Proceedings of the Edinburgh Mathematical Society (2005) **48**, 673–690 ©
DOI:10.1017/S0013091503000889 Printed in the United Kingdom

TRACIAL EQUIVALENCE FOR C^*-ALGEBRAS AND ORBIT EQUIVALENCE FOR MINIMAL DYNAMICAL SYSTEMS

HUAXIN LIN*

Department of Mathematics, East China Normal University, Shanghai, People's Republic of China

(Received 13 October 2003)

Abstract We introduce the notion of tracial equivalence for C^*-algebras. Let A and B be two unital separable C^*-algebras. If they are tracially equivalent, then there are two sequences of asymptotically multiplicative contractive completely positive linear maps $\phi_n : A \to B$ and $\psi_n : B \to A$ with a tracial condition such that $\{\phi_n \circ \psi_n\}$ and $\{\psi_n \circ \phi_n\}$ are tracially approximately inner. Let A and B be two unital separable simple C^*-algebras with tracial topological rank zero. It is proved that A and B are tracially equivalent if and only if A and B have order isomorphic ranges of tracial states. For the Cantor minimal systems (X_1, σ_1) and (X_2, σ_2), using a result of Giordano, Putnam and Skau, we show that two such dynamical systems are (topological) orbit equivalent if and only if the associated crossed products $C(X_1) \times_{\sigma_1} \mathbb{Z}$ and $C(X_2) \times_{\sigma_2} \mathbb{Z}$ are tracially equivalent.

Keywords: simple C^*-algebras; tracial equivalence; minimal dynamical systems

2000 *Mathematics subject classification:* Primary 46L05; 46L35

1. Introduction

Tracial topological rank for C^*-algebras was introduced in [**7**] (see also [**6**]). It plays an important role in the study of classification of amenable C^*-algebras. A C^*-algebra A has tracial topological rank zero if (roughly speaking) it can be approximated by finite-dimensional C^*-algebras in 'trace', or the part that cannot be approximated by finite-dimensional C^*-algebras has small trace. Many amenable C^*-algebras have been proved to have tracial topological rank zero. More recently, N. C. Phillips [**16**] has shown that if A is a unital separable simple C^*-algebra with tracial topological rank zero and if an automorphism α on A has finite order and has a 'tracially' Rokhlin property, then the resulting crossed product has tracial topological rank zero. It seems that 'tracial' versions of many notions in C^*-algebras should be exploited further. As was pointed out by Phillips, a notion of tracially approximated inner isomorphism may be useful. Moreover, there should be a 'tracial' version of 'isomorphisms'. In this short note we will discuss what should be an appropriate version of 'tracially isomorphism'. We will say

* Present address: Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA (hxlin@noether.uoregon.edu).

that two C^*-algebras are tracially equivalent if they are 'tracially isomorphic'. It turns out that this tracial equivalence is a rather interesting concept. It also further reinforces the usage of adjective 'tracial'. Let A be a separable C^*-algebra with non-trivial tracial state space $T(A)$. Denote by $\rho_A : K_0(A) \to \mathrm{aff}(T(A))$ the positive homomorphism from $K_0(A)$ into the affine continuous functions on $T(A)$. Suppose that A and B are two separable simple C^*-algebras with tracial topological rank zero. We show that A and B are tracially equivalent if and only if $\rho_A(K_0(A))$ and $\rho_B(K_0(B))$ are order isomorphic. Consequently, unital simple separable C^*-algebras with tracial topological rank zero are tracially equivalent to unital simple AF-algebras.

Let (X_1, ϕ_1) and (X_2, ϕ_2) be two dynamical systems. Suppose that X_1 and X_2 are the Cantor sets and ϕ_1 and ϕ_2 are minimal homeomorphisms. Giordano, Putnam and Skau [**5**] showed (among other things) that two such dynamical systems are topological orbit equivalent if and only if the resulting simple crossed products have the order isomorphic ranges of traces. Using this result, we show that (X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent if and only if the resulting simple crossed products are tracially equivalent.

2. Tracially approximately inner morphisms

Definition 2.1. Let A and B be C^*-algebras. Let $L_n : A \to B$ be a positive linear contraction, $n = 1, 2, \dots$. The sequence $\{L_n\}$ is said to be asymptotically multiplicative if

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \quad \text{for any pair } a, b \in A.$$

Note that, since each L_n is positive and linear, L_n is also $*$-preserving.

Definition 2.2. Let A be a unital simple separable C^*-algebra and let $\{\phi_n\}$ be a sequence of asymptotically multiplicative positive linear contractions from A to B. We say that $\{\phi_n\}$ is *tracially approximately inner*, if for any $\varepsilon > 0$, any non-zero $a \in A_+$, and any finite subset $\mathcal{F} \subset A$, there is an integer N, a sequence of unitaries $u_n \in A$ and a sequence of projections $p_n \in A$ such that

(1) $\|p_n x - x p_n\| < \varepsilon$ for all $x \in \mathcal{F}$ and for all $n \geqslant N$,

(2) $\|u_n^* p_n x p_n u_n - \phi_n(p_n x p_n)\| < \varepsilon$ for all $x \in \mathcal{F}$ and for all $n \geqslant N$, and

(3) $1 - p_n$ is equivalent to a projection in $\overline{aAa}$ for all $n \geqslant N$.

Remark 2.3. Suppose that A has at least one tracial state. Then, from (3), p can be chosen so that $\tau(p) < \sigma$ for all tracial states and for some previously given $\sigma > 0$. Thus that $\{\phi_n\}$ is tracially approximately inner implies that the 'part' of $\mathcal{F}$ on which $\{\phi_n\}$ is not approximately inner has arbitrarily small 'measure' (or in trace).

Remark 2.4. Definition 2.2 assumes A is simple. One can define tracially approximately inner morphisms for general unital C^*-algebras as follows.

Let A be a unital C^*-algebra and $\{\phi_n\}$ be a sequence of asymptotically multiplicative positive linear contractions from A to itself. We say that $\{\phi_n\}$ is tracially approximately inner if the following conditions hold: for any $\varepsilon > 0$, any finite subset set $\mathcal{F} \subset A$ and any non-zero element $b \in A_+$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exists an integer $N > 0$,

https://doi.org/10.1017/S0013091503000889 Published online by Cambridge University Press

a sequence of unitaries $u_n \in A$ and a sequence of projections $p_n \in A$ such that

(1) $\|p_n x - x p_n\| < \varepsilon$ for $x \in \mathcal{F}$ and for all $n \geqslant N$,

(2) $\|u_n^* p_n a p_n u_n - \phi_n(p_n a p_n)\| < \varepsilon$ for all $a \in \mathcal{F}$ and for all $n \geqslant N$, and

(3) $[f_{\sigma_2}^{\sigma_1}((1-p_n)b(1-p_n))] \leqslant [f_{\sigma_4}^{\sigma_3}(p_n b p_n)]$.

Here (3) means that there is $x_n \in A$ such that

$$x_n^* x_n = f_{\sigma_2}^{\sigma_1}((1-p_n)b(1-p_n)) \quad \text{and} \quad x_n x_n^* \in \mathrm{Her}(f_{\sigma_4}^{\sigma_3}(p_n b p_n)),$$

where f_α^β $(0 < \alpha < \beta)$ is a positive continuous function on $(0, \infty)$ such that $f_\alpha^\beta(t) = 0$ for $0 \leqslant t \leqslant \alpha$, $f_\alpha^\beta(t) = 1$ for $\beta \leqslant t$ and $f_\alpha^\beta(t)$ is linear in (α, β).

However, in this short note we will only consider the case that A is simple. The general case will be discussed elsewhere.

Definition 2.5. Let A and B be two unital C^*-algebras and $\{L_n\}$ be a sequence of asymptotically multiplicative positive linear contractions from A to B. Fix a projection $p \in A$. For sufficiently large n, there is a projection $q_n \in B$ such that

$$\|L_n(p) - q_n\| < \tfrac{1}{2}.$$

Note that if there is another projection q_n' such that

$$\|L_n(p) - q_n'\| < \tfrac{1}{2},$$

then q_n and q_n' are equivalent. We use $[L_n]|_p$ for $[q_n]$. Given a finite set $\mathcal{P}$ of projections in A, there is an integer $N > 0$ such that $[L_n]|_p$ is well defined for each $p \in \mathcal{P}$ and $n \geqslant N$. We will use $[L_n]|_{\mathcal{P}}$ for the finite set $[L_n]|_p$ with $p \in \mathcal{P}$. For the rest of this paper, whenever we write $[L_n]_{\mathcal{P}}$ we mean that n is large enough so that $[L_n]|_{\mathcal{P}}$ is well defined. Let $G(\mathcal{P})$ be the finitely generated subgroup in $K_0(A)$ which is generated by $\mathcal{P}$. With possibly even large N, $[L_n]$ defines a group homomorphism from $G(\mathcal{P})$ to $K_0(B)$. So, when we write $[L_n]|_{\mathcal{P}}$ we also mean that $[L_n]$ defines a group homomorphism. One should note that $[L_n]|_{\mathcal{P}}$ specifies the set of projections $\mathcal{P}$. It is possible that there is another projection $p' \in A$ which is equivalent to a projection $p \in \mathcal{P}$ such that $L_n(p')$ is also closed to a projection $q' \in B$ but q' may not be in $[L_n]_p$. However, in order to include $[L_n]_{p'}$ we should choose large N, so that the projection defined by $L_n(p)$ is equivalent to that of $L_n(p')$.

Definition 2.6. Let A be a unital simple C^*-algebra. Let $T(A)$ be the set of tracial states of A. We assume that $T(A) \neq \emptyset$. Then $T(A)$ is a compact convex space (a Choquet simplex; see, for example, [**1**]). Denote by $\mathrm{aff}(T(A))$ the set of continuous affine functions on $T(A)$. Define $\rho_A : K_0(A) \to \mathrm{aff}(T(A))$ to be the positive homomorphism given by $\rho_A([p]) = \hat{p}$, where $\hat{p}(t) = t(p)$ for $t \in T(A)$ and p is a projection in $M_k(A)$ for some integer $k > 0$.

Theorem 2.7. *Let A be a separable simple C^*-algebra and $\{\phi_n\}$ be a sequence of asymptotically multiplicative positive linear contractions from A to A. If $\{\phi_n\}$ is tracially approximately inner, then the following holds. There exists a sequence of finite subsets $\mathcal{G}_n$ of projections such that*

(1) $\operatorname{dist}(\rho_A(p), \rho_A(G(\mathcal{G}_n))) \to 0$ *as $n \to \infty$ for all projections $p \in A \otimes \mathcal{K}$, where $G(\mathcal{G}_n)$ is the subgroup of $K_0(A)$ generated by the equivalence classes determined by projections in $\mathcal{G}_n$, and*

(2) $[\phi_n]|_{\mathcal{G}_n} = \mathrm{id}|_{\mathcal{G}_n}$ *for all large n.*

Furthermore, for any $a \in A$,

$$\lim_{n\to\infty} \tau \circ \phi_n(a) = \tau(a)$$

uniformly on $T(A)$.

Proof. It suffices to show the following. For any $\varepsilon > 0$ and any finite subset $\mathcal{G} \subset \rho_A(K_0(A))$, there exists a finite subset $\mathcal{G}_0$ of projections in A such that

(1′) for any $g \in \mathcal{G}$, there exists $f \in G(\mathcal{G}_0)$, where $G(\mathcal{G}_0)$ is the subgroup generated by $\mathcal{G}_0$ such that
$$\|g - \rho_A(f)\| < \varepsilon,$$

(2′) $[\phi_n]|_{\mathcal{G}_0} = \mathrm{id}_{\mathcal{G}_0}$ for all large n.

For each k, let $\Phi_n^{(k)} = \phi_n \otimes \mathrm{id}_{M_k} : A \otimes M_k \to A \otimes M_k$ be the extension of ϕ_n to $A \otimes M_k$. It is clear that $\Phi_n^{(k)}$ are tracially approximately inner. From this, to simplify notation, we may assume that $\mathcal{G} = \{q_1, q_2, \dots, q_m\}$ are projections in A.

For each $\varepsilon > 0$, since A is simple, there exists a non-zero positive element $a \in A$ such that $\tau(a) < \varepsilon/2$ for all $\tau \in T(A)$. Choose a large N such that, for all $n \geqslant N$,

(i) $\|p_n q_i - q_i p_n\| < \frac{1}{8}$, $i = 1, 2, \dots, m$,

(ii) $\tau(1 - p_n) < \frac{1}{2}\varepsilon$ for all $\tau \in T(A)$, there are unitaries $u_n \in A$ such that

(iii) $\|\mathrm{ad}\, u_n \circ \phi_n(p_n q_i p_n) - p_n q_i p_n\| < \frac{1}{8}$, $i = 1, 2, \dots, m$.

From (i), one obtains a projection $e_i \leqslant p_n$ such that

$$\|e_i - p_n q_i p_n\| < \tfrac{1}{8}, \quad i = 1, 2, \dots, m.$$

Moreover, we may assume that there are projections $e_i' \in A$ such that

(iv) $\|\phi_n(p_n q_i p_n) - e_i'\| < \frac{1}{8}$, $i = 1, 2, \dots, m$.

It follows that

$$\|u_n^* e_i' u_n - e_i\| < \tfrac{1}{2}, \quad i = 1, 2, \dots, m.$$

Let $\mathcal{G}_0 = \{e_1, e_2, \dots, e_m\}$. It follows that

$$[\phi_n]|_{\mathcal{G}_0} = \mathrm{id}_{\mathcal{G}_0}.$$

We also note that

$$|\tau(q_i) - \tau(e_i)| < \varepsilon, \quad i = 1, 2, \dots, m.$$

This proves the first part of the theorem. To see the last part, we note that, for each $a \in A$,

$$\lim_{n\to\infty} [\tau \circ \phi_n(p_n a p_n) - \tau(p_n a p_n)] = 0$$

and it converges uniformly on $T(A)$. We also have $\tau(p_n a p_n) \to \tau(a)$ uniformly on $T(A)$ since $\tau(1-p_n) \to 0$ uniformly. The fact that $\{\phi_n\}$ is asymptotically multiplicative implies that

$$\lim_{n\to\infty} \|u_n \phi_n((1-p_n)a(1-p_n))u_n^* p_n\| = 0.$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty} \tau(\phi_n((1-p_n)a(1-p_n))) &= \lim_{n\to\infty} \tau(u_n \phi_n((1-p_n)a(1-p_n))u_n^*) \\
&= \lim_{n\to\infty} \tau(u_n \phi_n(1-p_n)a(1-p_n)u_n^*(1-p_n)) \\
&\leqslant \|a\|\tau(1-p_n) \\
&= 0.
\end{aligned}$$

Moreover, the convergence is uniform on $T(A)$. □

Corollary 2.8. *Let A be a unital separable C^*-algebra and let $h : A \to A$ be a homomorphism. Let $\phi_n = h$ for each n. If $\{\phi_n\}$ is tracially approximately inner, then h_{*0} induces the identity map on $\rho_A(K_0(A))$.*

Proof. Note that h_{*0} induces a positive homomorphism. In Theorem 2.7, let $\phi_n = h$ for all n. We then know that

$$\tau(h(a)) = \tau(a) \quad \text{for all } \tau \in T(A)$$

and all $a \in A$. Thus, h_{*0} induces the identity map on $\mathrm{aff}(T(A))$. In particular, it induces the identity map on $\rho_A(K_0(A))$. □

Definition 2.9 (Lin [7]). Recall that a unital simple C^*-algebra is said to have tracial topological rank zero (written $\mathrm{TR}(A) = 0$), if for any $\varepsilon > 0$, any finite subset $\mathcal{F}$ and a non-zero element $a \in A_+$ there exists a finite-dimensional C^*-subalgebra $C \subset A$ with $1_C = p$ such that

(1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2) $\mathrm{dist}(pxp, C) < \varepsilon$ for all $x \in \mathcal{F}$,

(3) $1 - p$ is equivalent to a projection in $\overline{aAa}$.

https://doi.org/10.1017/S0013091503000889 Published online by Cambridge University Press

Every simple C^*-algebra A with $\mathrm{TR}(A) = 0$ is a quasi-diagonal C^*-algebra and has real rank zero, stable rank one and weakly unperforated $K_0(A)$ (see [**6**]).

It is known all simple AH-algebras with real rank zero and with slow dimension growth have tracial topological rank zero (see [**4, 6**]). A slightly different version of it (see [**11**]) says that every simple AH-algebra with real rank zero, stable rank one and weakly unperforated K_0 has tracial topological rank zero. Kishimoto showed that certain simple crossed products associated with a shift have tracial topological rank zero. It is also proved (in [**12**]) that a simple C^*-algebra A with real rank zero, stable rank one, weakly unperforated $K_0(A)$, with countably many extremal tracial states which is an inductive limit of type-I C^*-algebras has $\mathrm{TR}(A) = 0$.

Theorem 2.10. *Let A be a separable simple C^*-algebra with $\mathrm{TR}(A) = 0$ and $\{\phi_n\}$ be a sequence of asymptotically multiplicative positive linear contractions from A to A. Then $\{\phi_n\}$ is tracially approximately inner if and only if the following hold.*

There exists a sequence of finite subsets $\mathcal{G}_n$ of projections such that

(1) $\mathrm{dist}(\rho_A(p), \rho_A(G(\mathcal{G}_n))) \to 0$ *as $n \to \infty$ for all projections $p \in A \otimes \mathcal{K}$, where $G(\mathcal{G}_n)$ is the subgroup generated by $\mathcal{G}_n$, and*

(2) $[\phi_n]|_{\mathcal{G}_n} = \mathrm{id}|_{\mathcal{G}_n}$ *for all large n.*

Proof. Let $\varepsilon > 0$, $\sigma > 0$ and $\mathcal{F}$ be a finite subset of A. Since A is a simple and $\mathrm{TR}(A) = 0$, there is a finite-dimensional C^*-subalgebra $B \subset A$ and a projection $p = 1_B$ such that

(i) $\|pa - ap\| < \frac{1}{4}\varepsilon$ for all $a \in \mathcal{F}$,

(ii) $pap \subset_{\varepsilon/4} B$ for all $a \in \mathcal{F}$, and

(iii) $\tau(1 - p) < \frac{1}{2}\sigma$ for all $\tau \in T(A)$.

It follows from Theorem 2.7 that it suffices to show the 'if' part of the theorem.

Write

$$B = M_{k(1)} \oplus M_{k(2)} \oplus \cdots \oplus M_{k(m)}.$$

Let $q_i \in M_{k(i)}$ be a minimal projection in $M_{k(i)}$, $i = 1, 2, \dots, m$. Since

$$\bigcup_{n=1}^{\infty} \rho_A(G(\mathcal{G}_n))$$

is dense in $\rho_A(K_0(A))$, we obtain projections $p_i \leqslant q_i$, $i = 1, 2, \dots, m$, such that $[p_i] \in G(\mathcal{G}_n)$ and

$$\tau(q_i - p_i) < \sigma/2m$$

for all $\tau \in T(A)$ and for all large n. We may assume that $G(\mathcal{G}_n)$ is generated by $[p_1], [p_2], \dots, [p_m]$. We may also assume that $\phi_n(p_i)$ are equivalent to p_i, $i = 1, 2, \dots$. Let $e_1 = q_1 - p_1$ and $d_i = 1_{M_{k(i)}}$, $i = 1, 2, \dots$. Consider the C^*-subalgebra

$$C = \bigoplus_{i=1}^{m} d_i A d_i.$$

The projection E_i (in d_iAd_i) with the form

$$E_i = \mathrm{diag}(e_i, e_i, \dots, e_i)$$

commutes with every element in $M_{k(i)}$. Define

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_m.$$

Then $Ex = xE$ for all $x \in B$. Let $B' = EBE$. Then B' is a finite-dimensional C^*-subalgebra of A (which is isomorphic to B). We note that we may identify $\mathcal{G}_n$ with $j_n(K_0(B'))$, where $j_n : B' \to A$ is the embedding.

Note that B' is semi-projective in the sense in [**14**]. Therefore, for any $\eta > 0$ and for all large n, there is an injective homomorphism $h_n : B' \to A$ such that

$$\|h_n - (\phi_n)|_{B'}\| < \eta.$$

For all large n, we have $[\phi_n]|_{\mathcal{G}_n} = \mathrm{id}|_{\mathcal{G}_n}$. It follows that (if η is sufficiently small)

$$(h_n)_{*0} = \mathrm{id}_{\mathcal{G}_n}.$$

It then follows that there is a unitary $w \in A$ such that

$$\mathrm{ad}\, w \circ h_n = \mathrm{id}_{B'}.$$

For each $a \in \mathcal{F}$, let $x \in B$ such that

$$\|pap - x\| < \tfrac{1}{4}\varepsilon.$$

Then

$$\|EaE - ExE\| = \|E(pap - x)E\| < \tfrac{1}{4}\varepsilon.$$

Note that $ExE \in B'$. We have that

$$w^* h_n(ExE)w = ExE.$$

Therefore,

$$\begin{aligned}
\|w^*\phi_n(EaE)w - EaE\| &\leqslant \|w^*\phi_n(EaE)w - w^*\phi_n(ExE)w\| \\
&\quad + \|w^*\phi_n(ExE)w - w^*h_n(ExE)w\| \\
&\quad + \|w^*h_n(ExE)w - ExE\| + \|ExE - EaE\| \\
&< \tfrac{1}{4}\varepsilon + \eta + 0 + \tfrac{1}{4}\varepsilon \\
&< \varepsilon
\end{aligned}$$

for all $a \in \mathcal{F}$ if $\eta < \frac{1}{2}\varepsilon$. We also have

$$\tau(1 - E) < \tau(1 - p) + \tau(p - E) < \tfrac{1}{2}\sigma + \tfrac{1}{2}\sigma = \sigma$$

for all $\tau \in T(A)$. Furthermore,

$$\begin{aligned}\|Ea - aE\| = \|Epa - apE\| &\leqslant \|E(pa - pap)\| \\ &\quad + \|Epap - Ex\| + \|xE - papE\| + \|(pap - ap)E\| \\ &< \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon \\ &= \varepsilon\end{aligned}$$

for all $a \in \mathcal{F}$. Thus $\{\phi_n\}$ is tracially approximately inner. □

3. Tracial equivalence

Definition 3.1. Let A be a separable C^*-algebra and B be another C^*-algebra. Let $\phi_t : A \to B$ be a family of completely positive linear contractions with $t \in [0, \infty)$ such that $\phi_t(a)$ is continuous on $C_0([0, \infty))$ for each a and

$$\lim_{t\to\infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$$

for all $a, b \in A$. Such a family will be called an *asymptotic morphism* from A into B.

Let $\{\phi_n\}$ be a sequence of asymptotically multiplicative completely positive linear contractions from A to B. We say that $\{\phi_n\}$ is a *tracially asymptotic morphism* if, in addition, for each projection $p \in M_k(A)$ (for any k),

$$\tau \circ \phi_n(p) \to \tau(q)$$

uniformly for $\tau \in T(B)$ for some projection $q \in B \otimes \mathcal{K}$.

Let $\{\phi_t\}$ be an asymptotic morphism and t_n be an increasing sequence such that $t_n \to \infty$. Let $\psi_n = \phi_{t_n}$. Then it is clear that $\{\psi_n\}$ is a tracially asymptotic morphism.

Definition 3.2. Let A and B be two unital separable (simple) C^*-algebras. We say that A and B are *tracially equivalent* if there is a tracially asymptotic morphism $\{\phi_n\}$ from A to B and there exists a tracial asymptotic morphism $\{\psi_n\}$ from B to A such that $\phi_n \circ \psi_n$ and $\psi_n \circ \phi_n$ are tracially approximately inner.

We say that A and B are *weakly tracially equivalent* if there is a sequence of asymptotically multiplicative positive linear contractions $\{\phi_n\}$ and there is $\{\psi_n\}$ from A to B and B to A, respectively, such that $\phi_n \circ \psi_n$ and $\psi_n \circ \phi_n$ are tracially approximately inner.

We say A and B are h-tracially equivalent if there are homomorphisms $h_1 : A \to B$ and $h_2 : B \to A$ such that $h_1 \circ h_2$ and $h_2 \circ h_1$ are both tracially approximately inner.

Lemma 3.3. *Let A be a unital separable simple C^*-algebra with $\mathrm{TR}(A) = 0$. Then there exists a sequence of finite-dimensional C^*-subalgebras $F_n \subset A$ and projections $p_n = 1_{F_n}$ such that*

(i) $\|p_n a - ap_n\| \to 0$ *for all* $a \in A$,

(ii) $\mathrm{dist}(p_n a p_n, F_n) \to 0$ *for all* $a \in A$, *and*

(iii) $\tau(1 - p_n) \to 0$ *uniformly on* $T(A)$ *and* $\tau(1 - p_n) > 0$ *for all* $\tau \in T(A)$.

https://doi.org/10.1017/S0013091503000889 Published online by Cambridge University Press

Proof. It is clear from the definition that (i), (ii) and the following hold.

(iii′) $\tau(1 - p_n) \to 0$ uniformly on $T(A)$.

What we need to show is that we can further assume that $\tau(1-p_n) > 0$ for all $\tau \in T(A)$.

Suppose that there is a tracial state τ such that $\tau(1 - p_{k(n)}) = 0$ for an increasing subsequence $\{k(n)\}$. Since A is simple, no non-zero positive element should have zero trace. Therefore, $p_{k(n)} = 1$, $n = 1, 2, \dots$. This implies that A is an AF-algebra.

Now we assume that A is an AF-algebra. Given a finite-dimensional C^*-subalgebra $F_n \subset A$, we may write that $F_n = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(l(n))}$. Let e_i be a minimal projection of $M_{r(i)}$, $i = 1, 2, \dots, l(n)$. Since A is simple, $\rho_A(K_0(A))$ is dense in $\mathrm{aff}(T(A))$. This implies, for any n, that there is a projection $c_i \leqslant e_i$ with $c_i \neq e_i$ such that

$$\tau(e_i - c_i) < 1/(r(i))2^n, \quad i = 1, 2, \dots, l(n),$$

for all $\tau \in T(A)$. Let $E_i^{(n)}$ be a projection with the form

$$\mathrm{diag}(c_i, c_i, \dots, c_i),$$

where c_i is repeated $r(i)$ times, $i = 1, 2, \dots, l(n)$. Put $E_n = E_1^{(n)} \oplus E_2^{(n)} \oplus \cdots \oplus E_{l(n)}^{(n)}$. Note that $E_n a = aE_n$ for all $a \in F_n$. Note also that

$$\tau(1 - E_n) < 1/2^n.$$

It is clear now that if we replace p_n by E_n, (i), (ii) and (iii) hold. □

Theorem 3.4. *Let A and B be two separable simple C^*-algebras with $\mathrm{TR}(A) = \mathrm{TR}(B) = 0$. Then A and B are tracially equivalent if and only if there is an order isomorphism from $\rho_A(K_0(A))$ onto $\rho_B(K_0(B))$ which maps $[1_A]$ to $[1_B]$.*

Proof. We first prove the 'if' part. To simplify notation, we may write that $\rho_A(K_0(A)) = \rho_B(K_0(B))$.

There exists a sequence of finite-dimensional C^*-subalgebras $F_n \subset A$ and projections $p_n = 1_{F_n}$ such that

(i) $\|p_n a - ap_n\| \to 0$ for all $a \in A$,

(ii) $\mathrm{dist}(p_n a p_n, F_n) \to 0$ for all $a \in A$, and

(iii) $\tau(1 - p_n) \to 0$ uniformly on $T(A)$.

There exists a sequence of finite-dimensional C^*-subalgebras $D_n \subset B$ and projections $q_n = 1_{D_n}$ such that

(iv) $\|q_n b - bq_n\| \to 0$ for all $a \in B$,

(v) $\mathrm{dist}(q_n b q_n, D_n) \to 0$ for all $b \in B$, and

(vi) $t(1 - q_n) \to 0$ uniformly on $T(B)$.

Denote the embedding by $j_n : F_n \to A$. Thus, $(j_n)_{*0}$ is a positive homomorphism from $K_0(F_n)$ into $K_0(A)$. Since $\rho_A \circ (j_n)_{*0}(K_0(F_n))$ is a finitely generated free group, by identifying $\rho_A(K_0(A))$ with $\rho_B(K_0(B))$, we obtain a positive homomorphism $\alpha_n : \rho_A \circ (j_n)_{*0}(K_0(F_n)) \to K_0(B)$ such that $\rho_B \circ \alpha_n = \mathrm{id}_{\rho_A \circ (j_n)_{*0}(K_0(F_n))}$. There is a monomorphism $h_n : F_n \to B$ such that $(h_n)_{*0} = \alpha_n \circ \rho_A \circ (j_n)_{*0}$.

For each $\delta_n > 0$, there is an integer $m(n)$ such that

(vii) $\|q_{m(n)}b - bq_{m(n)}\| < \delta_n$ for all b in a finite subset $\mathcal{G}_n$ of B which contains a set of matrix units of $h_n(F_n)$,

(viii) $\mathrm{dist}(q_{m(n)}bq_{m(n)}, D_{m(n)}) < \delta_n$ for all $b \in \mathcal{G}'_n$,

(ix) $\tau(1 - q_{m(n)}) < \delta_n$ for all $\tau \in T(B)$.

By Lemma 3.3, we may assume that $1 - q_{m(n)} \neq 0$. For any $\eta_n > 0$, with sufficiently small δ_n, there is a monomorphism $h'_n : h_n(F_n) \to D_{m(n)}$ such that

$$\|h'_n(b) - q_{m(n)}bq_{m(n)}\| < \eta_n \quad \text{for all } b \in \mathcal{G}_n \cap h_n(F_n).$$

Now there is a contractive completely positive linear map $s'_n : q_{m(n)}Bq_{m(n)} \to D_{m(n)}$ such that $s'_n|_{D_{m(n)}} = \mathrm{id}_{D_{m(n)}}$ (see, for example, Theorem 2.3.5 in [**9**]). Denote by $\bar{s}'_n : B \to D_{m(n)}$ the map defined by $\bar{s}'_n(b) = s'_n(q_{m(n)}bq_{m(n)})$ for all $b \in B$. Let $\imath_n : D_{m(n)} \to B$ be the embedding. There is positive homomorphism $\beta_n : \rho_B(\imath_{*0}(K_0(D_{m(n)}))) \to K_0(A)$ such that $\rho_A \circ \beta_n = \mathrm{id}_{\rho_B \circ \imath(D_{m(n)})}$, where we identify $\rho_A(K_0(A))$ with $\rho_B(K_0(B))$.

We write $F_n = M_{r(1)} \oplus \cdots \oplus M_{r(l(n))}$ and let e_i be a minimal projection in $M_{r(i)}$, $i = 1, 2, \ldots, l(n)$. Put $g_i = \beta_n \circ [h'_n \circ h_n(e_i)]$.

Note that

$$\tau(e_i) > \tau(h'_n \circ h_n(e_i))$$

for all $\tau \in T(B)$. Hence $g_i \leqslant [e_i]$. Thus, we obtain a projection $c_i \leqslant e_i$ in A such that $[c_i] = g_i$ in $K_0(A)$, $i = 1, 2, \ldots, l(n)$. Let $E_n^{(i)}$ be a projection in p_nAp_n with the form

$$E_n^{(i)} = \mathrm{diag}(c_i, c_i, \ldots, c_i),$$

where c_i is repeated $r(i)$ many times. Let $E_n = E_n^{(1)} \oplus E_n^{(2)} \oplus \cdots \oplus E_n^{(l(n))}$.

Note that

$$E_n a = aE_n \quad \text{for all } a \in F_n.$$

Define $F'_n = E_nF_nE_n$. Note $F'_n \cong F_n$ and denote by κ the isomorphism. There is a contractive completely positive linear map $s_n : E_nAE_n \to F'_n$ such that $s_n|_{F'_n} = \mathrm{id}_{F'_n}$ (see, for example, Theorem 2.3.5 in [**9**]). Denote by $\bar{s}_n : A \to F'_n$ the map defined by $\bar{s}_n(a) = s_n(E_naE_n)$. Note also that $\bar{s}_n$ is a contractive completely positive linear map. Define $L_n : A \to B$ by $L_n = h'_n \circ h_n \circ \kappa \circ \bar{s}_n$. For a fix projection $p \in A$, let $f \in \rho_B(K_0(B))$ such that $f = \hat{p}$ (here again we identify $\rho_A(K_0(A))$ with $\rho_B(K_0(B))$). It is easy to verify that

$$\tau \circ L_n(p) \to f(\tau)$$

uniformly on $T(B)$ by (i)–(iii) and (vii)–(ix).

There is a homomorphism $h_n'': D_{m(n)} \to A$ such that $(h_n'')_{*0} = \beta_n$. Define $\Psi_n' = h_n'' \circ \bar{s}_n'$. For each projection $q \in B$, let $g \in \rho_A(K_0(A))$ such that $g = \hat{q}$ (here again we identify $\rho_A(K_0(A))$ with $\rho_B(K_0(B))$). One checks

$$t \circ \Psi_n'(q) \to g(t)$$

uniformly on $T(B)$ by (vii)–(ix). So $\{L_n\}$ and $\{\Psi_n'\}$ are tracially asymptotic morphisms from A to B and B to A, respectively. In the following computation we note that $L_n(c_i) \subset D_{m(n)}$:

$$\begin{aligned}[\Psi_n' \circ L_n](c_i) &= [\Psi_n' \circ h_n' \circ h_n \circ \kappa](c_i) \\ &= [\Psi_n'] \circ [h_n' \circ h_n](e_i) \\ &= \beta_n \circ [h_n' \circ h_n(e_i)] = g_i = [c_i].\end{aligned}$$

Since $\mathrm{TR}(A) = 0$, A has stable rank one. Thus there is a unitary $w_n \in A$ such that $\operatorname{ad} w_n \circ \Psi_n' \circ L_n(c_i) = c_i$, $i = 1, 2, \dots, l(n)$. Define $\Psi_n = \operatorname{ad} w_n \circ \Psi_n'$. Thus, for each projection $e \in F_n'$,

$$[\Psi_n \circ L_n](e) = [e].$$

Denote also by $j_n : F_n' \to A$ and let G_n be the group generated by $j_n(K_0(F_n'))$.

Finally, let $\varepsilon > 0$ be positive be a finite subset of $\rho_A(K_0(A))$ and let $\mathcal{P} = \{d_1, \dots, d_m\}$ be a finite subset of projections in A. Let $N(\varepsilon) > 0$ be a positive number such that

$$\tau(1 - E_n) < \varepsilon \quad \text{for all } \tau \in T(A) \text{ if } n \geqslant N(\varepsilon).$$

Since $\|p_n d_i - d_i p_n\| \to 0$ as $n \to \infty$ for each i and $E_n a = a E_n$ for all $a \in F_n$, we may also assume that

$$\|E_n d_i - d_i E_n\| < \varepsilon, \quad i = 1, 2, \dots, m.$$

We obtain projections $d_i' \in F_n'$ such that

$$|\tau(d_i) - \tau(d_i')| < \varepsilon, \quad i = 1, 2, \dots, m.$$

Let $\mathcal{P}_n$ be the finite subset $\{d_1', \dots, d_m'\}$. Note we have shown that

$$[\Psi_n \circ L_n](d_i') = [d_i'], \quad i = 1, 2, \dots, m.$$

It follows from Theorem 2.10 that $\{\Psi_n \circ L_n\}$ is tracially approximately inner.

Let $d_i'' = L_n(d_i')$. Since $\Psi_n \circ L_n(c_i) = c_i$, we have that $[L_n \circ \Psi_n(d_i'')] = [d_i'']$, $i = 1, 2, \dots, m$. Since $\rho_A(K_0(A)) = \rho_B(K_0(B))$, we see that (1) and (2) in Theorem 2.10 also hold for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$ of projections in B. It follows that $\{L_{k(n)} \circ \Psi_n\}$ is tracially approximately inner. Therefore, A and B are tracially equivalent.

For the converse, suppose that $\{\phi_n\}$ from A to B and $\{\psi_n\}$ from B to A are two tracially asymptotic morphisms such that $\{\psi_n \circ \phi_n\}$ and $\{\phi_n \circ \psi_n\}$ are tracially approximately inner. There are positive homomorphisms $\sigma : \operatorname{aff}(T(A)) \to \operatorname{aff}(T(B))$ and $\gamma : \operatorname{aff}(T(B)) \to \operatorname{aff}(T(A))$ given by

$$\sigma(\hat{a})(t) = \lim_{n\to\infty} t(\phi_n(a)) \quad \text{for } a \in A_+$$

and

$$\gamma(\hat{b})(\tau) = \lim_{n\to\infty} \tau(\psi_n(b)) \quad \text{for } b \in B_+.$$

It follows from Theorem 2.7 that $\sigma \circ \gamma = \mathrm{id}_{\mathrm{aff}(T(B))}$ and $\gamma \circ \sigma = \mathrm{id}_{\mathrm{aff}(T(A))}$. Therefore, $\mathrm{aff}(T(A))$ and $\mathrm{aff}(T(B))$ are order isomorphic. Since $\{\phi_n\}$ is a tracially asymptotic morphism, $\sigma(\hat{p}) \in \rho_B(K_0(B))$ for all projections $p \in A$. This implies that σ maps $\rho_A(K_0(A))$ into $\rho_B(K_0(B))$. Similarly, γ maps $\rho_B(K_0(B))$ into $\rho_A(K_0(A))$. Since

$$\gamma \circ \sigma|_{\rho_A(K_0(A))} = \mathrm{id}_{\rho_A(K_0(A))} \quad \text{and} \quad \sigma \circ \gamma|_{\rho_B(K_0(B))} = \mathrm{id}_{\rho_B(K_0(B))},$$

we conclude that $\sigma|_{\rho_A(K_0(A))}$ and $\gamma|_{\rho_B(K_0(B))}$ are order isomorphisms. □

Remark 3.5. It is worth pointing out that Theorem 3.9 holds without assuming that A or B satisfy the universal coefficient theorem (UCT). Perhaps more importantly, neither A nor B are assumed to be amenable (nuclear). We also point out that, according to a result of Dadarlat [**2**] (see also [**6**]), a unital separable simple C^*-algebra with tracial topological rank zero need not be amenable.

Corollary 3.6. *Let A be a unital separable simple C^*-algebra with $\mathrm{TR}(A) = 0$. Then A is tracially equivalent to a unital simple AF-algebra.*

Proof. It follows from Theorem 6.11 in [**7**] that $K_0(A)$ is a countable weakly unperforated simple (partial) ordered group with the Riesz interpolation property. Consequently, $\rho_A(K_0(A))$ is a countable unperforated simple ordered group with the Riesz interpolation property. It follows from [**3**] that there is a unital simple AF-algebra B such that $K_0(B) = \rho_A(K_0(A))$. Since $\rho_A(K_0(A))$ has no infinitesimal elements, $\rho_B(K_0(B)) = \rho_A(K_0(A))$. By Theorem 3.4, A is tracially equivalent to B. □

The proof of the following theorem is virtually the same as that of Theorem 3.4. However, for completeness and for revealing the difference between the tracial equivalence and weakly tracial equivalence, we present a proof below.

Theorem 3.7. *Let A and B be two unital separable amenable simple C^*-algebras with $\mathrm{TR}(A) = \mathrm{TR}(B) = 0$. Then A and B are weakly tracially equivalent if there is a dense subset $G \subset \rho_A(K_0(A))$ and an order isomorphism $\alpha : \mathrm{aff}(T(A)) \to \mathrm{aff}(T(B))$ such that $\alpha(G) \subset \rho_B(K_0(B))$.*

Proof. There exists a sequence of finite-dimensional C^*-subalgebras $F_n \subset A$ and projections $p_n = 1_{F_n}$ such that

(i) $\|p_n a - a p_n\| \to 0$ for all $a \in A$,

(ii) $\mathrm{dist}(p_n a p_n, F_n) \to 0$ for all $a \in A$, and

(iii) $\tau(1 - p_n) \to 0$ uniformly on $T(A)$.

There exists a sequence of finite-dimensional C^*-subalgebras $D_n \subset A$ and there exist

https://doi.org/10.1017/S0013091503000889 Published online by Cambridge University Press

projections $q_n = 1_{D_n}$ such that

(iv) $\|q_n b - bq_n\| \to 0$ for all $a \in B$,

(v) $\mathrm{dist}(q_n b q_n, D_n) \to 0$ for all $b \in B$, and

(vi) $t(1 - q_n) \to 0$ uniformly on $T(B)$.

Denote the embedding by $j_n : F_n \to A$. By applying the argument in the proof of Lemma 3.3, we may assume that $\rho_A \circ (j_n)_{*0}(K_0(F_n)) \subset G$. Since $\rho_A \circ (j_n)_{*0}(K_0(F_n))$ is a finitely generated free group, we obtain a positive homomorphism $\alpha_n : \rho_A \circ (j_n)_{*0}(K_0(F_n)) \to K_0(B)$ such that $\rho_B \circ \alpha_n = \alpha|_{\rho_A \circ (j_n)_{*0}(K_0(F_n))}$. There is a monomorphism $h_n : F_n \to B$ such that $(h_n)_{*0} = \alpha_n \circ \rho_A \circ (j_n)_{*0}$.

For each $\delta_n > 0$, there is an integer $m(n)$ such that

(vii) $\|q_{m(n)} b - bq_{m(n)}\| < \delta_n$ for all b in a finite subset $\mathcal{G}_n$ of B which contains a set of matrix units of $h_n(F_n)$,

(viii) $\mathrm{dist}(q_{m(n)} b q_{m(n)}, D_{m(n)}) < \delta_n$ for all $b \in \mathcal{G}'_n$,

(ix) $\tau(1 - q_{m(n)}) < \delta_n$ for all $\tau \in T(B)$.

By Lemma 3.3, we may assume that $1 - q_{m(n)} \neq 0$. Thus for any $\eta_n > 0$, with sufficiently small δ_n, there is a monomorphism $h'_n : h_n(F_n) \to D_{m(n)}$ such that

$$\|h'_n(b) - q_{m(n)} b q_{m(n)}\| < \eta_n \quad \text{for all } b \in \mathcal{G}_n \cap h_n(F_n).$$

Now there is a contractive completely positive linear map $s'_n : q_{m(n)} B q_{m(n)} \to D_{m(n)}$ such that $s'_n|_{D_{m(n)}} = \mathrm{id}_{D_{m(n)}}$ (see, for example, Theorem 2.3.5 in [**9**]). Denote by $\bar{s}'_n : B \to D_{m(n)}$ the map defined by $\bar{s}'_n(b) = s'_n(q_{m(n)} b q_{m(n)})$ for all $b \in B$ Denote the embedding by $\imath_n : D_{m(n)} \to B$.

Again, by applying the argument in the proof of Theorem 2.7, we may assume that $\rho_B((\imath_n)_{*0}(K_0(D_{m(n)}))) \subset \alpha(G)$.

There is a positive homomorphism $\beta_n : \rho_B(\imath_{*0}(K_0(D_{m(n)}))) \to K_0(A)$ such that $\rho_A \circ \beta_n = \alpha^{-1}|_{\rho_B \circ \imath(D_{m(n)})}$.

We write $F_n = M_{r(1)} \oplus \cdots \oplus M_{r(l(n))}$ and let e_i be a minimal projection in $M_{r(i)}$, $i = 1, 2, \ldots, l(n)$. Put $g_i = \beta_n \circ h'_n \circ h_n(e_i)$. Now we will apply the argument in the proof of Theorem 2.10 again as follows. Note that

$$\hat{e}_i > \tau(h'_n \circ h_n(e_i))$$

for all $\tau \in T(B)$. Thus $g_i \leqslant [e_i]$. Thus, we obtain a projection $c_i \leqslant e_i$ in A such that $[c_i] = g_i$ in $K_0(A)$, $i = 1, 2, \ldots, l(n)$. Let $E_n^{(i)}$ be a projection in $p_n A p_n$ with the form

$$E_n^{(i)} = \mathrm{diag}(c_1, c_1, \ldots, c_1),$$

where c_i repeats $r(i)$ many times. Let $E_n = E_n^{(1)} \oplus E_n^{(2)} \oplus \cdots \oplus E_n^{(l(n))}$. Note that

$$E_n a = a E_n \quad \text{for all } a \in F_n.$$

Define $F'_n = E_n F_n E_n$. Note that $F'_n \cong F_n$ and denote by κ an isomorphism. There is a contractive completely positive linear map $s_n : E_n A E_n \to F'_n$ such that $s_n|_{F'_n} = \mathrm{id}_{F'_n}$ (see, for example, Theorem 2.3.5 in [**9**]). Denote by $\bar{s}_n : A \to F'_n$ the map defined by $\bar{s}_n(a) = s_n(E_n a E_n)$. Note that $\bar{s}_n$ is a contractive completely positive linear map.

Define $L_n : A \to B$ by $L_n = h'_n \circ h_n \circ \kappa \circ \bar{s}_n$. Note that $\{\tau(L_n(a))\}$ converges uniformly on $T(B)$ for each $a \in A$. There is a homomorphism $h''_n : D_{m(n)} \to A$ such that $(h''_n)_{*0} = \beta_n$. Define $\Psi'_n = h''_n \circ \bar{s}'_n$. We also have the result that $\{t(\Psi'_n(b))\}$ converges uniformly on $T(A)$ for each $b \in B$.

Note also that $\{L_n\}$ and $\{\Psi'_n\}$ are asymptotically multiplicative contractive completely positive linear maps from A to B and B to A, respectively. In the following computation we note that $L_n(c_i) \subset D_{m(n)}$:

$$\begin{aligned}[\Psi'_n \circ L_n](c_i) &= [\Psi'_n \circ h'_n \circ h_n \circ \kappa](c_i) = [\Psi'_n] \circ [h'_n \circ h_n](e_i) \\ &= \beta_n \circ h'_n \circ h_n(e_i) = g_i = [c_i].\end{aligned}$$

Since $\mathrm{TR}(A) = 0$, there is a unitary $w_n \in A$ such that $\mathrm{ad}\, w_n \circ \Psi'_n \circ L_n(c_i) = c_i$, $i = 1, 2, \dots, l(n)$. Define $\Psi_n = \mathrm{ad}\, w_n \circ \Psi'_n$. Thus for each projection $e \in F'_n$,

$$[\Psi_n \circ L_n](e) = [e].$$

Denote also by $j_n : F'_n \to A$. Let G_n be the group generated by $j_n(K_0(F'_n))$.

Let $\varepsilon > 0$ be positive and let $\mathcal{P} = \{d_1, \dots, d_m\}$ be a finite subset of projections in A. Let $N(\varepsilon) > 0$ be a positive number such that

$$\tau(1 - E_n) < \varepsilon$$

for all $\tau \in T(A)$ if $n \geqslant N(\varepsilon)$. We may also assume that

$$\|E_n d_i - d_i E_n\| < \varepsilon, \quad i = 1, 2, \dots, m.$$

We obtain projections $d'_i \in F'_n$ such that

$$|\tau(d_i) - \tau(d'_i)| < \varepsilon, \quad i = 1, 2, \dots, m.$$

Let $\mathcal{P}_n$ be the finite subset $\{d'_1, \dots, d'_m\}$. Note we have shown that

$$[\Psi_n \circ L_n](d'_i) = [d'_i], \quad i = 1, 2, \dots, m.$$

It follows from Theorem 2.10 that $\{\Psi_n \circ L_n\}$ is tracially approximately inner.

Let $d''_i = L_n(d'_i)$. Since $\Psi_n \circ L_n(c_i) = c_i$, we have that $[L_n \circ \Psi_n(d''_i)] = [d''_i]$, $i = 1, 2, \dots, m$. Since $\alpha : \mathrm{aff}(T(A)) \to \mathrm{aff}(T(B))$ is an order isomorphism and $\alpha(G)$ is dense in $\rho_B(K_0(B))$, we see Theorem 2.10 applies here. It follows that $\{L_{k(n)} \circ \Psi_n\}$ is tracially approximately inner. Therefore, A and B are weakly tracially equivalent. $\square$

Proposition 3.8. *Let A and B be two unital separable simple C^*-algebras. Suppose that A and B are h-tracially equivalent. There are then positive homomorphisms $\alpha : K_0(A) \to K_0(B)$ and $\beta : K_0(B) \to K_0(A)$ with $\alpha([1_A]) = [1_B]$ and $\beta([1_B]) = [1_A]$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ induce identity maps on $\rho_B(K_0(B))$ and $\rho_A(K_0(A))$, respectively.*

Proof. Let $h_1 : A \to B$ and $h_2 : B \to A$ be two homomorphisms such that $h_1 \circ h_2$ and $h_2 \circ h_1$ are tracially approximately inner. We first note that $(h_1)_{*0} : K_0(A) \to K_0(B)$ and $(h_2)_{*0} : K_0(B) \to K_0(A)$ are positive homomorphisms. The fact that $h_1 \circ h_2$ is tracially approximately inner implies (by Corollary 2.8) that $(h_1)_{*0} \circ (h_2)_{*0}$ induces the identity map on $\rho_B(K_0(B))$. It also implies that $(h_2)_{*0} \circ (h_1)_{*0}$ induces the identity map on $\rho_A(K_0(A))$. To complete the proof we need to show that h_1 and h_2 are unital. Clearly, $h_1(1_A) \leqslant 1_B$. However, $\tau(1_B - h_1(1_A)) = 0$ for all $\tau \in T(B)$. Since B is simple, this implies that $h_1(1_A) = 1_B$. The same argument shows that h_2 is also unital. □

We will use $\mathcal{N}$ for the class of separable C^*-algebras which satisfy the UCT (see [**15**]).

Theorem 3.9. *Let A and B be two unital separable amenable simple C^*-algebras in $\mathcal{N}$ with $\mathrm{TR}(A) = \mathrm{TR}(B) = 0$. Then A and B are h-tracially equivalent if and only if there are positive homomorphisms $\alpha : K_0(A) \to K_0(B)$ and $\beta : K_0(B) \to K_0(A)$ with $\alpha([1_A]) = [1_B]$ and $\beta([1_B]) = [1_A]$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ induce identity maps on $\rho_B(K_0(B))$ and $\rho_A(K_0(A))$, respectively.*

Proof. From the previous proposition, we only need to show the 'if' part of the theorem. It follows from [**4**,**13**] (see also [**10**]) that we may assume that $A = \lim_{n\to\infty}(A_n, \phi_n)$ and $B = \lim_{n\to\infty}(B_n, \psi_n)$, where B_n and A_n have the form $P_n(C(X_n) \otimes M_{k(n)})P_n$, where X is a finite simplex with dimension no more than three (of course, each X_n and k depends on B_n and A_n) and $P_n \in C(X) \otimes M_k$ is a projection, By [**4**] (or Theorem 4.6 in [**8**]) there are monomorphisms $h_1 : A \to B$ and $h_2 : B \to A$ such that $(h_1)_{*0} = \alpha$ and $(h_2)_{*0} = \beta$. Since $h_2 \circ h_1$ and $h_2 \circ h_1$ induce identity maps on $\rho_A(K_0(A))$ and $\rho_B(K_0(B))$, respectively, it follows from Theorem 2.10 that $h_2 \circ h_1$ and $h_1 \circ h_2$ are tracially approximately inner. Therefore, A and B are h-tracially equivalent. □

Corollary 3.10. *Let A and B be two separable amenable simple C^*-algebras in $\mathcal{N}$ with $\mathrm{TR}(A) = \mathrm{TR}(B) = 0$. Suppose that the following two short exact sequences split*

$$0 \to \ker \rho_A \to K_0(A) \to \rho_A(K_0(A)) \to 0$$

and

$$0 \to \ker \rho_B \to K_0(B) \to \rho_B(K_0(B)) \to 0.$$

If there is an order isomorphism from $\rho_A(K_0(A))$ onto $\rho_B(K_0(B))$ which maps $\rho_A([1_A])$ to $\rho_B([1_B])$, then there is a projection $e \in M_2(B)$ with $\rho_B(e) = \rho_B(1_B)$ such that $eM_2(B)e$ is h-tracially approximate equivalent to A.

Proof. This immediately follows from Theorem 3.9. □

Example 3.11. (1) The condition of h-tracially equivalent is stronger than that of tracially approximately equivalent.

https://doi.org/10.1017/S0013091503000889 Published online by Cambridge University Press

Let A be a unital amenable simple C^*-algebra with $\mathrm{TR}(A) = 0$. Suppose that

$$0 \to \ker \rho_A \to K_0(A) \to \rho_A(K_0(A)) \to 0$$

is not splitting. Let B be a unital amenable simple C^*-algebra with $\mathrm{TR}(B) = 0$ and $(K_0(B), K_0(B)_+, [1_B]) = (\rho_A(K_0(A)), \rho_A(K_0(A))_{++}, [1_A])$. Then, by Theorem 2.10, A and B are tracially equivalent. However, there is no $\beta : \rho_A(K_0(A)) \to K_0(A)$ such that $\rho_A \circ \beta = \mathrm{id}_{\rho_A(K_0(A))}$, since the above mentioned short exact sequence is not splitting.

(2) The weak tracial equivalence is weaker than tracial equivalence.

Let A be a simple AF-algebra with

$$(K_0(A), K_0(A)_+, [1_A]) = (\mathbb{Q} + \mathbb{Q}(\sqrt{2}), (\mathbb{Q} + \mathbb{Q}(\sqrt{2}))_+, 1)$$

and let B be a simple AF-algebra with

$$(K_0(B), K_0(B)_+, [1_B]) = (\mathbb{Q} + \mathbb{Q}(\sqrt{3}), (\mathbb{Q} + \mathbb{Q}(\sqrt{3}))_+, 1).$$

Note that $\mathrm{aff}(T(A)) = \mathbb{R} = \mathrm{aff}(T(B))$ and $\mathbb{Q}$ is dense in $\mathbb{R}$. It follows from Theorem 3.7 that A and B are weakly tracial equivalent. However, by Theorem 3.9, A and B are not tracially equivalent.

4. Dynamical systems

Let X be a compact metric space and $\sigma : X \to X$ be a homeomorphism such that

$$\mathrm{orbit}_{\sigma_1}(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$$

is dense in X for each $x \in X$. The result dynamical system is called *minimal*.

Let X be the Cantor set and $\sigma : X \to X$ be a homeomorphism such that (X, σ) is a minimal dynamical system. Such dynamical systems will be called Cantor minimal systems.

Definition 4.1. Let (X_1, σ_1) and (X_2, σ_2) be two dynamical systems. They are said to be (topological) orbit equivalent if there exists a homeomorphism $F : X_1 \to X_2$ such that

$$F(\mathrm{orbit}_{\sigma_1}(x)) = \mathrm{orbit}_{\sigma_2}(F(x))$$

for each $x \in X_1$.

Let (X_1, σ_1) and (X_2, σ_2) be two Cantor minimal systems.

In [**5**], Giordano, Putnam and Skau showed that, for Cantor minimal systems, orbit equivalence can be characterized by the K-theory of the associated crossed product C^*-algebra $C(X) \times_\sigma \mathbb{Z}$. Here we will use the tracial equivalences of crossed products $C(X) \times_\sigma \mathbb{Z}$ to describe orbit equivalences of Cantor minimal systems.

Theorem 4.2. *Let (X_1, σ_1) and (X_2, σ_2) be two Cantor minimal systems. Let $A = C(X_1) \times_{\phi_1} \mathbb{Z}$ and $B = C(X_2) \times_{\phi_2} \mathbb{Z}$. Then the following are equivalent.*

(1) (X_1, σ_1) *and* (X_2, σ_2) *are (topological) orbit equivalent.*

(2) *There is an order isomorphism between* $\rho_A(K_0(A))$ *and* $\rho_B(K_0(B))$ *which maps* $\rho_A([1_A])$ *to* $\rho_B([1_B])$.

(3) A *and* B *are tracially equivalent.*

Proof. The equivalence of (1) and (2) is established in [**5**]. It is known that for Cantor minimal systems, $C(X) \times_\sigma \mathbb{Z}$ is an $A\mathbb{T}$-algebra with real rank zero and stable rank one. Furthermore, $\mathrm{TR}(C(X) \times_\sigma \mathbb{Z}) = 0$. Thus the equivalence of (2) and (3) follows from Theorem 2.10. □

As mentioned above, for Cantor minimal systems (X, α) the associated simple crossed products $C(X) \times_\alpha \mathbb{Z}$ have tracial topological rank zero. By Corollary 3.6, they are tracially equivalent to some unital simple AF-algebras.

Corollary 4.3. *Let* (X_1, σ_1) *and* (X_2, σ_2) *be two Cantor minimal systems. Let* $A = C(X_1) \times_{\phi_1} \mathbb{Z}$ *and* $B = C(X_2) \times_{\phi_2} \mathbb{Z}$. *Suppose that* $\rho_A(K_0(A))$ *and* $\rho_B(K_0(B))$ *are finitely generated. Then the following are equivalent.*

(1) (X_1, σ_1) *and* (X_2, σ_2) *are (topological) orbit equivalent.*

(2) *There is positive homomorphism* $\alpha : K_0(A) \to K_0(B)$ *which maps* $[1_A]$ *to* $[1_B]$ *such that* α *induces an order isomorphism between* $\rho_A(K_0(A))$ *and* $\rho_B(K_0(B))$.

(3) A *and* $eM_2(B)e$ *are h-tracially equivalent for some projection* $e \in M_2(B)$ *with* $\rho_B(e) = \rho_B(1_B)$.

Proof. This follows from Corollary 2.8 the same way as Theorem 4.2 follows from Theorem 2.10. □

Acknowledgements. The note is a result of conversations with N. Chris Phillips. The author also acknowledges the support from an NSF grant (DMS 0355273) and Zhi-Jiang Professorship of East China Normal University and Shanghai Priority Academic Disciplines during the summer of 2003.

References

1. B. Blackadar and D. Handelman, Dimension functions and traces on C^*-algebras, *J. Funct. Analysis* **45** (1982), 297–340.
2. M. Dadalart, Nonnuclear subalgebras of AF algebras, *Am. J. Math.* **122**(3) (2000), 581–597.
3. E. G. Effros, D. Handelman and C.-L. Shen, Dimension groups and their affine representations, *Am. J. Math.* **102** (1980), 385–407.
4. G. A. Elliott and G. Gong, On the classification of C^*-algebras of real rank zero, II, *Ann. Math.* **144** (1996), 497–610.
5. T. Giordano, I. Putnam and C. Skau, Topological orbit equivalence and C^*-crossed products, *J. Reine Angew. Math.* **469** (1995), 51–111.
6. H. Lin, Tracially AF C^*-algebras, *Trans. Am. Math. Soc.* **353** (2001), 693–722.

7. H. Lin, Tracial topological rank of C^*-algebras, *Proc. Lond. Math. Soc.* **83** (2001), 199–234.
8. H. Lin, Embedding an AH-algebra into a simple C^*-algebra with prescribed KK-data, *K-Theory* **24** (2001), 135–156.
9. H. Lin, *An introduction to the classification of amenable C^*-algebras* (World Scientific, Singapore, 2001).
10. H. Lin, Classification of simple C^*-algebras and higher dimensional noncommutative tori, *Ann. Math.* **157** (2003), 521–544.
11. H. Lin, Simple AH-algebras, *Proc. Am. Math. Soc.* **131** (2003), 3813–3819.
12. H. Lin, Traces and simple C^*-algebras of tracial topological rank zero, *J. Reine Angew. Math.* **568** (2004), 99–137.
13. H. Lin, Classification of simple C^*-algebras with tracial topological rank zero, *Duke Math. J.*, **125** (2004), 91–119.
14. T. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs, vol. 8. (American Mathematical Society, Providence, RI, 1997).
15. J. Rosenberg and C. Schochet, The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized functor, *Duke Math. J.* **55** (1987), 431–474.
16. N. C. Phillips, Crossed products by finite cyclic groups actions with the tracial Rokhlin property, preprint, arXiv.math.OA/0306410.