PART IV

# SATELLITES AND RINGS

# REVIEW OF THE THEORIES OF MOTION OF THE NATURAL SATELLITES

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Most of the natural satellites of the planets of the solar system may be put into one of three main groups, according as to which of three main influences dominate the perturbation of their motion from Keplerian motion about the primary planet. The first of these is the attraction of the Sun, which governs the perturbations of the Moon's motion about the Earth, and those of the outer satellites of Jupiter (satellites VI to XIII), and Saturn's satellite Phoebe. The second is the departure of the gravitational field of the planet from that of a spherically symmetric body (the "figure terms"), and this governs the perturbations of the two satellites of Mars, Jupiter's satellite Amalthea (V), Neptune's satellite Triton, is probably the most important influence on Uranus' satellites, and is important, though not dominant, for the inner satellites of Saturn. The third influence is the mutual attraction of the satellites themselves. An order of magnitude argument suggests that periodic perturbations from this cause could scarcely be expected to be measureable from Earth, were it not that the frequent appearance of small-integer near-commensurabilities of pairs of orbital periods, and the consequent argumentation of the associated perturbations by a variety of types of resonance effects, in the systems of Jupiter and Saturn, causes mutual perturbations to dominate the orbital theories of three of the four great satellites of Jupiter, and six of the nine satellites of Saturn, and enables the masses of most of the satellites involved to be determined with otherwise unexpected relative precision (in some favourable cases, of the order of one per-cent) from Earth based data. Let us now consider the satellite systems of each of the outer planets in a little more detail.

The two small satellites of Mars, Phobos and Deimos, were discovered by Hall at Washington in 1877. Their orbits were found to be very small (Phobos has an orbital period of about 7 2/3 hours), and very nearly circular. Interest was aroused by Sharpless (1945) whose analysis of the positional observations indicated an acceleration of the longitude of Phobos, of an amount very difficult to reconcile with either resistive drag or tidal friction. Wilkins (1968) found that the observations could be fitted adequately without the accelerative term, and Sinclair (1972), analysing observations from 1877 to 1969, found the value obtained for the acceleration to be very sensitive to the way in which the observations



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were selected, so that its reality was not conclusively established. Shor (1975), including also more recent data, concluded that the observations were better fitted with the acceleration than without it. Positional data obtained by the spacecraft Mariner IX (Born and Duxbury, 1975) in general confirmed the orbital parameters obtained by Sinclair and Shor.

Turning now to Jupiter's system, the innermost satellite Amalthea (satellite V) was discovered by Barnard at the Lick Observatory in 1892. Its orbit is very nearly circular, and Van Woerkom (1950), in his discussion of observations made between 1892 and 1921, and in 1949, concluded that the eccentricity was so small that the apse motion could not be determined. P.V. Sudbury (1969), who used also observations made in 1954, 1955, and in 1967, found a lack of continuity between the expressions representing the node longitude for the two periods 1892 to 1921, and 1949 to 1967, for which observational data were available. Let us now consider the outer satellites. There is a group of four satellites, consisting of VI and VII, discovered by Perrine at the Lick Observatory in 1904 and 1905, respectively, X, discovered by Nicholson at Mount Wilson in 1938, and XIII, discovered by Kowal at the Hale Observatories in 1974. These move in very similar direct orbits, under the influence of strong solar perturbations. It was shown by Ross (1906, 1907) and Bobone (1935, 1937) that Delaunay's lunar theory may be successfully applied to these orbits. Hansen's method was applied to satellite X by Charnow, Musen, and Maury (1968). The outermost group of Jupiter's satellites comprises VIII, discovered by Melotte at Greenwich in 1908, and IX, XI, and XII, discovered by Nicholson in 1914, 1938, and 1951, respectively. These move in retrograde orbits at very similar distances from Jupiter. The solar perturbations are so great that these orbits show substantial departures from ellipses, and numerical integrations (Cowell and Crommelin, 1908, Grosch, 1947, and Herget, 1968) show that the instantaneous Keplerian elliptic parameters vary widely within a single period. Brown (1923) investigated the use of Delaunay's method, but later (1930, also Brown and Brouwer, 1937) developed a method using the true longitude as independent variable, and the reciprocal of the distance from Jupiter as one of the dependent variables, for application to satellite VIII. This method was applied by Hori (1957, 1958) to satellite IX. An approach using multiple Fourier analysis was made by Kovalevsky (1959) to satellite VIII.

Before considering the four great satellites, which offer the most intricate resonance situation in the solar system, let us turn to the system of Saturn, in which there are a number of resonance cases each involving just two satellites. The first satellite found was Titan, discovered by Huygens in 1655. The satellites Iapetus, Rhea, Tethys, and Dione were found by Cassini, in 1671, 1672, 1684, and 1684, respectively. Mimas and Enceladus were found by W. Herschel in 1789, Hyperion by the Bonds and Lassell within two days in 1848, and Phoebe was found by Pickering in 1898. A long series of positional measurements was carried out at Pulkova, and later at Babelsberg, by H. Struve and G. Struve, who analysed the data and constructed orbital theories of Mimas, Enceladus, Tethys, Dione, Rhea, Titan, and Iapetus (H. Struve; 1888, 1898, G. Struve; 1926, 1933). More recently, Jeffreys (1953, 1954) made improvements to the theories and reestimated the masses of most of the satellites, and further improvements, including the addition of later observations, by Kozai (1955, 1956, 1957, 1976; all satellites but the most recent three found), Garcia (1972: Tethys, Dione, Rhea and Titan), Rapaport (1973, 1976: Enceladus and Dione), and Sinclair (1974: Iapetus, 1977: all satellites but the most recent three). It is good to be able to report on the renewed activity in recent years in systematic positional measures of Saturn's satellites; in some cases the interval during which no observations had been made had grown to be longer than the period for which such observations were available. We have seen the effect of such a lacuna in the case of Jupiter V.

Let us now consider the particular types of orbital motion exhibited in Saturn's system. The case of Titan shows no peculiar features; the important perturbations are the precessions of the apse and of the orbit plane arising from the effect of Saturn's figure, the attraction of Iapetus and of the Sun. Iapetus, unlike the other satellites, does not remain near the plane of Saturn's equator and rings, and it could not, since the perturbations on it due to Titan and to the Sun are comparable. To first order, the secular part of the disturbing function is constant, and this implies that the normal to the orbit plane describes an approximately elliptic cone, the period of the motion being about 3000 years. The plane normal to the axis of this cone is called the Laplacian plane, and use of it as reference plane considerably simplifies the theory of the perturbations. The motion of Phoebe is governed largely by solar perturbations, and a Delaunay type theory was applied by Ross (1905) and revised by Zadunaisky (1954).

The first resonance case identified in Saturn's system was that involving Titan and Hyperion, first recognized as such by Newcomb (1891), who showed that the observed retrograde motion of the apse is reconciled with perturbation theory since the argument $\theta=4\lambda_H-3\lambda_T-\tilde{\omega}_H$, of what is in normal cases a periodic term of the disturbing function, in this case does not change monotonically, but oscillates, or "librates", about the value 180°. An alternative way of describing this is to say that the forced oscillations, due to the attraction of Titan, in the radial distance and longitude, are in this case, because of the argumentation due to the resonance, of larger amplitude than the free oscillations usually identified with the eccentricity of the orbit, so that the observed apses are the maxima and minima of the radial distance under the action of Titan, and the mean eccentricity is given by the amplitude of this forced oscillation. Motion of this type with no free oscillation would, if Titan's orbit were circular, correspond to a periodic solution of Poincaré's second sort in the restricted problem of three bodies, of such a type that, at each conjunction of Hyperion with Titan, Hyperion is at aposaturnium. In fact there is a free oscillation, giving mainly a libration in longitude of about 9° amplitude, and of period about 21 months. Also the eccentricity of Titan's orbit introduces an additional forced long-period oscillation, of period equal to that of the relative motion of the two apses (about 18 3/4 years), and of amplitude about 13°.8 in the

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apse longitude of Hyperion, and 0.024 in the eccentricity. The observations made between 1887 and 1922 were reduced in the treatment given by Woltjer (1928), who found that the mean apse motion and period of libration in longitude gave inconsistent estimates of the mass of Titan. Jeffreys (1953) suggested that terms in the square of Titan's eccentricity, which Woltjer did not calculate, are probably significant, and I have found that, to reconcile the observational data, it is in fact necessary to construct a second-order perturbation theory. I have done this using a Lie series transformation, of the type developed by Hori (1966), to separate the long-period and short-period parts of the motions, and I hope soon to be able to give an improved estimate of the mass of Titan.

In the case of the pair Enceladus and Dione, each produces significant perturbations on the other's motion, (so that the general, rather than the restricted, problem of three bodies must be used as a model), and there are two critical arguments, $\theta=2\lambda_{Di}-\lambda_{En}-\tilde{\omega}_{En}$ and $\theta'=2\lambda_{Di}-\lambda_{En}-\tilde{\omega}_{Di}$, of the disturbing functions. However the resonance is not so close relative to the disturbing mass-ratios as in the case of Titan's action on Hyperion, and this has the consequence that a good approximation to the long-period motion is provided by the equations linear in the rectangular-type variables $h=e_{En}\sin\theta$, $k=e_{En}\cos\theta$, $h'=e_{Di}\sin\theta'$, and $k'=e_{Di}\cos\theta'$. There are two independent free oscillations about the appropriate periodic orbit of Poincaré's second sort in the general problem of three bodies. In the case of Enceladus, the observed eccentricity corresponds to a forced oscillation, as in the case of Hyperion, while that of Dione is a free oscillation, as in the case usually encountered in orbital motion.

A further type of resonance is exhibited by Mimas and Tethys, whose motion is a free oscillation about a periodic solution of the general problem of three bodies of Poincaré's third sort, that is, near to commensurability, and not coplanar. The critical argument is $\theta=4\lambda_{Te}-2\lambda_{Mi}-\Omega_{Te}-\Omega_{Mi}$, and librates about 0°. In addition to the mutual perturbations, there is appreciable precession of the orbit planes due to the oblateness of Saturn. The appropriate periodic solution is one in which the orbit planes precess at such a rate that conjunctions and oppositions of the two satellites always occur when they are at their furthest from the plane perpendicular to their total angular momentum (which is a constant) since, with this plane as reference plane, $\Omega_{Te}=\Omega_{Mi}+180°$, so that then $\theta=4(\lambda_{Te}-\lambda_{Mi})+2(\lambda_{Mi}-\Omega_{Mi})-180°$.

The case of Rhea is an example of a libration not associated with a resonance of orbital periods. Woltjer (1922) gave a treatment of the libration of the apse of Rhea about that of Titan, and this was rediscussed by Hagihara (1927), who showed that this was not dominated by the near 2:7 resonance of orbital period with Titan. Here the amplitude of the forced oscillation exceeds that of the free one in the theory of the secular variations for Rhea.

Let us now return to the four great satellites of Jupiter, which were discovered by Galileo (and perhaps independently by Marius) in 1610.

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There are here two resonances of the 2:1 type ($n_1=2n_2$, and $n_2=2n_3$), but in addition we have the relation $n_1-2n_2=n_2-2n$ satisfied to at least observational accuracy. Laplace showed that the critical angle $\lambda_1-3\lambda_2+2\lambda_3$ could librate about 180°. Observations have shown that the amplitude of the libration is certainly very small. Sampson (1921) derived a theory based on a system of cylindrical polar co-ordinates, and discussed the available observational material, and produced tables which have been of long use in predicting the positions of the satellites, their eclipses and occultations by Jupiter, and transits across its disc. In recent years modification of some of the time constants have been made to maintain the agreement with observation (see Peters, 1973), and Lieske (1974, 1977) has undertaken a program of larger scope of improvement of the theory, including the 3:7 near-commensurability between satellites III and IV. A fundamentally different approach was made by de Sitter (1918, 1925) who proposed the use of an intermediary orbit which, for the satellites I, II, and III, consists of a periodic solution of the appropriate four-body problem, constructed by taking an exact solution of the perturbation equations in which the disturbing functions retain their secular and critical terms only, and in which the free oscillations in both the eccentricity and orbit plane variables are absent, and are taken account of in the calculation of the variations from the intermediary orbit. The angular arguments chosen are such that no small divisors arise in the treatment of the periodic perturbations. Sinclair (1975) showed how the broad features of the long-period motions could be described in an illuminating way by the use of the four critical arguments $\theta_1=2\lambda_2-\lambda_1-\tilde{\omega}_1$, $\theta_2=2\lambda_2-\lambda_1-\tilde{\omega}_2$, $\theta_3=2\lambda_3-\lambda_2-\tilde{\omega}_2$, and $\theta_4=2\lambda_3-\lambda_2-\tilde{\omega}_3$. The two 2:1 commensurabilities are not too close to prevent a good approximation being given by the use of the linear equations for rectangular variables of the type used for Enceladus and Dione, and the equations for the set $h_1=e_1 \sin\theta_1$, $k_1=e_1 \cos\theta_1$, $h_2=e_2 \sin\theta_2$, $k_2=e_2 \cos\theta_2$, $h_3=e_2 \sin\theta_3$, $k_3=e_2 \cos\theta_3$, $h_4=e_3 \sin\theta_4$, and $k_4=e_3 \cos\theta_4$, with the main secular variation terms included from the outset, show readily the periodic solution proposed as intermediary by de Sitter, and enable the equations for the variations to be set up easily in linear form. De Sitter (1931) analysed the data provided by Gill's heliometer measurements, as well as visual and photometric observations of eclipses, and found values for the coefficients of the main long-period terms, and frequencies and amplitudes of the larger free oscillations. The resulting equations for the masses of the satellites and the oblateness of Jupiter are far from consistent, however. Aksnes and Franklin (1974, 1975, 1976) have analysed the rich data provided by photometric observations of mutual occultations during the passage of the Earth through the plane of the orbits in 1973. Aksnes has recently begun the reformulation of the theory on the lines proposed and begun by de Sitter, giving hope that the potential advantages of that promising approach may be reaped.

Coming now to the system of Uranus, the satellites Titania and Oberon were discovered by W. Herschel in 1787, and Ariel and Umbriel by Lassell at Liverpool in 1851, and the fainter satellite Miranda by Kuyper at the McDonald Observatory in 1948. Dunham (1971) analysed the photographic observations made from 1905 to 1916, and from 1948 to 1966.

The earlier visual observations did not add to the determinations of the orbits. The orbits of the four brighter satellites are all very nearly circular, and very nearly coplanar with Uranus' equator. The apse motion of Ariel alone is properly determinable; that of Titania barely so. Consequently the masses are unknown, apart from a very rough estimate of that of Titania. The approximate relation $n_A - n_U - 2n_T + n_0 = 0^\circ.00341$ per day is not of the type leading to libration, though the relation $n_M - 3n_A + 2n_0 = -0^\circ.08$ per day may give rise to detectable perturbations of Miranda of period 12.5 years (Whitaker and Greenberg, 1973, Greenberg, 1975).

Neptune's large satellite, Triton, was discovered by Lassell at Liverpool in 1846. The perturbations of its orbit are dominated by the oblateness of Neptune, and Eichelberger and Newton (1926), improving Newcomb's theory, found that the normal to its orbit plane describes a circular cone, the semi-vertical angle being found by Gill and Gault (1968) to be 18.86 degrees, and the period of description 580.83 years. The second satellite, Nereid, was discovered by Kuyper in 1949, and its orbit has the high eccentricity of 0.76.

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## DISCUSSION

Szebehely: Referring to the general versus the restricted problem, I assume you speak about the system, planet + satellite + satellite.

Message: Yes.

Szebehely: It is my understanding that the restricted problem is not applicable to the Neptune-Triton-Sun system.

Message: I agree.

Marsden: Would you care to make any comment about the alleged tenth and eleventh satellites of Saturn?
Message: I understand that their existence has not yet been confirmed.

Kozai: Regarding the question by Prof. Szebehely, the three-body problem of satellite case is more than the three-body problem as we cannot assume that the planet is a point mass.