Examples in the use of Involution.—The following examples do not usually find a place in text-books dealing with involution, and so may be interesting to those teaching the subject. They are simply special cases of the two well-known theorems: (1) that a straight line cuts the six sides of a complete quadrilateral in six points that form an involution, (2) that if any point in the plane of the quadrilateral be joined to the six vertices, the six rays so formed form a pencil in involution.

As a case of (1) consider the line cutting the sides of the quadrilateral to be the line at infinity. Then if O be any point in the plane of the quadrilateral and straight lines be drawn from O parallel to the sides, these lines cut the line at infinity in the same points as the sides of the quadrilateral. Since two pairs of rays completely determine an involution, if these are rectangular, the other pair of corresponding rays is also rectangular. Thus, if two pairs of sides of a complete quadrilateral intersect at right angles, so must the remaining pair. This immediately gives us the concurrency of the altitudes of a triangle. For if ABC be a triangle and CF and BE altitudes intersecting in H, we can look on HBCA as a quadrilateral which has two pairs of sides, HB, CH, respectively perpendicular to CA and AB. Consequently the third pair AH and BC must be rectangular.

Consider a conic passing through ABCH. The line at infinity cuts the conic and the quadrilateral in six points in involution. Let it cut AB and CH in R and R_1 (of course, points at infinity), BC and AH in P and P_1, and the conic is Q and Q_1. If any point be joined to these, the rays to R and R_1 and to P and P_1 are rectangular pairs. The involution pencil is therefore rectangular, and the lines to Q and Q_1 must be at right angles. But these give the directions of the points at infinity on the conic, therefore the asymptotes of the conic are at right angles; and we have that any conic passing through the vertices of a triangle and its orthocentre must be a rectangular hyperbola.

https://doi.org/10.1017/S1757748900000876 Published online by Cambridge University Press

As an example in the use of (2), we can consider one of the sides of the quadrilateral to be the line at infinity. Let ABC be a triangle, H its orthocentre, and u the line at infinity. Let AB, BC, CA cut u in P, P_1, E. Then A, B, C, P, P_1, E are the six vertices of the quadrilateral, and their joins with H form a pencil in involution. HP and HC are at right angles; so also are HA and HP_1. Thus the pencil from H is rectangular, and consequently HE and HB are at right angles, *i.e.* CA and HB are at right angles—another proof of the concurrency of the altitudes of a triangle.

G. PHILIP.

Note on Right-Angled Triangles.—It frequently happens in teaching that one wishes to draw a right-angled triangle with rational sides, and at the same time, for the sake of variety if nothing else, one desires to avoid the hackneyed one whose sides are 3, 4, 5.

The following simple rules, easily remembered, enable one to do this with little trouble.

1. If we select any odd number for one side, the approximate halves of its square are the other two sides.

(By "approximate halves" is meant the actual halves diminished and increased respectively by $\frac{1}{2}$.)

e.g. 13, 84, 85 $\left(i.e.\ 13,\ \frac{169}{2}-\frac{1}{2},\ \frac{169}{2}+\frac{1}{2}\right)$.

2. Choose any three consecutive numbers. The middle number is the first side. Half the product of the other two is the second side, while the third side is the second side + 1.

e.g. from 10, 11, 12 we derive the sides 11, 60, 61

$$\left(i.e.\ 11,\ \frac{10\times 12}{2},\ 60+1.\right)$$