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Note on the Inequality Theorems that lead up to the Exponential Limit

$$\underset{x=\infty}{\mathrm{L}} \left(1+\frac{1}{x}\right)^x = e.$$

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The object of the paper is to suggest a rearrangement of these theorems, which, it is contended, is simpler than, but as effective as, that usually adopted. The slightly more general theorems thus obtained suggest themselves readily, but the writer has not endeavoured to investigate other than the most elementary theorems in question.

THEOREM I. If $a_1, a_2, \ldots a_n$ be n positive quantities then

$$\frac{\Sigma a}{n} \not< \sqrt[n]{\Pi a}.$$

The theorem is true for any two quantities a_1, a_2, or for three a_1, a_2, a_3. Assume it true for any $n-1$ quantities $a_1, a_2, \ldots a_{n-1}$.

Arrange $a_1 \ldots a_n$ in order of magnitude, a_n being greatest, and let $\sqrt[n-1]{a_1 \ldots a_{n-1}} = b$, so that $b < a_n$.

$$\therefore \quad \frac{a_1 + \ldots + a_{n-1}}{n-1} \not< b,$$

$$\therefore \quad a_1 + \ldots a_{n-1} \not< (n-1)b.$$

$$\text{Hence} \quad a_1 + \ldots + a_n \not< a_n + (n-1)b,$$

$$\therefore \quad \frac{a_1 + \ldots + a_n}{n} \not< b + \frac{(a_n - b)}{n};$$

$$\therefore \quad \left(\frac{a_1 + \ldots + a_n}{n}\right)^n \not< \left(b + \frac{a_n - b}{n}\right)^n$$

$$\not< b^n + {}_nC_1 b^{n-1} \frac{a_n - b}{n} + \ldots$$

$$\not< a_n b^{n-1} + \ldots$$

$$\not< \Pi a + \ldots$$

$$\therefore \quad \frac{\Sigma a}{n} \not< \sqrt[n]{\Pi a} \qquad \therefore \text{ etc.}$$

(*Vide* Weber's Algebra.)

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Theorem II. If a, m, and n be positive then

$$\frac{ma^x + na^y}{m+n} > a^{\frac{mx+ny}{m+n}} \quad \text{(i.)}$$

and

$$\frac{ma^x - na^y}{m-n} < a^{\frac{mx-ny}{m-n}} \quad \text{(ii.)}$$

(i.) is a direct consequence of I. as found by putting m a's equal to a^x, n of them equal to a^y. This proves the theorem when m and n are positive integers; but from the manner in which m and n occur it is clear that we can substitute for them any fractions so that the theorem is true when m and n are not integers.

From (i.) we deduce, when $m-n$ is positive,

$$\frac{(m-n)a^{\frac{mx-ny}{m-n}} + na^y}{(m-n)+n} > a^{\frac{(mx-ny)+ny}{(m-n)+n}}$$

$$> a^x;$$

$$\therefore \quad (m-n)a^{\frac{mx-ny}{m-n}} > ma^x - na^y,$$

$$\therefore \quad \frac{ma^x - na^y}{m-n} < a^{\frac{mx-ny}{m-n}}.$$

Since x and y are *any* quantities it is clearly immaterial whether $m-n$ or $n-m$ be positive.

(This theorem was suggested by Professor Chrystal, though not in this form.)

Theorem III. If a be positive and distinct from unity, and $m-n$ positive, then

$$\frac{a^m - 1}{m} > \frac{a^n - 1}{n}.$$

This theorem is of considerable importance in the borderland between the elementary analysis and the infinitesimal calculus.

(a) Let m and n be both positive.

In (ii.) of the preceding put $x=n$, $y=m$, so that $mx-ny=0$. Then (ii.) becomes

$$\frac{ma^n - na^m}{m-n} < 1,$$

$$\therefore \quad ma^n - na^m < m-n, \quad \because \ m-n \text{ is positive.}$$

$$\therefore \quad m(a^n-1) < n(a^m-1),$$

$$\therefore \quad \frac{a^m-1}{m} > \frac{a^n-1}{n},$$

which proves the theorem when m and n are positive.

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(β) Since a is any positive quantity, we deduce

$$\frac{\left(\frac{1}{a}\right)^m - 1}{m} > \frac{\left(\frac{1}{a}\right)^n - 1}{n},$$

$$\therefore \quad \frac{a^{-m} - 1}{-m} < \frac{a^{-n} - 1}{-n},$$

$$i.e. \quad \frac{a^{\nu} - 1}{\nu} < \frac{a^{\mu} - 1}{\mu},$$

if μ and ν be both negative and $\mu - \nu = (-n) - (-m) = m - n$ be positive.

(γ) Finally if we put $x = -n$, $y = m$ in II., we obtain

$$\frac{ma^{-n} + na^m}{m+n} > 1,$$

$$\therefore \quad ma^{-n} + na^m > m + n,$$

$$\therefore \quad \frac{a^m - 1}{m} > \frac{1 - a^{-n}}{n}, \quad i.e. > \frac{a^{-n} - 1}{-n}.$$

The theorem is therefore true in all cases.

Cor. 1. If a be any positive quantity $\dfrac{a^x - 1}{x}$ constantly increases with x.

Cor. 2. $\dfrac{a^x - 1}{x} \lesseqgtr a - 1$ according as $x \lesseqgtr 1$.

THEOREM IV.

$$ma^{m-1}(a-1) > a^m - 1 > m(a-1)$$

for all values of m except those for which $0 < m < 1$, when the signs of inequality are reversed.

By theorem III. we have in all cases

$$\frac{a^m - 1}{m} > \frac{a^{m-1} - 1}{m-1}$$

Hence if $m(m-1)$ be positive we deduce

$$(m-1)(a^m - 1) > m(a^{m-1} - 1),$$

$$i.e. \quad ma^{m-1}(a-1) > a^m - 1.$$

But if m be positive and less than unity, $m(m-1)$ is negative and

$$(m-1)a^{m-1} < m(a^{m-1} - 1),$$

so that

$$ma^{m-1}(a-1) < a^m - 1.$$

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To prove the second part of the inequality we note that when m is positive

$$\frac{a^m - 1}{m} \lesseqgtr \frac{a-1}{1} \quad \text{according as } m \lesseqgtr 1\,;$$

and if m be negative

$$\frac{a^m - 1}{m} < \frac{a-1}{1}$$

$$\therefore \quad a^m - 1 > m(a-1) \quad \because \quad m \text{ is negative.}$$

Hence in all cases

$$ma^{m-1}(a-1) \gtreqless a^m - 1 \gtreqless m(a-1)$$

according as m does not, or does, lie between 0 and 1.

THEOREM V. $\underset{x=\infty}{\mathrm{L}} \left(1+\frac{1}{x}\right)^x$ is finite.

If x and y be positive; $\frac{x}{y}$ positive and greater than 1; and $1 \pm \frac{1}{x}$, $1 \pm \frac{1}{y}$ be positive, we deduce

$$\frac{\left(1 \pm \frac{1}{x}\right)^{\frac{x}{y}} - 1}{\frac{x}{y}} > \frac{\left(1 \pm \frac{1}{x}\right) - 1}{1},$$

$$\therefore \quad \left(1 \pm \frac{1}{x}\right)^{\frac{x}{y}} - 1 > \pm \frac{1}{y},$$

$$\therefore \quad \left(1 \pm \frac{1}{x}\right)^{\frac{x}{y}} > 1 \pm \frac{1}{y}.$$

$$\text{Hence} \quad \left(1 \pm \frac{1}{x}\right)^{x} > \left(1 \pm \frac{1}{y}\right)^{y}$$

$$\text{and} \quad \left(1 \pm \frac{1}{x}\right)^{-x} < \left(1 \pm \frac{1}{y}\right)^{-x}$$

$$\text{Hence,} \quad \left(1 + \frac{1}{x}\right)^{x} > \left(1 + \frac{1}{y}\right)^{y}$$

$$\text{and} \quad \left(1 - \frac{1}{x}\right)^{-x} < \left(1 - \frac{1}{y}\right)^{-y}, \text{ if } x > y\,;$$

and we have the conclusion that

$\left(1+\frac{1}{x}\right)^x$ constantly increases with x; and $\left(1-\frac{1}{x}\right)^{-x}$ constantly decreases as x increases.

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On the other hand, when y is > 1,

$$\frac{\left(1+\frac{1}{x}\right)^{-\frac{x}{y}}-1}{-\frac{x}{y}} < \left(1+\frac{1}{x}\right)-1$$

$$\therefore \left(1+\frac{1}{x}\right)^{-\frac{x}{y}}-1 > -\frac{1}{y}$$

$$\left(1+\frac{1}{x}\right)^{-x} > \left(1-\frac{1}{y}\right)^{y}$$

$$\therefore \left(1+\frac{1}{x}\right)^{x} < \left(1-\frac{1}{y}\right)^{-y}$$

Hence $\left(1+\frac{1}{x}\right)^{x}$ has some definite upper limit A,

and $\left(1-\frac{1}{x}\right)^{-x}$ a definite lower limit B, where A $\not>$ B

As a matter of fact, $A = B$ for

$$\underset{x=\infty}{\mathrm{L}} \frac{\left(1-\frac{1}{x}\right)^{-x}}{\left(1+\frac{1}{x}\right)^{x}} = \mathrm{L}\left(1-\frac{1}{x^2}\right)^{-x} = \mathrm{L}\left[\left(1-\frac{1}{x^2}\right)^{-x^2}\right]^{\frac{1}{x}}$$

$$= B^0 = 1.$$

Even if this paper had no other interest, it would draw the attention to the remarkable fact that the *order* in which these theorems of inequalities require to be demonstrated is somewhat arbitrary.

Thus Theorem V. may be utilised to prove Theorem I. (*vide* Chrystal's *Algebra*), and yet the inequality

$$\left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{y}\right)^{y}, \quad x > y,$$

must be very perfect, for, from the inequality

$$\left(\frac{a+c}{a-c}\right)^{a} < \left(\frac{b+c}{b-c}\right)^{b}, \quad a > b > c > 0,$$

may readily be concluded that

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} < \left(1+\frac{1}{y}\right)^{y+\frac{1}{2}}$$

It may be added that the process of differentiation may be successfully applied to III., and to the inequalities of V.