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Complemented Subspaces of Linear Bounded Operators

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Abstract. We study the complementation of the space $W(X, Y)$ of weakly compact operators, the space $K(X, Y)$ of compact operators, the space $U(X, Y)$ of unconditionally converging operators, and the space $CC(X, Y)$ of completely continuous operators in the space $L(X, Y)$ of bounded linear operators from X to Y. Feder proved that if X is infinite-dimensional and $c_0 \hookrightarrow Y$, then $K(X, Y)$ is uncomplemented in $L(X, Y)$. Emmanuele and John showed that if $c_0 \hookrightarrow K(X, Y)$, then $K(X, Y)$ is uncomplemented in $L(X, Y)$. Bator and Lewis showed that if X is not a Grothendieck space and $c_0 \hookrightarrow Y$, then $W(X, Y)$ is uncomplemented in $L(X, Y)$. In this paper, classical results of Kalton and separably determined operator ideals with property $(*)$ are used to obtain complementation results that yield these theorems as corollaries.

Introduction

Throughout this paper X and Y will denote real Banach spaces and X^* will denote the continuous linear dual of X. An operator $T\colon X \to Y$ will be a continuous and linear function. The set of all bounded linear operators from X to Y will be denoted by $L(X, Y)$, and the compact (resp., weakly compact, unconditionally converging, completely continuous) operators will be denoted by $K(X, Y)$ (resp., $W(X, Y)$, $U(X, Y)$, $CC(X, Y)$). An operator $T\colon X \to Y$ is unconditionally converging if T maps weakly unconditionally converging series into unconditionally converging series. An operator $T\colon X \to Y$ is called completely continuous (or Dunford–Pettis) if T maps weakly Cauchy sequences to norm convergent sequences.

If A is a subset of X, then $[A]$ denotes the closed linear span of A. Let (e_n) be the Schauder basis of c_0, (e_n^*) be the basis of ℓ_1, and (e_n^2) the basis of ℓ_2. The reader is referred to Diestel [7] or Dunford–Schwartz [11] for undefined notation and terminology.

For many years mathematicians have been interested in the problem of whether an operator ideal is complemented in the space $L(X, Y)$ of all bounded linear operators between X and Y; see Thorp [29], Arteburn and Whitley [2], Emmanuele [12, 13], John [23], Feder [16, 17], Emmanuele and John [15], and Kalton [24]. In this note we will present results related to the complementability of $W(X, Y)$, $U(X, Y)$, and $CC(X, Y)$ in $L(X, Y)$.

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1 The Uncomplemented Spaces $W(X, Y)$ and $U(X, Y)$

We begin this section with a characterization of spaces X so that $W(X, c_0)$ is complemented in $L(X, c_0)$. In the process, we extend the characterizations of *Grothendieck space* given in [9]. (A Banach space X is called a Grothendieck space if weak* and weak convergence of sequences in X^* coincide.) See [3, Theorem 4] for a related result.

Theorem 1.1 *Let X be a Banach space. The following are equivalent:*

(i) *X is a Grothendieck space;*
(ii) *$L(X, c_0) = W(X, c_0)$;*
(iii) *$W(X, c_0)$ is complemented in $L(X, c_0)$.*

Proof (i) $\Rightarrow$ (ii). Since X is a Grothendieck space, every bounded linear operator $T\colon X \to c_0$ is weakly compact (see [9, p. 179]).

(ii) $\Rightarrow$ (iii) is clear. Hence, it suffices to verify that (iii) $\Rightarrow$ (i).

Suppose that X is not a Grothendieck space. Choose a w^*-null sequence (x_n^*) in X^* with no weakly null subsequence. Let X_0 be a separable subspace of X such that the natural restriction map M from $[x_n^* : n \in \mathbb{N}]$ to X_0^* is an isometry. Thus $(M(x_n^*))$ is w^*-null, and no subsequence of $(M(x_n^*))$ converges weakly.

Define $T\colon \ell_\infty \to L(X, c_0)$ by

$$T(b)(x) = \sum b_n x_n^*(x) e_n, \quad b = (b_n) \in \ell_\infty, x \in X.$$

Let $S\colon c_0 \to \ell_\infty$ be the inclusion. Let $P\colon L(X, c_0) \to W(X, c_0)$ be a projection, and let $R\colon L(X, c_0) \to L(X_0, c_0)$ be the natural restriction map. Then $SRPT\colon \ell_\infty \to W(X_0, \ell_\infty)$ is an operator such that $SRPT(e_n) = T(e_n)|_{X_0}$ for each $n \in \mathbb{N}$. Proposition 5 of Kalton [24] produces an infinite subset M of $\mathbb{N}$ such that $SRPT(\chi_M) = T(\chi_M)|_{X_0}$. Hence $T(\chi_M)|_{X_0}$ is weakly compact. However, $(T(\chi_M)|_{X_0})^*(e_n^*) = M(x_n^*)$ for each $n \in M$, which is a contradiction. ■

Remark Emmanuele [13, Theorem 2] showed that if X has the Dunford–Pettis property, the Gelfand–Phillips property, and does not have the Schur property, then $W(X, Y)$ is not complemented in $L(X, Y)$, whenever $c_0 \hookrightarrow Y$. Bator and Lewis [3, Theorem 4] improved this result by only assuming that X is not a Grothendieck space and $c_0 \hookrightarrow Y$. If X is a *separable* Grothendieck space, it is readily seen that X is reflexive and $W(X, Y) = L(X, Y)$ for any Banach space Y. Thus, we have the following result.

Corollary 1.2 *Suppose X is a separable Banach space and $c_0 \hookrightarrow Y$. The following are equivalent:*

(i) *X is a Grothendieck space;*
(ii) *$L(X, Y) = W(X, Y)$;*
(iii) *$W(X, Y)$ is complemented in $L(X, Y)$.*

Theorem 1.3 *If X is non-reflexive, then $W(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$.*

Proof Let X_0 be a separable subspace of X that is not a Grothendieck space. Choose a w^*-null sequence (y_n^*) in X_0^* such that no subsequence of it converges weakly to a

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point in X_0^*. For each $n \in \mathbb{N}$, let $x_n^* \in X^*$ be a Hahn-Banach extension of y_n^*. Define $T\colon \ell_\infty \to L(X, \ell_\infty)$ by

$$T(b)(x) = (b_n x_n^*(x)), \quad b = (b_n) \in \ell_\infty,\ x \in X.$$

Note that the operator T is well defined and $T(e_n) = x_n^* \otimes e_n$ for each $n \in \mathbb{N}$.

Suppose that $W(X, \ell_\infty)$ is complemented in $L(X, \ell_\infty)$. Let $P\colon L(X, \ell_\infty) \to W(X, \ell_\infty)$ be a projection, and let $R\colon L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ be the natural restriction map. Define $\psi\colon \ell_\infty \to L(X_0, \ell_\infty)$ by $\psi(b) = RT(b)$. Then $RPT\colon \ell_\infty \to W(X_0, \ell_\infty)$ is an operator so that $RPT(e_n) = y_n^* \otimes e_n = \psi(e_n)$ for each $n \in \mathbb{N}$. Proposition 5 of Kalton [24] produces an infinite subset M of $\mathbb{N}$ such that $RPT(\chi_M) = \psi(\chi_M)$. Hence $\psi(\chi_M)$ is weakly compact. However, $(\psi(\chi_M))^*(e_n^*) = x_n^*|_{X_0} = y_n^*$, $n \in M$, which is a contradiction. ■

Corollary 1.4 *Suppose that X is a Banach space and $\ell_\infty \hookrightarrow Y$. The following are equivalent:*

(i) *X is reflexive;*
(ii) *$L(X, Y) = W(X, Y)$;*
(iii) *$W(X, Y)$ is complemented in $L(X, Y)$.*

Proof (iii) $\Rightarrow$ (i). Suppose that X is non-reflexive. It is known that if $\ell_\infty \hookrightarrow Y$, then $\ell_\infty \overset{c}{\hookrightarrow} Y$ (since ℓ_∞ is injective). Since $W(X, Y)$ is complemented in $L(X, Y)$, $W(X, \ell_\infty)$ is complemented in $W(X, Y)$, and thus in $L(X, Y)$. Now $W(X, \ell_\infty) \subseteq L(X, \ell_\infty) \subseteq L(X, Y)$, hence $W(X, \ell_\infty)$ is complemented in $L(X, \ell_\infty)$, which is a contradiction with Theorem 1.3. ■

Now we turn our attention to the complementation of the space of unconditionally converging operators $U(X, Y)$ in $L(X, Y)$. A key step in the proof of [24, Theorem 4], [12, Theorem 2], and [17, Theorem 1] involves the unconditional pointwise convergence of a series of compact operators to an operator which is not compact. Analogous hypotheses guarantee that $U(X, Y)$ is not complemented in $L(X, Y)$.

Theorem 1.5 *Suppose that E is an infinite-dimensional separable and complemented subspace of X so that $U(E, Y) = K(E, Y)$. If (T_i) is a sequence from $U(X, Y)$, $T \in L(X, Y)$, $\sum T_i(x)$ converges unconditionally to $T(x)$ for each $x \in X$, and $T|_E$ is not unconditionally converging, then $U(X, Y)$ is not complemented in $L(X, Y)$.*

Proof Let $J\colon E \to X$ be the natural inclusion, and note that $\sum T_i J(x)$ converges unconditionally to $TJ(x)$ for each $x \in E$. However, since TJ is not unconditionally converging, $\sum T_i J$ does not converge in the norm topology. Without loss of generality, suppose that $\inf_i \|T_i J\| > 0$.

Now suppose that $U(X, Y)$ is complemented in $L(X, Y)$ and let $\Gamma\colon L(X, Y) \to U(X, Y)$ be a projection. If $P\colon X \to E$ is a projection and $R\colon L(X, Y) \to L(E, Y)$ is the natural restriction map, then $Q\colon L(E, Y) \to U(E, Y)$, $Q(T) = R\Gamma(TP)$ defines a projection from $L(E, Y)$ onto $U(E, Y)$. Define $\phi\colon \ell_\infty \to L(E, Y)$ by

$$\phi(b) = \sum b_i T_i J \quad \text{(strong operator topology)}$$

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for $b = (b_i) \in \ell_\infty$. Then $\phi(e_n) = T_n J$, and thus $\phi(e_n) \in U(E, Y)$ for each $n \in \mathbb{N}$. Further, $\{\phi(b)(x) : b \in \ell_\infty, x \in E\}$ is separable. Apply [24, Lemma 2] to obtain an infinite subset M of $\mathbb{N}$ so that $\phi(b) \in U(E, Y)$ for each $b \in \ell_\infty(M)$.

Since $\inf_{i\in M} \|T_i J\| > 0$, $\sum_{i\in M} T_i J$ is not unconditionally convergent. However, since $\sum_{i\in M} T_i J(x)$ is unconditionally convergent for each $x \in E$, the Uniform Boundedness Principle shows that $\sum_{i\in M} T_i J$ is weakly unconditionally convergent. Thus $c_0 \hookrightarrow U(E, Y) = K(E, Y)$, and [12, Theorem 2] provides a contradiction which finishes the proof. ■

Corollary 1.6 *If $c_0 \overset{c}{\hookrightarrow} X$ and $c_0 \hookrightarrow Y$, then $U(X, Y)$ is not complemented in $L(X, Y)$.*

Proof Suppose that E is a complemented isomorphic copy of c_0 in X and (x_n) is a copy of (e_n) in E. Let $\widehat{P}\colon X \to E$ be a projection, $I\colon E \to c_0$ be an isomorphism with $I(x_n) = e_n$ for each $n \in \mathbb{N}$, and $A = I\widehat{P}$. For each $n \in \mathbb{N}$, let $x_n^* = e_n^* A$ and let (y_n) be a copy of (e_n) in Y. Define $T\colon X \to Y$ by $T(x) = \sum x_n^*(x)\, y_n$, $x \in X$. Let $T_n = x_n^* \otimes y_n$, $n \in \mathbb{N}$. Then each T_n is a rank one operator, $\sum T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$, and $T(x_n) = y_n$ for each $n \in \mathbb{N}$. Hence, $T|_E$ is not unconditionally converging. Apply Theorem 1.5. ■

Bator and Lewis [3, Theorem 2] showed that if X is separable and there is an operator $T\colon X \to Y$ which is not unconditionally converging, then $U(X, Y)$ is not complemented in $L(X, Y)$. Now we present a generalization of this theorem.

Corollary 1.7 *Suppose that B_{X^*} is w^*-sequentially compact and Y is a Banach space. The following are equivalent:*

(i) $c_0 \not\hookrightarrow X$ *or* $c_0 \not\hookrightarrow Y$;
(ii) $L(X, Y) = U(X, Y)$;
(iii) $U(X, Y)$ *is complemented in* $L(X, Y)$.

Proof (i) $\Rightarrow$ (ii). If $A\colon X \to Y$ is an operator which is not unconditionally converging, then there is a subspace H of X isomorphic to c_0 such that $A|_H$ is an isomorphism [7, p. 54]. Hence $c_0 \hookrightarrow X$ and $c_0 \hookrightarrow Y$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Suppose that $c_0 \hookrightarrow X$ and $c_0 \hookrightarrow Y$. Let (x_n) be a basic sequence in X such that $(x_n) \sim (e_n)$ and let (x_n^*) be a sequence of biorthogonal coefficients in X^*. Since B_{X^*} is w^*-sequentially compact, we may suppose without loss of generality that (x_n^*) is w^*-convergent. If $(x_n^*) \overset{w^*}{\longrightarrow} x^*$, then $(x_n^* - x^*)(x_n) = 1 - x^*(x_n) \to 1$, hence (x_n) is not limited. By a result of Schlumprecht [30, p. 36], (x_n) has a subsequence (c_n) such that $[c_n] \overset{c}{\hookrightarrow} X$. This provides a contradiction to Corollary 1.6. ■

Theorem 1.8 *If $c_0 \hookrightarrow X$, then $U(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$.*

Proof Let (x_n) be a copy of (e_n) in X, and let $X_0 = [x_n]$. Suppose that (x_n^*) is the associated sequence of coefficient functionals and (f_n^*) is a sequence of Hahn–Banach extensions in X^*. Define $T\colon \ell_\infty \to L(X, \ell_\infty)$ by

$$T(b)(x) = (b_n\, f_n^*(x)), \quad b = (b_n) \in \ell_\infty,\ x \in X.$$

Note that the operator T is well defined and $T(e_n) = f_n^* \otimes e_n$ for each $n \in \mathbb{N}$.

Suppose that $P\colon L(X, \ell_\infty) \to U(X, \ell_\infty)$ is a projection and $R\colon L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ is the natural restriction map. Then $RPT\colon \ell_\infty \to U(X_0, \ell_\infty)$ is an operator so that $RPT(e_n) = T(e_n)|_{X_0} = x_n^* \otimes e_n$ for each $n \in \mathbb{N}$. By [24, Proposition 5], there is an infinite subset M of $\mathbb{N}$ such that $RPT(\chi_M) = T(\chi_M)|_{X_0}$. Hence $T(\chi_M)|_{X_0}$ is unconditionally converging. However, $(T(\chi_M)|_{X_0})(x_n) = e_n$, $n \in M$, which is a contradiction. ■

Corollary 1.9 *Suppose that X is a Banach space and $\ell_\infty \hookrightarrow Y$. The following are equivalent:*

(i) $c_0 \not\hookrightarrow X$;

(ii) $L(X, Y) = U(X, Y)$;

(iii) *$U(X, Y)$ is complemented in $L(X, Y)$.*

Proof (i) $\Rightarrow$ (ii). If $L(X, Y) \neq U(X, Y)$, then $c_0 \hookrightarrow X$ (see [7, p. 54]).

(iii) $\Rightarrow$ (i). Suppose that $c_0 \hookrightarrow X$. Since $\ell_\infty \hookrightarrow Y$, $\ell_\infty \overset{c}{\hookrightarrow} Y$. If $U(X, Y) \overset{c}{\hookrightarrow} L(X, Y)$, then $U(X, \ell_\infty) \overset{c}{\hookrightarrow} L(X, \ell_\infty)$, which is a contradiction with the previous theorem. ■

2 Separably Determined Operator Ideals

In the previous section we made an investigation of the subspaces $W(X, Y)$ and $U(X, Y)$ of $L(X, Y)$. In this section we generalize some of these results to arbitrary operator ideals. We say that $\mathcal{O}$ is an operator ideal if for all Banach spaces X, Y, Z, and W, the following hold.

- $\mathcal{O}(X, Y)$ is a subspace of $L(X, Y)$.
- If $S \in L(Z, X)$, $T \in \mathcal{O}(X, Y)$, and $R \in L(Y, W)$, then $RTS \in \mathcal{O}(Z, W)$.

Lemma 2.1 *Let $\mathcal{O}$ be a non-trivial operator ideal and X, Y be Banach spaces. Then every finite rank operator from X to Y is in $\mathcal{O}(X, Y)$.*

Proof Let $x_0^* \in X^*$, $y_0 \in Y$, and let $\varphi \in \mathcal{O}(X, Y)$, $\varphi \neq 0$. Choose $x_0 \in X$ and $z^* \in B_{Y^*}$ such that $\varphi(x_0) = z \neq 0$ and $z^*(z) = 1$. Define $T\colon X \to X$ and $S\colon Y \to Y$ by $T(x) = x_0^*(x)\, x_0$, $x \in X$ and $S(y) = z^*(y)\, y_0$, $y \in Y$. Note that $S\varphi T = x_0^* \otimes y_0$, and thus $x_0^* \otimes y_0 \in \mathcal{O}(X, Y)$. ■

In the following results $\mathcal{O}$ is a closed operator ideal. We consider conditions that yield copies of c_0 and ℓ_∞ in $\mathcal{O}(X, Y)$. If X is infinite dimensional and $c_0 \hookrightarrow L(X, Y)$, then $\ell_\infty \hookrightarrow L(X, Y)$ (see [24, 25]). Part (iii) of the following corollary generalizes this result. It is known that $\ell_\infty \hookrightarrow L(\ell_2, \ell_2)$ (see [24]). Moreover, if X has an unconditional basis, then $\ell_\infty \hookrightarrow L(X, X)$ (see [25]). Part (iv) of the following corollary generalizes this result. The Banach space X has the Dunford–Pettis property (DPP) if every weakly compact operator $T\colon X \to Y$ is completely continuous.

Corollary 2.2 (i) *If $\ell_\infty \hookrightarrow X^*$ or $\ell_\infty \hookrightarrow Y$, then $\ell_\infty \hookrightarrow \mathcal{O}(X, Y)$.*

(ii) *Suppose that X and Y are infinite-dimensional Banach spaces and $\mathcal{O}(X, Y)$ is complemented in $L(X, Y)$. If $c_0 \hookrightarrow X^*$ or $c_0 \hookrightarrow Y$, then $\ell_\infty \hookrightarrow \mathcal{O}(X, Y)$.*

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(iii) *Suppose that X and Y are infinite-dimensional Banach spaces and $\mathcal{O}(X,Y)$ is complemented in $L(X,Y)$. Then $\ell_\infty \hookrightarrow \mathcal{O}(X,Y)$ if and only if $c_0 \hookrightarrow \mathcal{O}(X,Y)$.*

(iv) *Suppose X, Y are infinite-dimensional Banach spaces satisfying the following assumption: there exist a Banach space G with an unconditional basis (g_n) and biorthogonal coefficients (g_n^*), and two operators $R\colon G \to Y$ and $S\colon G^* \to X^*$ such that $(R(g_n))$ and $(S(g_n^*))$ are seminormalized and either $(R(g_n))$ or $(S(g_n^*))$ is a basic sequence. If $\mathcal{O}(X,Y)$ is complemented in $L(X,Y)$, then $\ell_\infty \hookrightarrow \mathcal{O}(X,Y)$.*

(v) *Assume that X has the DPP and X^* does not have the Schur property, and there is an operator $T\colon \ell_2 \to Y$ such that the sequence $(T(e_n^2))$ is seminormalized. If $\mathcal{O}(X,Y)$ is complemented in $L(X,Y)$, then $\ell_\infty \hookrightarrow \mathcal{O}(X,Y)$.*

Proof (i) Observe that X^* and Y embed in the finite rank operators from X to Y. Apply Lemma 2.1.

(ii) If $c_0 \hookrightarrow Y$ (or $c_0 \hookrightarrow X^*$), then $c_0 \hookrightarrow \mathcal{O}(X,Y)$. Apply [20, Theorem 1].

(iii) Apply [20, Theorem 1].

(iv) Lemma 3.2 in [18] shows that $(S(g_n^*)\otimes R(g_n)) \sim (e_n)$, and thus $c_0 \hookrightarrow \mathcal{O}(X,Y)$. Apply [20, Theorem 1].

(v) Since X has the *DPP* and X^* does not have the Schur property, $\ell_1 \hookrightarrow X$ (see [8,22]). Then $L^1 \hookrightarrow X^*$ (by a result in [26]), hence $\ell_2 \hookrightarrow X^*$ (see [7]). Apply (iv). ■

The following result concerns operators on abstract continuous function spaces. We refer the reader to [1, 5] for a complete discussion of this setting. We recall that if $T\colon C(K,X) \to Y$ is an operator with representing measure m and semivariation $\tilde{m}$, then T is called *strongly bounded* if $(\tilde{m}(A_n)) \to 0$, whenever (A_n) is a pairwise disjoint sequence of Borel subsets of K. The Banach space X has property (V) if every unconditionally converging operator on X is weakly compact [26].

Theorem 2.3 *Suppose there exists an operator ideal $\mathcal{O}(X,Y)$ so that $c_0 \not\hookrightarrow \mathcal{O}(X,Y)$. Then the following assertions hold.*

(i) *Every operator $T\colon C(K,X) \to Y$ is strongly bounded.*

(ii) *If X is reflexive, then every operator $T\colon C(K,X) \to Y$ is weakly compact.*

(iii) *If $\ell_1 \not\hookrightarrow X$ and X has property (u), then every operator $T\colon C(K,X) \to Y$ is weakly compact.*

(iv) *Every operator $T\colon C(K,X) \to Y$ has an unconditionally converging adjoint.*

Proof (i) Suppose that $T\colon C(K,X) \to Y$ is an operator which is not strongly bounded. Then T is not unconditionally converging by results in [5, 10]. It follows that T is an isomorphism on a copy of c_0 [7, p. 54], and $c_0 \hookrightarrow Y$. Therefore c_0 embeds in the rank one operators from X to Y, hence in $\mathcal{O}(X,Y)$.

(ii) Part (i) and [5, Theorem 4.1] show that every operator $T\colon C(K,X) \to Y$ is weakly compact.

(iii) By results in [6, 31], $C(K,X)$ has property (V). Hence, by part (i), every operator $T\colon C(K,X) \to Y$ is unconditionally converging, and thus weakly compact.

(iv) If $T\colon C(K,X) \to Y$ is an operator and $T^*\colon Y^* \to C(K,X)^*$ is not unconditionally converging, then T^* is an isomorphism on a copy of c_0 (see [7, p. 54]). Hence $c_0 \hookrightarrow C(K,X)^*$, and thus $\ell_1 \overset{c}{\hookrightarrow} C(K,X)$ (see [4]). Then $\ell_1 \overset{c}{\hookrightarrow} X$ (see [28]), and thus

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$c_0 \hookrightarrow X^*$. Therefore $c_0 \hookrightarrow \mathcal{O}(X, Y)$, which is a contradiction which concludes the proof. ■

We use the following notation. Let $A\colon X \to \ell_\infty$ be an operator and M be a nonempty subset of $\mathbb{N}$. We define $A_M\colon X \to \ell_\infty$ by $A_M(x) = (y_n)$, where $y_n = e_n^*(A(x))$, $n \in M$, and $y_n = 0$ otherwise.

Suppose that $\mathcal{O}$ is a closed operator ideal. We say that $\mathcal{O}$ *has property* $(*)$ if whenever X is a Banach space and $A \notin \mathcal{O}(X, \ell_\infty)$, there is an infinite subset M_0 of $\mathbb{N}$ such that $A_M \notin \mathcal{O}(X, \ell_\infty)$ for all infinite subsets M of M_0. If M is a subset of $\mathbb{N}$, then $\mathcal{P}_\infty(M)$ denotes the infinite subsets of M.

A closed operator ideal $\mathcal{O}$ is said to be separably determined provided that for each pair of Banach spaces X and Y, an operator $T\colon X \to Y$ belongs to $\mathcal{O}(X, Y)$ if and only if $T|_S \in \mathcal{O}(S, Y)$ for each separable subspace S of X.

Lemma 2.4 *If $T\colon F \to E^*$ is an operator and $T^*|_E$ is (weakly) compact, then T is (weakly) compact.*

Proof (*compact*) Let $S = T^*|_E$. Suppose $x^{**} \in B_{E^{**}}$ and choose a net (x_α) in B_E which is w^*-convergent to x^{**}. Then $(T^*(x_\alpha)) \xrightarrow{w^*} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_E)$, which is a relatively compact set. Then $(T^*(x_\alpha)) \to T^*(x^{**})$. Hence $T^*(B_{E^{**}}) \subseteq \overline{S(B_E)}$, which is relatively compact. Therefore $T^*(B_{E^{**}})$ is relatively compact, and thus T is compact. ■

Lemma 2.5 *The ideal of (weakly) compact operators has property* $(*)$.

Proof (*compact*) Suppose that $A\colon X \to \ell_\infty$ is not compact and let $x_n^* = A^*(e_n^*)$ for each $n \in \mathbb{N}$. Then $A^*\colon \ell_\infty^* \to X^*$ is not compact. It follows that $A^*|_{\ell_1}$ is not compact, and thus (x_n^*) is not relatively compact. Without loss of generality suppose that (x_n^*) has no convergent subsequence. Let $M_0 = \mathbb{N}$ and let M be an infinite subset of M_0. Note that $A_M^*(e_n^*) = x_n^*$, $n \in M$ and $A_M^*(e_n^*) = 0$, otherwise. Then $(A_M^*(e_n^*))$ is not relatively compact, and thus A_M is not compact. ■

Theorem 2.6 *Let $\mathcal{O}$ be a separably determined operator ideal with property $(*)$. Suppose that U has an unconditional and seminormalized basis (u_n) with biorthogonal coefficients (u_n^*). If $L(X, U) \neq \mathcal{O}(X, U)$ and $U \hookrightarrow Y$, then $\mathcal{O}(X, Y)$ is not complemented in $L(X, Y)$.*

Proof Let $A\colon X \to U$ such that $A \notin \mathcal{O}(X, U)$. Let $J\colon U \to \ell_\infty$ be an isometric embedding. Then $JA \notin \mathcal{O}(X, \ell_\infty)$. Let X_0 be a separable subspace of X such that $B = JA|_{X_0} \notin \mathcal{O}(X_0, \ell_\infty)$. Choose $M_0 \in \mathcal{P}_\infty(\mathbb{N})$ such that $B_M \notin \mathcal{O}(X_0, \ell_\infty)$ for each $M \in \mathcal{P}_\infty(M_0)$. Note that $\sum b_n u_n^*(A(x)) u_n$ converges unconditionally in U for each $x \in X$ and $b = (b_n) \in \ell_\infty$. Define $T\colon \ell_\infty \to L(X, U)$ by

$$T(b)(x) = \sum b_n u_n^*(A(x)) u_n,\ x \in X.$$

Let $i\colon U \to Y$ be a linear embedding and let $y_n = i(u_n)$ for each $n \in \mathbb{N}$. Use the injectivity of ℓ_∞ to select an operator $S\colon Y \to \ell_\infty$ so that $S(y_n) = e_n$ for each $n \in \mathbb{N}$. Suppose that $\mathcal{O}(X, Y) \overset{c}{\hookrightarrow} L(X, Y)$. Let $P\colon L(X, Y) \to \mathcal{O}(X, Y)$ be a projection, and

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let $R\colon L(X,\ell_\infty) \to L(X_0,\ell_\infty)$ be the natural restriction map. Consider the operators $\varphi\colon \ell_\infty \to L(X_0,\ell_\infty)$ and $\Gamma\colon \ell_\infty \to \mathcal{O}(X_0,\ell_\infty)$ defined by $\varphi(b) = RSiT(b)$ and $\Gamma(b) = RSPiT(b)$. Since $T(e_n)$ is a rank one operator, $\Gamma(e_n) = \varphi(e_n)$ for each $n \in \mathbb{N}$. By [24, Proposition 5], there is an infinite subset M of M_0 such that $\Gamma(b) = \varphi(b)$, $b \in \ell_\infty(M)$. Hence, $\varphi(\chi_M) \in \mathcal{O}(X_0,\ell_\infty)$. However, $\varphi(\chi_M) = B_M \notin \mathcal{O}(X_0,\ell_\infty)$, which is a contradiction. ■

The following result generalizes Theorem 1.1.

Corollary 2.7 *Let $\mathcal{O}$ be a separably determined operator ideal with property $(*)$. The following are equivalent:*

(i) $L(X,c_0) = \mathcal{O}(X,c_0)$;

(ii) $\mathcal{O}(X,c_0)$ *is complemented in* $L(X,c_0)$.

Theorem 2.6 generalizes [17, Corollary 4]. We remark that if X is an infinite-dimensional Banach space and $c_0 \hookrightarrow Y$, then the Josefson–Nissenzweig theorem guarantees that $L(X,Y) \neq K(X,Y)$.

Corollary 2.8 ([17, Corollary 4]) *If X is an infinite dimensional Banach space and $c_0 \hookrightarrow Y$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof Use the Josefson–Nissenzweig theorem to choose a w^*-null and normalized sequence (x_n^*) in X^*. Define $S\colon X \to c_0$ by $S(x) = (x_n^*(x))$. Clearly $(S^*(e_n^*)) = (x_n^*)$ is not relatively compact, hence S is not compact. Apply Theorem 2.6 and Lemma 2.5. ■

Corollary 2.9 ([13, Theorems 2 and 3]; [3, Theorem 4]) *If X is not a Grothendieck space and $c_0 \hookrightarrow Y$, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof Let (x_n^*) be a w^*-null sequence in X^* with no weakly convergent subsequence. Then the operator $T\colon X \to c_0$ defined by $T(x) = (x_n^*(x))$ is not weakly compact. Apply Theorem 2.6 and Lemma 2.5. ■

Corollary 2.10 ([13, Corollary 4]) *Assume that X contains a complemented copy of c_0 and $c_0 \hookrightarrow Y$. Then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof If $c_0 \overset{c}{\hookrightarrow} X$ and $W(X,Y) \overset{c}{\hookrightarrow} L(X,Y)$, then $W(c_0,Y) \overset{c}{\hookrightarrow} L(c_0,Y)$. Apply Corollary 2.9. ■

The next theorem is motivated by results in [3, 13, 14, 24].

Theorem 2.11 (i) *Suppose that U has an unconditional and seminormalized basis (u_i) with biorthogonal coefficients (u_i^*), $U \overset{c}{\hookrightarrow} X$, and $T\colon U \to Y$ is an operator such that $(T(u_i))$ is not relatively compact in Y. Let $S(X,Y)$ be a closed linear subspace of $L(X,Y)$ which properly contains $K(X,Y)$ such that $\phi(b) \in S(U,Y)$ for all $b \in \ell_\infty$, where $\phi(b)(u) = \sum b_i u_i^*(u) T(u_i)$, $u \in U$. Then $K(X,Y)$ is not complemented in $S(X,Y)$.*

(ii) *Suppose that U has an unconditional and seminormalized basis (u_i) with biorthogonal coefficients (u_i^*), $U \overset{c}{\hookrightarrow} X$, and $T\colon U \to Y$ is an operator such that $(T(u_i))$ is* *not relatively weakly compact in Y. Let $S(X,Y)$ be a closed linear subspace of $L(X,Y)$ which properly contains $W(X,Y)$ such that $\phi(b) \in S(U,Y)$ for all $b \in \ell_\infty$, where $\phi(b)(u) = \sum b_i u_i^*(u)T(u_i)$, $u \in U$. Then $W(X,Y)$ is not complemented in $S(X,Y)$.*

Proof (i) Note that $\sum b_j u_j^*(u)T(u_j)$ converges unconditionally in Y for each $u \in U$ and $b = (b_i) \in \ell_\infty$, by the unconditionality of the basis (u_i). Let $J\colon [(T(u_i)] \to \ell_\infty$ be a linear isometry, and let $A\colon Y \to \ell_\infty$ be a continuous linear extension of J. Now suppose that $K(X,Y)$ is complemented in $S(X,Y)$. Then $K(U,Y)$ is complemented in $S(U,Y)$. Let $P\colon S(U,Y) \to K(U,Y)$ be a projection. Consider the operators $AP\phi\colon \ell_\infty \to K(U,\ell_\infty)$ and $A\phi\colon \ell_\infty \to S(U,\ell_\infty)$. Since $\phi(e_j) = u_j^* \otimes T(u_j)$, $\phi(e_j)$ is a rank one operator, thus compact. Then $AP\phi(e_j) = A\phi(e_j)$ for each $j \in \mathbb{N}$. Proposition 5 of Kalton [24] produces an infinite subset M of $\mathbb{N}$ such that

$$AP\phi(b) = A\phi(b),\ b \in l_\infty(M).$$

Therefore $A\phi(\chi_M)$ is compact. But $\phi(\chi_M)(u_j) = T(u_j)$, $j \in M$, and

$$\{T(u_j) : j \in M\}$$

is not relatively compact. Therefore $\phi(\chi_M)$ is not compact. However, this is a contradiction, since $A|_{[T(u_i)]}$ is an isometry.

(ii) The proof is essentially the same as the proof of (i), replacing "relatively weakly compact" with "relatively compact". ■

Corollary 2.12 ([14, Lemma 3]) *Let Y be a Banach space without the Schur property. If $\ell_1 \overset{c}{\hookrightarrow} X$, then $K(X,Y)$ is not complemented in $W(X,Y)$.*

Proof Let (y_n) be a weakly null normalized basic sequence in Y and $S(X,Y) = W(X,Y)$. Define $T\colon \ell_1 \to Y$ by $T(x) = \sum x_n y_n$, $x = (x_n) \in \ell_1$. If $\phi\colon \ell_\infty \to L(\ell_1,Y)$ is defined as in the previous theorem, then $\phi(b)(x) = \sum b_n x_n y_n$, $x = (x_n) \in \ell_1$. Note that ϕ is well defined and since $\phi(b)(e_n^*) = (b_n y_n)$ is weakly null, $\phi(b)$ is weakly compact for each $b \in \ell_\infty$. Apply Theorem 2.11. ■

The next result contains [24, Lemma 3].

Corollary 2.13 *Suppose that $\ell_1 \overset{c}{\hookrightarrow} X$ and Y is infinite-dimensional. Then $K(X,Y)$ is not complemented in $CC(X,Y)$ and $K(X,Y)$ is not complemented in $U(X,Y)$. Consequently, $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof Let (y_n) be a normalized basic sequence in Y. Define $T\colon \ell_1 \to Y$ by $T(x) = \sum x_n y_n$, $x = (x_n) \in \ell_1$. Let $\phi\colon \ell_\infty \to L(\ell_1,Y)$, $\phi(b)(x) = \sum b_n x_n y_n$, $x = (x_n) \in \ell_1$. Note that $\phi(b)$ is completely continuous for each $b \in \ell_\infty$, since ℓ_1 has the Schur property. Let $S(X,Y) = CC(X,Y)$ and apply Theorem 2.11. Furthermore, $\phi(b)$ is unconditionally converging for each $b \in \ell_\infty$. By Theorem 2.11, $K(X,Y)$ is not complemented in $U(X,Y)$. ■

Corollary 2.14 ([3, Theorem 3]) *If $\ell_1 \overset{c}{\hookrightarrow} X$ and Y is non-reflexive, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

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Proof Let (y_n) be a sequence in B_Y with no weakly convergent subsequence and $S(X,Y) = L(X,Y)$. Define $T\colon \ell_1 \to Y$ by $T(x) = \sum x_n y_n$, $x = (x_n) \in \ell_1$. Let $\phi\colon \ell_\infty \to L(\ell_1, Y)$, $\phi(b)(x) = \sum b_n x_n y_n$, $x = (x_n) \in \ell_1$. Apply Theorem 2.11. ∎

We recall that if $\ell_\infty \not\hookrightarrow X$, then every operator $T\colon \ell_\infty \to X$ is weakly compact (see [27]). Furthermore, if $c_0 \not\hookrightarrow X$, then every operator $T\colon C(K) \to X$ is weakly compact (see [19, 21]). The next result contains [13, Theorem 5].

Corollary 2.15 *If $L(X,\ell_1) \neq K(X,\ell_1)$ and Y is non-reflexive, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof Let $T\colon X \to \ell_1$ be a noncompact operator. Then $T^*\colon \ell_\infty \to X^*$ is not weakly compact. By results of [21, 27], $c_0 \hookrightarrow X^*$, and thus $\ell_1 \overset{c}{\hookrightarrow} X$ (see [4]). Apply the previous corollary. ∎

Corollary 2.16 ([12, Theorem 2]) *If $c_0 \hookrightarrow K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof By Corollaries 2.8 and 2.13 we may assume that $c_0 \not\hookrightarrow X^*$ and $c_0 \not\hookrightarrow Y$. Thus, by [24, Theorem 4], $\ell_\infty \not\hookrightarrow K(X,Y)$. If $K(X,Y)$ is complemented in $L(X,Y)$, then Corollary 2.2 provides a contradiction which concludes the proof. ∎

In the next results we investigate the complementation of the space of completely continuous operators $CC(X,Y)$ in $L(X,Y)$. A Banach space X has the Schur property if every weakly null sequence in X is norm null. Corollary 3.10 of [22] shows that if X is a separable Banach space, then X has the Schur property if and only if $L(X,c_0) = CC(X,c_0)$.

Theorem 2.17 *The ideal of completely continuous operators has property* $(*)$.

Proof Let $A\colon X \to \ell_\infty$ be an operator which is not completely continuous. Let (x_n) be a weakly null sequence in X and $\delta > 0$ such that $\|A(x_n)\| > \delta$ for each $n \in \mathbb{N}$. Let $n_1 = 1$ and choose $N_1 \in \mathbb{N}$ such that $|e^*_{N_1}(A(x_{n_1}))| > \delta$. Now $(A(x_n))$ is weakly null. Choose $n_2 > n_1$ so that $|e^*_k(A(x_n))| < \delta$ for $n \geq n_2$ and $1 \leq k \leq N_1$. Choose $N_2 > N_1$ such that $|e^*_{N_2}(A(x_{n_2}))| > \delta$. Continuing this process we obtain a subsequence (x_{n_i}) of (x_n) and an increasing sequence (N_i) of natural numbers such that $|e^*_{N_i}(A(x_{n_i}))| > \delta$ for each $i \in \mathbb{N}$. Let $M_0 = \{N_i : i = 1, 2, \dots\}$. Note that M_0 is an infinite subset of $\mathbb{N}$ and $\|A_{M_0}(x_{n_i})\| \geq \delta$ for each $i \in \mathbb{N}$. If M is an infinite subset of M_0, then A_M is not completely continuous. Therefore the ideal of completely continuous operators has property $(*)$. ∎

Corollary 2.18 *Let X be a Banach space. The following are equivalent:*

(i) $L(X,c_0) = CC(X,c_0)$*;*

(ii) $CC(X,c_0)$ *is complemented in* $L(X,c_0)$*.*

If X is separable, then (i) *and* (ii) *are equivalent to*

(iii) *X has the Schur property.*

Proof Apply Corollary 2.7, Theorem 2.17, and [22, Corollary 3.10]. ∎

Corollary 2.19 *If $L(X, c_0) \neq CC(X, c_0)$ and $c_0 \hookrightarrow Y$, then $CC(X, Y)$ is not complemented in $L(X, Y)$.*

Proof Apply Theorem 2.6 and Theorem 2.17. ■

A bounded subset S of X is called a limited subset of X if each w^*-null sequence (x_n^*) in X^* tends to 0 uniformly on S, and X is said to have the Gelfand–Phillips property if every limited subset of X is relatively compact.

Corollary 2.20 *Suppose that X has the Gelfand–Phillips property and does not have the Schur property. If $c_0 \hookrightarrow Y$, then $CC(X, Y)$ is not complemented in $L(X, Y)$.*

Proof Let (x_n) be a normalized weakly null sequence in X. Then (x_n) is not limited. Choose a w^*-null sequence (x_n^*) in X^* such that $x_n^*(x_m) = \delta_{nm}$ (see [30, Theorem 1.3.1]). Define $T\colon X \to c_0$ by $T(x) = (x_n^*(x))$. Then T is not completely continuous since $\|T(x_n)\| \geq |x_n^*(x_n)| = 1$ for each $n \in \mathbb{N}$. Apply Corollary 2.19. ■

Theorem 2.21 *If X does not have the Schur property, then $CC(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$.*

Proof Suppose that (x_n) is a normalized weakly null basic sequence in X. Let $X_0 = [x_n]$ and let (x_n^*) be the associated sequence of coefficient functionals. For each $n \in \mathbb{N}$, let $f_n^* \in X^*$ be a Hahn–Banach extension of x_n^*. Define $T\colon \ell_\infty \to L(X, \ell_\infty)$ by $T(b)(x) = (b_n f_n^*(x))$, $b = (b_n) \in \ell_\infty$, $x \in X$. Note that the operator T is well defined and $T(e_n) = f_n^* \otimes e_n$ for each $n \in \mathbb{N}$.

Suppose that $CC(X, \ell_\infty)$ is complemented in $L(X, \ell_\infty)$. Let $P\colon L(X, \ell_\infty) \to CC(X, \ell_\infty)$ be a projection, and let $R\colon L(X, \ell_\infty) \to L(X_0, \ell_\infty)$ be the natural restriction map. Define $\psi\colon \ell_\infty \to L(X_0, \ell_\infty)$ by $\psi(b) = RT(b)$. Then $RPT\colon \ell_\infty \to CC(X_0, \ell_\infty)$ is an operator such that $RPT(e_n) = x_n^* \otimes e_n = \psi(e_n)$ for each $n \in \mathbb{N}$. Proposition 5 of Kalton [24] produces an infinite subset M of $\mathbb{N}$ such that $RPT(\chi_M) = \psi(\chi_M)$. Hence $\psi(\chi_M)$ is completely continuous. However, $\psi(\chi_M)(x_n) = e_n$, $n \in M$, which is a contradiction. ■

Corollary 2.22 *Suppose that X is a Banach space and $\ell_\infty \hookrightarrow Y$. The following are equivalent:*

(i) *X has the Schur property;*
(ii) *$L(X, Y) = CC(X, Y)$;*
(iii) *$CC(X, Y)$ is complemented in $L(X, Y)$.*

Proof (iii) $\Rightarrow$ (i). Suppose that X does not have the Schur property. If $\ell_\infty \hookrightarrow Y$, then $\ell_\infty \overset{c}{\hookrightarrow} Y$. Since $CC(X, Y) \overset{c}{\hookrightarrow} L(X, Y)$, $CC(X, \ell_\infty) \overset{c}{\hookrightarrow} L(X, \ell_\infty)$, which is a contradiction with the previous theorem. ■

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