Weak Amenability of a Class of Banach Algebras

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Abstract. We show that, if a Banach algebra \( A \) is a left ideal in its second dual algebra and has a left bounded approximate identity, then the weak amenability of \( A \) implies the \((2m+1)\)-weak amenability of \( A \) for all \( m \geq 1 \).

In a recent paper [2] Dales, Ghahramani and Grobæk have introduced the concept of \( n \)-weak amenability for Banach algebras. They point out the fact that, for \( n \geq 1 \), \((n+2)\)-weak amenability always implies \( n \)-weak amenability, and prove further that if a Banach algebra \( \mathcal{A} \) is an ideal in \( \mathcal{A}^{**} \), then the weak amenability of \( \mathcal{A} \) also implies the \((2m+1)\)-weak amenability of \( \mathcal{A} \) for all \( m > 0 \). As to the general case, they have raised an open question: Does weak amenability imply 3-weak amenability? This question has been answered in negative by the author in [5]. In this note we consider the Banach algebras which are one sided ideals in their second dual algebras, and discuss sufficient conditions under which weak amenability will imply \((2m+1)\)-weak amenability for \( m > 0 \). We shall also consider an example to show the use of our result.

Let \( \mathcal{A} \) be a Banach algebra and \( X \) be a Banach \( \mathcal{A} \)-bimodule. A linear mapping \( D: \mathcal{A} \to X \) is a derivation if \( D(ab) = a \cdot D(b) + D(a) \cdot b \) for \( a, b \in \mathcal{A} \). For any \( x \in X \), the mapping \( \delta_x: a \mapsto ax - xa, a \in \mathcal{A} \), is a continuous derivation, called an inner derivation. Let \( B^1(\mathcal{A}, X) \) be the space of all continuous derivations from \( \mathcal{A} \) into \( X \) and let \( Z^1(\mathcal{A}, X) \) be the space of all inner derivations from \( \mathcal{A} \) into \( X \). Then the first cohomology group of \( \mathcal{A} \) with coefficients in \( X \) is the quotient space \( H^1(\mathcal{A}, X) = B^1(\mathcal{A}, X)/Z^1(\mathcal{A}, X) \).

For each \( n \geq 1 \), \( \mathcal{A}^{(n)} \), the \( n \)-th conjugate space of \( \mathcal{A} \), is a Banach \( \mathcal{A} \)-bimodule, with the module actions defined inductively by

\[
\langle u, F \cdot a \rangle = \langle a \cdot u, F \rangle, \quad \langle u, a \cdot F \rangle = \langle u \cdot a, F \rangle, \quad F \in \mathcal{A}^{(n)}, u \in \mathcal{A}^{(n-1)}, a \in \mathcal{A}.
\]

A Banach algebra \( \mathcal{A} \) is called \( n \)-weakly amenable if \( H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\} \). Usually, 1-weakly amenable Banach algebras are called weakly amenable.

Recall that for a Banach algebra \( \mathcal{A} \), its second dual \( \mathcal{A}^{**} \) is a Banach algebra when equipped with the first Arens product which is given by the following formula

\[
\langle f, \Phi \Psi \rangle = \langle \Psi f, \Phi \rangle, \quad f \in \mathcal{A}^{**}, \quad \Phi, \Psi \in \mathcal{A}^{**},
\]
where $\Psi f \in \mathcal{A}^*$ is defined by

$$\langle a, \Psi f \rangle = \langle fa, \Psi \rangle, \quad a \in \mathcal{A}.$$  

We refer to Arens’ original paper [1] and the survey paper [3] for properties and references about Arens products. In this note, for $m \geq 1$, we always equip $\mathcal{A}^{(2m)}$ with the first Arens product.

For a Banach space $X$ we will denote by $\hat{X}$ (resp. $\hat{x}$) the image of $X$ (resp. $x \in X$) in $X^{(2m)}$ under the canonical mapping. But if no confusion may occur we will keep using $X$ to denote this image. For $m > 0$, the subspace of $X^{(2m+1)}$ annihilating $\hat{X}$ will be denoted by $X^\perp$, i.e., $X^\perp = \{ F \in X^{(2m+1)} ; F|_X = 0 \}$. Concerning the Banach algebra $\mathcal{A}^{(2m)}$ we have:

**Lemma 1** Suppose that $\mathcal{A}$ is a left, right or two sided ideal in $\mathcal{A}^{**}$. Then it is also a left, right or two sided ideal in $\mathcal{A}^{(2m)}$ for all $m \geq 1$.

**Proof** Assume that $\mathcal{A}$ is a left ideal of $\mathcal{A}^{(2m)}$, $m \geq 1$. We prove that it is also a left ideal of $\mathcal{A}^{(2m+2)}$. First we have the following $\mathcal{A}$-bimodule direct sum decompositions

$$\mathcal{A}^{(2m+2)} = (\mathcal{A}^*)^\perp + (\mathcal{A}^{**})^* \quad (1)$$

and

$$\mathcal{A}^{(2m+1)} = (\mathcal{A})^\perp + (\mathcal{A}^*)^* \quad (2)$$

For any $F \in \mathcal{A}^{(2m+1)}$, let $F = f_1 + \hat{f}_2$, $f_1 \in \mathcal{A}^\perp$, $f_2 \in \mathcal{A}^*$. Then $af_1 = 0$ for $a \in \mathcal{A}$, since $\mathcal{A}$ is a left ideal in $\mathcal{A}^{(2m)}$. So

$$aF = af_2 = (af_2)^*.$$

For any $\Phi \in \mathcal{A}^{(2m+2)}$, let $\Phi = \phi + \hat{u}$, $\phi \in (\mathcal{A}^*)^\perp$, $u \in \mathcal{A}^{**}$. Then

$$\langle F, \Phi a \rangle = \langle (af_2)^*, \phi + \hat{u} \rangle = \langle (af_2)^*, \hat{u} \rangle = \langle F, (ua)^* \rangle.$$  

This shows that $\Phi a = (ua)^* \in \hat{\mathcal{A}}$ for $a \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{(2m+2)}$. Therefore $\mathcal{A}$ is a left ideal of $\mathcal{A}^{(2m+2)}$. So the lemma is true when $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$. For the other two cases the proof is similar.

It is known that for a Banach algebra $\mathcal{A}$ with a bounded approximate identity (b.a.i. in short), if $X$ is a Banach $\mathcal{A}$-bimodule in which $\mathcal{A}$ acts trivially on one side, then $\mathcal{H}^1(\mathcal{A}, X^*) = \{ 0 \}$ (see [4, Proposition 1.5]). The following lemma can be viewed as an extension of this result.

**Lemma 2** Suppose that $\mathcal{A}$ is a Banach algebra with a left (right) b.a.i.. Suppose that $X$ is a Banach $\mathcal{A}$-bimodule and $Y$ is a weak$^*$ closed submodule of the dual module $X^*$. If the left (resp. right) $\mathcal{A}$-module action on $Y$ is trivial, then $\mathcal{H}^1(\mathcal{A}, Y) = \{ 0 \}$. 

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Proof The proof is quite standard. Here we give the proof in the case that \( \mathcal{A} \) has a left b.a.i. and \( \mathcal{A} \) acts trivially on the left in \( Y \). Suppose that \( D: \mathcal{A} \to Y \) is a continuous derivation. Let \( (e_i) \) be a left b.a.i. of \( \mathcal{A} \), and \( f \in Y \) be a weak* cluster point of \( \{D(e_i)\} \).

Since \( \mathcal{A}Y = \{0\} \), we have
\[
D(a) = \lim D(e_i a) = f a = f a - a f, \quad a \in \mathcal{A}.
\]

Hence \( D \) is inner. This shows that \( \mathcal{H}^1(\mathcal{A}, Y) = \{0\} \).

With the preceding two lemmas, we can now prove a partial converse to [2, Proposition 1.2] as follows.

**Theorem 3** Suppose that \( \mathcal{A} \) is a weakly amenable Banach algebra. If \( \mathcal{A} \) has a left (right) b.a.i. and is a left (resp. right) ideal in \( \mathcal{A}^{**} \), then \( \mathcal{A} \) is \((2m+1)\)-weakly amenable for \( m \geq 1 \).

**Proof** We give the prove in the case that \( \mathcal{A} \) has a left b.a.i. and is a left ideal in \( \mathcal{A}^{**} \).

The proof for the other case is similar. First, from the \( \mathcal{A} \)-bimodule decomposition (2) we have the cohomology group decomposition
\[
\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(2m+1)}) = \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) + \mathcal{H}^1(\mathcal{A}, \mathcal{A}^\perp).
\]

If \( \mathcal{A} \) is weakly amenable, we have \( \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\} \). \( \mathcal{A}^\perp \) is clearly weak* closed submodule of \( \mathcal{A}^{(2m+1)} \). Since \( \mathcal{A} \) is a left ideal in \( \mathcal{A}^{**} \), it is a left ideal in \( \mathcal{A}^{(2m)} \) from Lemma 1. It follows that the left \( \mathcal{A} \)-module action on \( \mathcal{A}^\perp \) is trivial. Then Lemma 2 leads to \( \mathcal{H}^1(\mathcal{A}, \mathcal{A}^\perp) = \{0\} \). As a consequence we have \( \mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(2m+1)}) = \{0\} \), i.e., \( \mathcal{A} \) is \((2m+1)\)-weakly amenable.

Now let us consider an example. Suppose that \( S \) is an infinite set and \( s_0 \) a fixed element in \( S \). Define an algebra product in \( \ell^1(S) \) in the following way.

\[
(3) \quad a \cdot b = a(s_0)b, \quad a, b \in \ell^1(S).
\]

It is easily verified that with this product \( \ell^1(S) \) is a Banach algebra. We shall denote it by \( (\ell^1(S), \cdot) \), or \( \ell^1(S) \) in short. It has a left identity \( e_0 \) defined by
\[
e_0(s) = \begin{cases} 1 & \text{if } s = s_0 \\ 0 & \text{if } s \neq s_0. \end{cases}
\]

But it has no right approximate identity. \( \ell^1(S) \) is also a left ideal in \( \ell^1(S)^{**} \). In fact, for \( u \in \ell^1(S)^{**} \), \( u = \text{wk}^* \lim a_n \), with \( (a_n) \) a bounded net in \( \ell^1(S) \), we have
\[
u \cdot a = \text{wk}^* \lim a_n \cdot a = \lim a_n(s_0) a \in \ell^1(S), \quad a \in \ell^1(S).
\]

Here we have used the fact that \( \lim a_n(s_0) \) exists. It is also easy to see that \( \ell^1(S) \) is not a right ideal of \( \ell^1(S)^{**} \). The \( \ell^1(S) \)-bimodule actions on the dual module \( \ell^1(S)^* = \ell^\infty(S) \) are in fact formulated as follows.

\[
(4) \quad a \cdot f = (a, f) e_0^s, \quad f \cdot a = a(s_0) f, \quad a \in \ell^1(S), f \in \ell^\infty(S),
\]

where \( e_0^s \) is the element of \( \ell^\infty(S) \) satisfying \( e_0^s(s_0) = 1 \), and \( e_0^s(s) = 0 \) for \( s \neq s_0 \).
Assertion 1  The Banach algebra $\ell^1(S, \cdot)$ is weakly amenable.

Proof  Suppose that $D: \ell^1(S) \to \ell^\infty(S)$ is a derivation. Then for $a, b \in \ell^1(S)$, from equations (3) and (4),

$$a(s_0)D(b) = D(a \cdot b) = a \cdot D(b) + D(a) \cdot b$$

$$= \langle a, D(b) \rangle e_0^* + b(s_0)D(a).$$

By taking $b = a$, we see $\langle a, D(a) \rangle = 0$ for all $a \in \ell^1(S)$. This in turn implies that

$$\langle a, D(b) \rangle = -\langle b, D(a) \rangle, \quad a, b \in \ell^1(S).$$

So

$$D(a) = D(e_0 \cdot a) = \langle e_0, D(a) \rangle e_0^* + a(s_0)D(e_0)$$

$$= -\langle a, D(e_0) \rangle e_0^* + a(s_0)D(e_0)$$

$$= D(e_0) \cdot a - a \cdot D(e_0), \quad a \in \ell^1(S).$$

Therefore $D$ is inner. This shows that $\ell^1(S, \cdot)$ is weakly amenable and the proof is complete.

By using Theorem 3, Assertion 1 induces immediately the following:

Assertion 2  For $m \geq 0$, $\ell^1(S, \cdot)$ is $(2m + 1)$-weakly amenable.

Note  The algebra $\ell^1(S, \cdot)$ is not $2m$-weakly amenable for any $m \geq 1$.

Proof  From [2, Proposition 1.2] it suffices to show that $\ell^1(S, \cdot)$ is not 2-weakly amenable. Let $E = \{e_0^*\}^\perp \subset \ell^1(S)^{**}$. Then for every $u \in E$ and every $a \in \mathfrak{A}$, from equation (4), $u \cdot a = 0$. This implies that any linear mapping from $\ell^1(S)$ into $E$ is a derivation. Especially $D: a \mapsto a(s_1)u$ for some nonzero $u \in E$ and $s_1 (\neq s_0) \in S$ is a continuous non-inner derivation from $\ell^1(S)$ into $\ell^1(S)^{**}$. Therefore $\ell^1(S, \cdot)$ is not 2-weakly amenable.

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References


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