an element $\gamma_A(T) \in A[[T]]$ with $f(\gamma'_A(T), \gamma_A(T)) = 0$.

If $\gamma$ is a differential ($k$-)arc and if $(A, \mathfrak{m}_A)$ is a local $k$-algebra with nilpotent maximal ideal $\mathfrak{m}_A$ and residue field $A/\mathfrak{m}_A$ isomorphic to $k$, that is, a *test-ring*, a *differential $A$-deformation* of $\gamma$ is an $A$-point of $\mathcal{L}^{\delta}_{\infty}(C)_{\gamma}$, namely, an element $\gamma_A(T) \in A[[T]]$ with $f(\gamma'_A(T), \gamma_A(T)) = 0$ and $\gamma_A(T) = \gamma(T) \pmod{\mathfrak{m}_A[[T]]}$.

As a corollary of Theorem 1.2, we obtain a description of the space of differential formal deformations as a one-parameter space, which constitutes, to the best of our knowledge, an original statement in the study of (algebraic) differential equations.

COROLLARY 1.3. *Let $k$ be a field of characteristic zero. Let $f \in k[X, Y]$ be a nonconstant polynomial. Let $z(T) \in k[[T]]$ be a power series satisfying $f(z'(T), z(T)) = 0$. Let $z(U, T)$ be a differential formal deformation of $z(T)$. Then there exists a unique power series $a(U) \in k[[U]]$ with $a(0) = 0$ and such that $z(T, U) = z(T + a(U))$.*

Indeed, for every integer $n \geq 1$, the quotient $k[[U]]/\langle U^n\rangle$ is a test-ring. We may apply Theorem 1.2 to the image of $z(T, U)$ in $k[[U]]/\langle U^n\rangle[[T]]$, which gives the existence of a unique element $a_n(U) \in \langle U\rangle/\langle U^n\rangle$ such that $z(T, U) = z(T + a_n(U)) \pmod{U^n}$. By uniqueness of the $a_n(U)$, there exists a unique power series $a(U) \in k[[U]]$ such that $a(U) = a_n(U) \pmod{U^n}$ for every integer $n \geq 1$. Such an element $a(U)$ has the required property.

## 2. Proof of our statement

**2.1. A first remark.** For every $c = (c_1, c_0) \in C(k)$, we denote by $\mathcal{L}^{\delta}_{\infty,c}(C)(A)$ the set of differential $A$-arcs $\gamma_A$ such that $\gamma_A(0) = c_0$ and $\gamma'_A(0) = c_1$. Let us begin with the following algebraic formulation of the Cauchy–Lipschitz theorem in our context.

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Proposition 2.1. *Keep the notation of Theorem 1.2. Let $c = (c_1, c_0) \in C(k)$. If $\partial_1(f)(c) \neq 0$, then there exists a family $(c_i)_{i\geq 2}$ of elements of $k$ such that $[f(Y_1, Y_0)]_\delta + \langle Y_0 - c_0, Y_1 - c_1\rangle = \langle (Y_i - c_i)_{i\geq 0}\rangle$. In particular, we have $\mathcal{L}^\delta_{\infty,c}(C) \cong \operatorname{Spec}(k)$.*

Proof. Let us set $c_2 = -\partial_1(f)(c)^{-1}\partial_2(f)(c)c_1$. Thus, there exist two polynomials $Q_0, Q_1 \in k[Y_0, Y_1, Y_2]$ such that $Y_2 - c_2 = \delta(f(Y_1, Y_0)) + (Y_0 - c_0)Q_0 + (Y_1 - c_1)Q_1$. The same argument, applied, for every integer $i \geq 3$, to $\delta^{(i)}(f(Y_1, Y_0))$, gives rise to the existence of the elements $c_i$ and polynomials $Q_{i,j} \in k[Y_0, \ldots, Y_i]$ such that

$$Y_i - c_i = \delta^{(i+1)}(f(Y_1, Y_0)) + \sum_{j=0}^{i-1}(Y_j - c_j)Q_{i,j}.$$

Hence, we deduce that $\langle (Y_i - c_i)_{i\geq 0}\rangle = [f(Y_1, Y_0)]_\delta + \langle Y_0 - c_0, Y_1 - c_1\rangle$. □

**2.2. The proof of Theorem 1.2.** The second assertion formally comes from the first one. Indeed, the natural bijection $\operatorname{Spf}(k[[X]])(A) \to \mathcal{L}^\delta_\infty(C)_\gamma(A)$, functorial in the test-ring $(A, \mathfrak{m}_A)$ and defined by $a \in \mathfrak{m}_A \mapsto \gamma(T + a)$, determines an isomorphism of formal $k$-schemes. Let us prove the first assertion, which splits into different cases. Up to a translation, we may assume that $\gamma(0) = 0$.

*Case 1.* We assume that $\partial_1(f)(\gamma'(0), \gamma(0)) \neq 0$. To prove Theorem 1.2, we state the following lemma.

Lemma 2.2. *Let $(A, \mathfrak{m}_A)$ be a test-ring. Let $P \in \mathfrak{m}_A[[T]]$, $\alpha \in k^\times$. Then, for every $b \in \mathfrak{m}_A$, the equation $b = \alpha X + X^2 P(X)$ admits a unique solution $a \in \mathfrak{m}_A$.*

Proof. Multiplying by $\alpha^{-1}$, one may assume that $\alpha = 1$. Let us show the uniqueness. Indeed, let us assume that there exist $a_1, a_2 \in \mathfrak{m}_A$ which satisfy

$$\begin{aligned} b &= a_1 + (a_1)^2 P(a_1) \\ b &= a_2 + (a_2)^2 P(a_2). \end{aligned} \tag{2.1}$$

Set $Q := (X + Y)P(X) + Y^2(P(X) - P(Y)) \in X\,A[[X, Y]] + Y\,A[[X, Y]]$. From (2.1),

$$(a_1 - a_2)\cdot(1 + Q(a_1, a_2)) = 0. \tag{2.2}$$

Since in (2.2) the element $1 + Q(a_1, a_2)$ is invertible in $A$, we conclude that $a_1 = a_2$.

We now prove the existence part of the statement by induction on the nilpotence degree of $\mathfrak{m}_A$ in $A$, that is, the smallest integer $n \in \mathbf{N}$ such that $\mathfrak{m}_A^n = 0$. For $n = 2$, the assertion is clear since we have $a = b$. Let us assume that the assertion holds for every integer $n \geq 2$. By assumption, we know that there exists a unique element $a' \in \mathfrak{m}_A$ such that

$$\bar{b} = \bar{a}' + (\bar{a}')^2 P(\bar{a}') \pmod{\mathfrak{m}_A^n}. \tag{2.3}$$

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By (2.3), there exists $b_0 \in \mathfrak{m}_A^n$ such that

$$b + b_0 = a' + (a')^2 P(a') \tag{2.4}$$

in the ring $A$. Set $a_0 := -b_0$. Then (2.4) coincides with

$$b = (a' + a_0) + (a')^2 P(a'),$$

or, equivalently, with

$$b = (a' + a_0) + (a' + a_0)^2 P(a' + a_0) \tag{2.5}$$

since $a_0^2 = 0$ in the ring $A$. We set $a := a' + a_0$. This element has the required property thanks to (2.5). □

Let $(A, \mathfrak{m}_A)$ be a test-ring. Let $\eta_A(T) = \gamma(T) + \sum_{i\geq 0} \eta_{A,i} T^i$, with $\eta_{A,i} \in \mathfrak{m}_A$ for every integer $i \in \mathbf{N}$, be a differential $A$-deformation of $\gamma$. By assumption,

$$f(\eta_A'(T), \eta_A(T)) = 0. \tag{2.6}$$

The action of $\partial_T$ on (2.6) provides

$$\eta_A''(T)\partial_1(f)(\eta_A'(T), \eta_A(T)) = -\eta_A'(T)\partial_2(f)(\eta_A'(T), \eta_A(T)). \tag{2.7}$$

Furthermore, by the Taylor expansion of (2.6),

$$\eta_{A,1}\partial_1(f)(\gamma', \gamma) + \eta_{A,0}\partial_2(f)(\gamma', \gamma) = 0 \pmod{\langle T\rangle}. \tag{2.8}$$

Let $\gamma_A(T) = \gamma(T) + \sum_{i\geq 0} \gamma_{A,i} T^i$, with $\gamma_{A,i} \in \mathfrak{m}_A$ for every integer $i \in \mathbf{N}$, be a differential $A$-deformation of $\gamma$. Let us exhibit a candidate for the element $a \in \mathfrak{m}_A$. We apply (2.7) to the particular case $\eta_A(T) = \gamma(T)$ and deduce that there exists $u \in (k[[T]])^\times$ such that $\gamma(T) = Tu(T)$, since $\partial_1(f)(\gamma'(T), \gamma(T)) \in (k[[T]])^\times$ and $\gamma(0) = 0$ by assumption. Then, we conclude by Lemma 2.2 that there exists a *unique* element $a \in \mathfrak{m}_A$ such that $\gamma_{A,0} = au(a)$.

We are going to show that this element $a$ has the required property. Set $\tau_a(T) := \gamma(T + a) := \gamma(T) + \sum_{n\geq 0} \tau_{a,i} T^i$. Let us prove, by induction on the integer $n \in \mathbf{N}$, that $\gamma_{A,n}$ coincides with $\tau_{a,n}$. The relation $\gamma_{A,0} = au(a) = \tau_{a,0}$ implies that this assertion holds for $n = 0$. Since $\partial_1(f)(\gamma', \gamma) \in (A[[T]])^\times$, we deduce from (2.8) applied to $\gamma_A$ and $\tau_a$ that

$$\begin{aligned}\gamma_{A,1} &= -\gamma_{A,0}\partial_2(f)(\gamma', \gamma)/\partial_1(f)(\gamma', \gamma) \pmod{\langle T\rangle}\\ &= -\tau_{a,0}\partial_2(f)(\gamma', \gamma)/\partial_1(f)(\gamma', \gamma) \pmod{\langle T\rangle}\\ &= \tau_{a,1}.\end{aligned}$$

In the same way, (2.7) applied to $\gamma_A$ and $\tau_a$ implies, by induction, that $\gamma_{A,n} = \tau_{a,n}$, for every integer $n \in \mathbf{N}$.

*Case 2.* We assume that $\partial_1(f)(\gamma'(0), \gamma(0)) = 0$. The basic idea here is to reduce to the previous case. Let $A$ be a test-ring and $\gamma_A$ a differential $A$-deformation of $\gamma$.

First, let us note that $A \subset A[[u]]_u$. In addition, the $k$-algebra $A[[u]]_u$ is a test-ring with maximal ideal $\mathfrak{m}_A[[u]]_u$ and residue field $k((u))$. In this way, the differential

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$A$-deformation $\gamma_A$ can be seen as a differential $A[[u]]_u$-deformation of $\gamma$. Let us denote by $\gamma_u$ the differential arc $(\gamma'(T+u), \gamma(T+u)) \in k((u))[[T]]^2$. Since the arc $\gamma$ is not constant, we observe that $\partial_2(f)(\gamma_u(0)) = \partial_2(f)(\gamma(u)) \neq 0$ in $k((u))$, and we can apply, in this context, the last argument.

So, by the arguments used in the first case, we conclude that there exist a nonnegative integer $n$ and a power series $a(u) \in \mathfrak{m}_A[[u]]$ such that

$$\gamma_A(T+u) = \gamma_u(T + a(u)/u^n) = \gamma(T + u + a(u)/u^n). \tag{2.9}$$

We are going to show that $a(u)$ lies in $u^n\mathfrak{m}_A[[u]]$. Let us assume that the converse holds (in particular, $n$ is positive). Thus we may write $a(u) = \sum_{0\le i\le n-1} a_i\, u^i + u^n\tilde{a}(u)$ with $\tilde{a}(u) \in \mathfrak{m}_A[[u]]$ and we may assume that $a_0 \neq 0$. Let us set $\bar{a}(u) = \sum_{0\le i\le n-1} a_i u^i$ and $b(u) = u + \tilde{a}(u) \in A[[u]]$. Let $m$ be the smallest integer such that $\mathfrak{m}_A^{m+1} = 0$. By a Taylor expansion in (2.9),

$$\gamma_A(T+u) = \gamma(T + b(u) + \bar{a}(u)/u^n) = \gamma(T + b(u)) + \sum_{j=1}^{m} (\bar{a}(u))^j \gamma^{(j)}(T + b(u))/j!u^{jn}. \tag{2.10}$$

Since $a_0 \neq 0$, there exists a positive integer $\nu$ such that $\nu \le m$ and $a_i \in \mathfrak{m}_A^\nu$ for every integer $i \in \{0, \ldots, n-1\}$. Moreover, there exists an integer $i_0 \in \{0, \ldots, n-1\}$ such that $a_{i_0} \in \mathfrak{m}_A^\nu \setminus \mathfrak{m}_A^{\nu+1}$. Let us replace $A$ by $A' := A/\mathfrak{m}_A^{\nu+1}$. Then the image of (2.10) in $A'[[T, u]]$, upon changing the integer $n$, now reads

$$\gamma_A(T+u) = \gamma(T + b(u)) + \bar{a}(u)\gamma'(T + b(u))/u^n$$

with $\bar{a}(0) \neq 0$. Multiplying by $u^n$ and specialising $u$ to 0, one obtains the relation $\bar{a}(0)\gamma'(T) = 0$, which is a contradiction since the arc $\gamma(T)$ is not constant.

At the end, we have the relation $\gamma_A(T+u) = \gamma_u(T + a(u))$ with $a \in \mathfrak{m}_A[[u]]$ uniquely determined. By specialising $u$ to 0, we conclude that there exists $\alpha := a_0 \in \mathfrak{m}_A$ such that $\gamma_A(T) = \gamma(T + \alpha)$, and $\alpha$ is unique.

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