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DERIVATIONS FROM TOTALLY ORDERED SEMIGROUP ALGEBRAS INTO THEIR DUALS

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ABSTRACT. For a well-behaved measure μ, on a locally compact totally ordered set X, with continuous part μ_c, we make $L^p(X,\mu_c)$ into a commutative Banach bimodule over the totally ordered semigroup algebra $L^p(X,\mu)$, in such a way that the natural surjection from the algebra to the module is a bounded derivation. This gives rise to bounded derivations from $L^p(X,\mu)$ into its dual module and in particular shows that if μ_c is not identically zero then $L^p(X,\mu)$ is not weakly amenable. We show that all bounded derivations from $L^1(X,\mu)$ into its dual module arise in this way and also describe all bounded derivations from $L^p(X,\mu)$ into its dual for $1 < p < \infty$ in the case that X is compact and μ continuous.

1. **Introduction.** A Banach algebra is said to be weakly amenable (WA) if all bounded derivations from it into its dual module are inner. Weak amenability of L^1-algebras has been considered by several authors recently. In [8], B. E. Johnson completed some previous partial results of his by showing that $L^1(G)$ is WA for all locally compact groups. In [5], N. Groenbæk described all bounded derivations from $L^1(\mathbb{R}_+, w)$ into its dual, showing that it is not WA for any weight w. In [3], the author considered the weak amenability of $\ell^1(S)$ for several types of discrete semigroup S. In particular he showed that for commutative S there is, at least often, a coincidence in S being regular or not and $\ell^1(S)$ being WA or not. These results perhaps suggest that whether an L^1-algebra is WA or not is dependent on the algebraic properties of the underlying topological semigroup, such as whether its elements are invertible or not, rather than any topological properties of the semigroup.

Here we consider the weak amenability of L^p-algebras whose underlying semigroups are totally ordered semigroups which are locally compact in their order topology. For such a semigroup, X, we write $M_+(X)$ for the set of σ-finite, regular, positive Borel measures on X and $\bar{M}(X)$ for the set of measures in $M_+(X)$ which are supported on the whole of X. For $\mu \in M_+(X)$, with the order convolution defined below, $L^1(X,\mu)$ is a Banach algebra, and if μ is finite, $L^p(X,\mu)$ is a Banach algebra for $1 < p < \infty$ also (see [2]). These algebras have been considered in [7], [11] and [2]. Now $\mu \in M_+(X)$ has a unique decomposition, $\mu = \mu_c + \mu_d$, where μ_c is continuous, μ_d is discrete and both are in $M_+(X)$. In Section 3 we show that for each $1 \leq p < \infty$ and $\mu \in \bar{M}(X)$, $L^p(X,\mu)$ is WA if and only if $\mu_c \equiv 0$ (where of course we assume μ finite for $1 < p < \infty$). Here then

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the weak amenability of $L^1(X,\mu)$ reflects the topological, not the algebraic, structure of X. In his article in [10], Lau expressed an interest in the weak amenability of $M(S)$ for locally compact, semitopological semigroups S (Problem 24). Our results imply that if X is discrete then $M(X)$ is WA, and if there is a non-discrete measure μ in $\mathcal{M}(X)$ then $M(X)$ contains a subalgebra which is not WA $\bigl(L^1(X,\mu)\bigr)$.

In Section 4 we go on to describe all bounded derivations from $L^1(X,\mu)$ into its dual and in Section 5 by a (necessarily in the cases $1<p<\infty$) different method we describe all the bounded derivations from $L^p(X,\mu)$ into its dual when X is compact and μ continuous. Thus these sections describe fully the 1-dimensional cohomology group of $L^p(X,\mu)$ with coefficients in $L^p(X,\mu)^*$ for their respective cases.

2. **Preliminaries.** Our basic objects are totally ordered sets with their order topology (as described in [9]), which is Hausdorff. For such sets we use the usual interval notation. We shall require these sets to be order complete so that the compact subsets are the closed and bounded ones and we have local compactness. With maximum multiplication these spaces become topological semigroups. We refer to such semigroups as totally ordered semigroups. If X is a totally ordered semigroup and $\mu\in\mathcal{M}(X)$ (defined in the introduction) then, as in [2], the order convolution of two μ-integrable functions, f and g, is given by

$$f*g(x)=f(x)\int_{(-\infty,x]}g(u)\,d\mu(u)+g(x)\int_{(-\infty,x)}f(u)\,d\mu(u).$$

As noted in the introduction it was shown in [2], that taking multiplication to be convolution, $L^1(X,\mu)$ is a commutative Banach algebra and $L^p(X,\mu)$ is a Banach algebra for $1<p<\infty$ if (and only if) μ is finite. Thus when considering $L^p(X,\mu)$ for $1<p<\infty$ we will assume that μ is finite.

For definitions relating to Banach algebras not given here we refer the reader to [6]. The notion of weak amenability of a commutative Banach algebra was introduced in [1]. Such an algebra, A, is said to be weakly amenable (WA), if each bounded derivation from A into any commutative Banach bimodule is 0. It was shown in the same paper that a commutative Banach algebra, A, is WA if (and only if) all bounded derivations from A into its dual module are zero. That is if each bounded linear map, D, from A to A^*, which satisfies the derivation equality,

$$D(ab)(c)=D(a)(bc)+D(b)(ca),$$

is zero. Since this later characterisation has a natural extension to non-commutative Banach algebras it is the way weak amenability is normally studied.

We note that the dual of $L^p(X,\mu)$ is $L^q(X,\mu)$, where $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, and since we are only considering σ-finite measures the dual of $L^1(X,\mu)$ is $L^\infty(X,\mu)$. The σ-finiteness also means that we are free to use Fubini's theorem.

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3. **Weak amenability of totally ordered semigroup algebras.** We reiterate that unless otherwise stated X is a totally ordered semigroup and μ a σ-finite (finite when we are considering $L^p(X,\mu)$ for $1 < p < \infty$), regular, Borel measure supported on the whole of X. For such a measure, μ, the measure of any μ-measurable set can be approximated from below by the measures of compact subsets. That is to say μ is compact regular. We begin this section by stating two results which are special cases of remarks given in [2]. For a fixed measure μ we use $\|\cdot\|_p$ to denote the norm in $L^p(X,\mu)$ and for a fixed p we use $\|\cdot\|_\mu$ to denote the norm in $L^p(X,\mu)$.

PROPOSITION 3.1. *If $f \in L^p(X,\mu)$ for any $1 < p < \infty$ then $f \in L^1(X,\mu)$ and $\|f\|_1 \le \mu(X)^{\frac{1}{q}}\|f\|_p$, where $\frac{1}{p}+\frac{1}{q} = 1$.*

PROPOSITION 3.2. *For $1 \le p < \infty$ and $f, g \in L^p(X,\mu)$,*

$$\int_{(-\infty,x)} f * g(u)\,d\mu(u) = \int_{(-\infty,x)} f(u)\,d\mu(u) \int_{(-\infty,x)} g(u)\,d\mu(u).$$

As noted in the introduction a measure $\mu \in M_+(X)$ has a decomposition, $\mu = \mu_c + \mu_d$, where μ_c is continuous, μ_d discrete and both are in $M_+(X)$. We make $L^p(X,\mu_c)$ into a commutative Banach bimodule over $L^p(X,\mu)$. So for $f \in L^p(X,\mu)$ and $g \in L^p(X,\mu_c)$ we define right and left actions,

$$f \cdot g(x) = g \circ f(x) = g(x) \int_{(-\infty,x)} f(u)\,d\mu(u).$$

Now

$$\begin{aligned}\int_X \left| g(x) \int_{(-\infty,x)} f(u)\,d\mu(u)\right|^p d\mu_c(x) &\le \int_X |g(x)|^p \left(\int_X |f(u)|\,d\mu(u)\right)^p d\mu_c(x)\\ &\le C\|f\|_\mu^p\,\|g\|_{\mu_c}^p\end{aligned}$$

where $C = 1$ if $p = 1$ and $C = \mu(X)^{\frac{p}{q}}$ if $p > 1$, by Proposition 3.1. Thus $\cdot$ and $\circ$ are bounded bilinear maps from respectively $L^p(X,\mu) \times L^p(X,\mu_c)$ and $L^p(X,\mu_c) \times L^p(X,\mu)$ into $L^p(X,\mu_c)$. Also for $f, g \in L^p(X,\mu)$ and $h \in L^p(X,\mu_c)$ we have by Proposition 3.2,

$$\begin{aligned}(f * g)\cdot h(x) &= \int_{(-\infty,x)} f(u)\,d\mu(u) \int_{(-\infty,x)} g(u)\,d\mu(u) h(x)\\ &= \int_{(-\infty,x)} f(u)\,d\mu(u)(g\cdot h)(x) = f\cdot(g\cdot h)(x).\end{aligned}$$

Similarly $h \circ (f * g) = (h \circ f) \circ g$. Finally for $f, h \in L^p(X,\mu)$ and $g \in L^p(X,\mu_c)$,

$$(f \cdot g) \circ h(x) = \int_{(-\infty,x)} f(u)\,d\mu(u) g(x) \int_{(-\infty,x)} h(u)\,d\mu(u) = f \cdot (g \circ h)(x).$$

Hence $L^p(X,\mu_c)$ is in this way a commutative Banach bimodule over $L^p(X,\mu)$.

Now let $i\colon L^p(X,\mu) \longrightarrow L^p(X,\mu_c)$ take the equivalence class containing f to the equivalence class containing f. Then $i(f*g)(x) - \big(i(f)\circ g + f\cdot i(g)\big)(x) = f(x)g(x)\mu(\{x\})$. Since $\mu(\{x\}) > 0$ for at most countably many $x \in X$, we have that $i(f*g) = i(f)\circ g + f\cdot i(g)$ and i is a bounded derivation. If μ_c is non-zero it will be non-zero on a compact subset K of X. The characteristic function of K is in $L^p(X,\mu)$ and so i will be non-zero. This proves one half of

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THEOREM 3.3. *For a totally ordered semigroup X and $\mu \in M(X)$, with continuous part μ_c, the totally ordered semigroup algebra $L^p(X,\mu)$ is WA if and only if $\mu_c \equiv 0$.*

PROOF. We need to prove that if $\mu_c \equiv 0$ then $L^p(X,\mu)$ is WA. Now if $\mu_c \equiv 0$ then μ will be concentrated on the countable set $P = \{x \in X : \mu(\{x\}) > 0\}$. Thus, denoting the point mass at $x \in X$ by x also, it is enough to show that $D(x)(y) = 0$ for all derivations D from $L^p(X,\mu)$ to its dual and all $x, y \in P$. If $x \leq y \in P$, then $x * y = \mu(\{x\})y$ and the derivation equality, gives that $\mu(\{y\})D(x)(z) = 0$. Hence $D(x)(z) = 0$ whenever $x \leq z \in P$. Putting $x = y \geq z \in P$ gives that $D(x)(z) = 0$ for $x \geq z$ also. ∎

4. **The bounded derivations from $L^1(X,\mu)$ to $L^\infty(X,\mu)$.** As noted in the preliminaries, weak amenability is normally studied by considering bounded derivations from a Banach algebra into its dual. Following the construction given in [1] the derivation from $L^p(X,\mu)$ into $L^p(X,\mu_c)$ displayed in the last section gives rise to bounded derivations from $L^p(X,\mu)$ into its dual, $L^q(X,\mu)$, of the form,

$$D_\lambda(f)(g) = \lambda\big(g \cdot i(f)\big) = \int_X \lambda(x) f(x) \int_{(-\infty,x)} g(u)\, d\mu(u)\, d\mu_c(x)$$

for each $\lambda \in L^q(X,\mu_c)$, $f, g \in L^p(X,\mu)$.

In this section we show that all the bounded derivations from $L^1(X,\mu)$ into $L^\infty(X,\mu)$ are of this form. Similarly to [8] we are able to write a bounded derivation as a double integral.

Thus we start with an arbitrary bounded derivation $D: L^1(X,\mu) \longrightarrow L^\infty(X,\mu)$. Then defining $\Lambda\big((f,g)\big) = D(f)(g)$ gives a bounded bilinear map from $L^1(X,\mu) \times L^1(X,\mu)$ to $\mathbb{C}$. This then 'extends' to a bounded linear functional (which we also denote Λ) on $L^1(X,\mu)\hat{\otimes} L^1(X,\mu)$. Now by Grothendieck's well-known result this tensor product is isometrically isomorphic to $L^1(X \times X,\ \mu \times \mu)$, so that Λ can be thought of as being in $L^\infty(X \times X,\ \mu \times \mu)$. Then

$$D(f)(g) = \Lambda\big((f,g)\big) = \Lambda(f \otimes g) = \int_X \int_X \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(x)\, d\mu(y).$$

Hence for $f, g, h \in L^1(X,\mu)$,

$$\begin{aligned} D(f * g)(h) = {} & \int_X \int_X \int_{(-\infty,x]} \Lambda\big((x,y)\big) h(y) f(x) g(u)\, d\mu(u)\, d\mu(x)\, d\mu(y) \\ & + \int_X \int_X \int_{(-\infty,x)} \Lambda\big((x,y)\big) h(y) g(x) f(u)\, d\mu(u)\, d\mu(x)\, d\mu(y) \end{aligned}$$

and we have similar expressions for $D(f)(g * h)$ and $D(g)(f * h)$.

Now taking $c < d \leq a < b$ the derivation equality implies that

$$\int_{[d,a]} h(u)\, d\mu(u) \int_{[a,b)} \int_{(c,d)} \Lambda\big((x,y)\big) g(x) f(y)\, d\mu(x)\, d\mu(y) = 0$$

for all $f, g, h \in L^1(X,\mu)$ (consider f, g, h vanishing almost everywhere outside of $[a,b)$, (c,d) and $[d,a]$ respectively). Thus

(1) if $\mu([d,a]) > 0$ then $\Lambda\big((x,y)\big) = 0$ almost everywhere on $(c,d) \times [a,b)$.

We wish to show that this implies that Λ is zero almost everywhere on the set $L = \{(x,y) \in X \times X : x < y\}$. In doing this we need to take care near the diagonal and so we define $M = \{(x,y) \in L : (x,y) \neq \emptyset\}$ (note we are using (x,y) to denote a point in $X \times X$ and an interval in X), and $N = \{(x,y) \in L : (x,y) = \emptyset$ and $\mu(\{y\}) > 0\}$. We note that these sets are measurable and that $L \setminus (M \cup N)$ has measure zero. It is straightforward using (1) to find an open neighbourhood of each point of M and N on which Λ is zero almost everywhere. For example take $(x,y) \in M$ such that there is $u \in (x,y)$ with $(x,u) \neq \emptyset$, say $v \in (x,u)$. Then if (x,v) and (v,u) are empty we have $\mu(\{v\}) > 0$ and we can take $(c,v) \times (v,u)$ (or $[x,v) \times (v,u)$ if x is minimal) as our neighbourhood. If $(v,u) \neq \emptyset$ then $\mu([v,u]) > 0$ and our neighbourhood can be taken to be $(c,v) \times (u,b)$. Finally for this example if $(x,v) \neq 0$ take $w \in (x,v)$, then $v \in (w,u)$ so that $\mu([w,u]) > 0$ and our neighbourhood can be taken to be $(c,w) \times (u,b)$.

The existence of such neighbourhoods of each point and the compact regularity of μ then means that Λ is zero almost everywhere on L which implies that

$$\int_X \int_{(-\infty,y)} \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(x)\, d\mu(y) = 0 \tag{2}$$

and so

$$\begin{aligned} D(f)(g) &= \int_X \int_{[y,\infty)} \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(x)\, d\mu(y) \\ &= \int_X \int_{(-\infty,x]} \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(y)\, d\mu(x). \end{aligned} \tag{3}$$

Next taking f, g and h to be zero outside of $[a,b)$, $[d,a]$ and $(c,d]$ respectively, the derivation equality together with (2) and the easy fact that if $\mu(\{x\}) > 0$ then $\Lambda(x,x) = 0$, imply that

$$\begin{aligned} &\int_{(c,d]} \int_{[a,b)} \int_{[d,a]} \Lambda\big((x,y)\big) g(u) f(x) h(y)\, d\mu(u)\, d\mu(x)\, d\mu(y) \\ &\qquad = \int_{(c,d]} \int_{[a,b)} \int_{[d,a]} \Lambda\big((x,u)\big) g(u) f(x) h(y)\, d\mu(u)\, d\mu(x)\, d\mu(y) \end{aligned}$$

and so defining $\bar{\Lambda}\big((y,u,x)\big) = \Lambda\big((x,y)\big) - \Lambda\big((x,u)\big)$, we have

$$\bar{\Lambda}\big((y,u,x)\big) = 0 \text{ almost everywhere on } (c,d] \times [d,a] \times [a,b). \tag{4}$$

Again considering separate cases it is straightforward, using (4), to find an open neighbourhood of each point of $\{(y,u,x) \in X \times X \times X : y < u < x\}$ on which $\bar{\Lambda}$ is zero almost everywhere.

Thus using the compact regularity of μ again we get

$$\int_X \int_{(-\infty,x)} \int_{(y,x)} \bar{\Lambda}\big((y,u,x)\big) f(u) g(y) h(x)\, d\mu(u)\, d\mu(y)\, d\mu(x) = 0$$

for all $f, g, h \in L^1(X,\mu)$ which implies that

$$\int_{(-\infty,x)} \int_{(y,x)} \bar{\Lambda}\big((y,u,x)\big) f(u) g(y)\, d\mu(u)\, d\mu(y) = 0 \tag{5}$$

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for almost all $x \in X$. Let R be the set of measure zero for which (5) does not hold for $x \in R$. Then for $x \notin R$ there is a set R_x of measure zero for which whenever $y \in (-\infty, x) \setminus R_x$, $\int_{(y,x)} \overline{\Lambda}\big((y,u,x)\big) f(u)\, d\mu(u) = 0$. Then for $x \notin R$, $y \in (-\infty, x) \setminus R_x$, there is a set $R_{(x,y)}$ of measure zero for which whenever $u \in (y,x) \setminus R_{(x,y)}$, $\overline{\Lambda}\big((y,u,x)\big) = 0$, that is $\Lambda\big((x,y)\big) = \Lambda\big((x,u)\big)$.

We now define λ from X to $\mathbb{C}$, a measurable bounded almost everywhere function. We first define it on $X \setminus R$ and for now assume that X has not got a minimal element. Then $\mu\big((-\infty,x)\big) > 0$ for all $x \in X \setminus R$ and there is $y_x \in (-\infty, x) \setminus R_x$ with $\mu([y_x, x)) > 0$. For $x \in X \setminus R$ put $\lambda(x) = \Lambda\big((x, y_x)\big)$ and $U_x = \big\{u \in (-\infty,x) \setminus R : \lambda(u) \neq \Lambda\big((x,u)\big)\big\}$. Then U_x is measurable and we show that it has measure zero. Take K to be a compact subset of U_x which then has a minimal element k and a maximal element k'. Since $\mu\big((-\infty,k)\big) > 0$ there is $y \in (-\infty,k) \setminus R_x$ (so that $\mu\big((y,x)\big) > 0$). Then we have

$$\Lambda\big((x,u)\big) = \Lambda\big((x,y)\big) \text{ for all } u \in (y,x) \setminus R_{(x,y)}$$

and

$$\Lambda\big((x,u)\big) = \Lambda\big((x,y_x)\big) \text{ for all } u \in (y_x,x) \setminus R_{(x,y_x)}.$$

Taking $u_x \in (y,x) \cap (y_x,x) \setminus (R_{(y,x)} \cup R_{(x,y_x)})$ gives $\lambda(x) = \Lambda\big((x,u_x)\big) = \Lambda\big((x,y)\big)$ and thus $\lambda(x) = \Lambda\big((x,u)\big)$ for all $u \in (y,x) \setminus R_{(x,y)}$ and so certainly for $u \in [k,k'] \setminus R_{(x,y)}$. Thus K and hence U_x has measure zero. Associating with λ its element in $L^\infty(X,\mu)$ then gives

$$\int_X \int_{(-\infty,x)} \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(y)\, d\mu(x) = \int_X \int_{(-\infty,x)} \lambda(x) f(x) g(y)\, d\mu(y)\, d\mu(x)$$

for all $f, g \in L^1(X,\mu)$.

If X has a minimal element, x' say, then put $X' = X \setminus \{x'\}$. If $\mu(\{x'\}) = 0$ then X' has no minimal element so that λ can be defined as above on X' and extended to X arbitrarily. If x' is such that $\mu(\{x'\}) > 0$ then define λ by $\lambda(x) = \Lambda\big((x,x')\big)$. Since $x' \notin R_x$ for any $x \in R$ we have $\lambda(x) = \Lambda\big((x,y)\big)$ for all $y \in (x',x) \setminus R_{(x,x')}$ and hence for all $y \in (-\infty,x) \setminus R_{(x,x')}$.

Thus (3) together with the fact that $\Lambda\big((x,x)\big) = 0$ if $\mu(\{x\}) > 0$ implies that

$$D(f)(g) = \int_X \int_{(-\infty,x)} \lambda(x) f(x) g(y)\, d\mu(y)\, d\mu(x).$$

Finally again using the derivation equality, this time with functions f, g and h vanishing outside of $\{a\}$, $\{a\}$ and $[c,a)$ respectively, it can be seen that if $\mu(\{a\}) > 0$ then the map $y \longmapsto \Lambda\big((a,y)\big)$ is zero almost everywhere on compact subsets of $(-\infty,a)$ and hence on $(-\infty,a)$. Thus

$$\int_X \int_{(-\infty,x)} \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(y)\, d\mu_d(x) = 0,$$

where μ_d is the discrete part of μ. Thus we have,

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THEOREM 4.1. *For X a totally ordered semigroup and $\mu \in M(X)$, the bounded derivations from $L^1(X,\mu)$ into $L^\infty(X,\mu)$ are of the form*

$$D(f)(g) = \int_X \int_{(-\infty,x)} \lambda(x) f(x) g(y)\, d\mu(y)\, d\mu_c(x)$$

where $\lambda \in L^\infty(X,\mu)$.

REMARK 4.2. Reversing the order of integration in the statement of Theorem 4.1 gives that $D(f)(y) = \int_{(y,\infty)} \lambda(x) f(x)\, d\mu_c(x)$ for almost all y. Thus D maps $L^1(X,\mu)$ into the space of bounded continuous functions from X to $\mathbb{C}$.

5. **The bounded derivations from $L^p([0,1],m)$ into its dual for $1 \leq p < \infty$.** Throughout this section m will denote Lebesgue measure on $[0,1]$ and we write dx for $dm(x)$. We denote the monomial $x \longmapsto x^n$ by x^n, so that $1 = x^0$. In this section we describe the bounded derivations from $L^p([0,1],m)$ into its dual, $L^q([0,1],m)$, for all relevant p. This gives as a corollary the bounded derivations from $L^p(X,\mu)$ into its dual for all compact X and continuous μ. Central to the last section was that to a bounded linear map T from $L^1(X,\mu)$ into $L^\infty(X,\mu)$ is associated a $\Lambda \in L^\infty(X \times X, \mu \times \mu)$ for which $T(f)(g) = \int_X \int_X \Lambda\big((x,y)\big) f(x) g(y)\, d\mu(x)\, d\mu(y)$, for all $f, g \in L^1(X,\mu)$. For $p > 1$ no similar map may exist for bounded linear maps from $L^p(X,\mu)$ into its dual. It is finding a map to play the role that λ played in the last section that is the main problem here. We proceed with a series of lemmas. In the statements of these D is a bounded derivation from $L^p([0,1],m)$ into $L^q([0,1],m)$.

LEMMA 5.1. *For $l, n \in \mathbb{N}_0$, $D(x^l)(x^n) = \frac{l+n+1}{n+1} D(1)(x^{l+n})$.*

PROOF. The proof is a straightforward proof by induction on l. ∎

For the rest of this section if $f \in L^p(X,\mu)$, then F will denote the function $x \longmapsto \int_{(-\infty,x)} f(u)\, d\mu(u)$ and if f is equal almost everywhere to a differentiable function then we denote the derivative of the differentiable function in the equivalence class containing f by f'.

LEMMA 5.2. *For polynomials f and g, $D(f)(g) = D(1)\big((fG)'\big)$.*

PROOF. We have for $l \in \mathbb{N}_0$ and $r \in \mathbb{N}$,

$$D(x^l)\big((x^r)'\big) = rD(x^l)(x^{r-1}) = (l+r)D(1)(x^{l+r-1}) = D(1)\big((x^{l+r})'\big).$$

Thus for a polynomial g we have $D(x^l)(g') = D(1)\big((x^l g)'\big) - g(0)D(1)\big((x^l)'\big)$. Since a polynomial h can be written as the derivative of another polynomial g for which $g(0) = 0$ we get $D(x^l)(h) = D(1)\big((x^l H)'\big)$. The result follows since D, $D(1)$ and differentiation are linear maps. ∎

LEMMA 5.3. *Define ψ by $\psi(x) = xD(1)(x)$. Then there exists $\theta \in L^q([0,1],m)$ such that ψ is equal almost everywhere to a function, σ, for which $\sigma(x) = \int_0^x \theta(u)\, du$ and $\sigma(1) = 0$.*

PROOF. We show that any function, φ, satisfying $\left|\int_0^1 \varphi(u) f'(u)\, du\right| \leq K\|f\|_p$ for some $K \in \mathbb{R}$ and all polynomials f, satisfies the conclusion of the lemma. Under the initial condition on φ, $f \longmapsto \int_0^1 \varphi(u)f'(u)\,du$ is a bounded linear functional on the polynomials, which then has a continuous linear extension to a function, ϕ say, in $L^q([0,1],m)$. Hence

$$\int_0^1 \varphi(u)f'(u)\,du = \int_0^1 \phi(u)f(u)\,du$$

for all polynomials f. Put I equal to the absolutely continuous function given by $I(x) = \int_0^x \phi(u)\,du$. Integrating by parts gives,

$$I(1)f(1) = \int_0^1 \phi(u)f(u)\,du + \int_0^1 I(u)f'(u)\,du = \int_0^1 \big(\varphi(u) + I(u)\big)f'(u)\,du,$$

for all polynomials f. Since changing a polynomial by a constant does not effect its derivative we get then that $I(1) = 0$ and

$$\int_0^1 \varphi(u)f'(u)\,du = \int_0^1 \big(-I(u)\big)f'(u)\,du$$

for all polynomials f. Hence $\varphi(u) = -I(u) = \int_0^u -\phi(u)\,du$.

We now need to show that ψ as defined in the statement satisfies the condition given on φ above. This follows since Lemma 5.2 and the product rule give $D(f)(1) = D(1)\big((xf)'\big) = D(1)(xf') + D(1)(f)$ for all polymonials f so that $\left|\int_0^1 D(1)(u)uf'(u)\,du\right| \leq 2\|D\|\,\|f\|_p$. ∎

LEMMA 5.4. *For polynomials f and g, there exists a λ such that $D(f)(g) = \int_0^1 \lambda(u)f(u)G(u)\,du$ and $\lambda G \in L^q([0,1],m)$.*

PROOF. Let ψ and σ be as in the statement of Lemma 5.3. Define $\phi(x) = \sigma(x)/x$ for $x \in (0,1]$ and let ϕ_n be the restriction of ϕ to $[\frac{1}{n},1]$. Then ϕ_n is absolutely continuous and so we can integrate by parts to get,

$$\int_{\frac{1}{n}}^1 \phi_n(x)(fG)'(x)\,dx = \phi_n(1)f(1)G(1) - \phi_n\Big(\frac{1}{n}\Big)f\Big(\frac{1}{n}\Big)G\Big(\frac{1}{n}\Big) - \int_{\frac{1}{n}}^1 \phi_n'(x)fG(x)\,dx. \tag{6}$$

Now $\phi_n(1) = 0$ for each n and writing $G(x) = xh(x)$ for some polynomial h we have $\phi_n(\frac{1}{n})f(\frac{1}{n})G(\frac{1}{n}) = \sigma(\frac{1}{n})f(\frac{1}{n})h(\frac{1}{n}) \longrightarrow 0$. Also since each ϕ_n is differentiable almost everywhere on $[\frac{1}{n},1]$, ϕ is differentiable almost everywhere on $[0,1]$, and the product rule gives $\sigma'(x) = x\phi'(x) + \phi(x)$ for almost all $x \in [0,1]$. Now σ' and ϕ can both be regarded as elements of $L^q([0,1],m)$ and so $x \longmapsto x\phi'(x)$ can also, and in particular it is integrable. Then since fh is bounded, $\int_0^1 \phi'(x)fG(x)\,dx = \int_0^1 \phi'(x)xf(x)h(x)\,dx$ is finite and so the RHS of (6) converges to it. Similarly the finiteness of $\int_0^1 \phi(x)(fG)'(x)\,dx$ will ensure that the LHS of (6) will converge to it. Hence we have

$$D(f)(g) = D(1)\big((fG)'\big) = \int_0^1 \phi(x)(fG)'(x)\,dx = \int_0^1 -\phi'(x)fG(x)\,dx.$$

Thus we put $\lambda = -\phi'$. We know that $x \longmapsto x\phi'(x)$ is in $L^q([0,1],m)$ and so again writing for a polynomial g, $G(x) = xh(x)$ for some polynomial h, we get that $\lambda G \in L^q([0,1],m)$ for each polynomial g. ∎

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Now for a Borel measurable function λ, a sufficient condition that

$$D(f)(g) := \int_0^1 \lambda(x)f(x)G(x)\,dx \tag{7}$$

defines a bounded derivation from $L^p([0,1],m)$ into $L^q([0,1],m)$ is that there is $K \in \mathbb{R}$ such that for all $g \in L^p([0,1],m)$, $\lambda G \in L^q([0,1],m)$ and $\|\lambda G\|_q \leq K\|g\|_p$. We denote the set of all such Borel measurable λ by $\mathcal{L}(p)$. In extending Lemma 5.4 to apply to all $f, g \in L^p([0,1],m)$ we show that λ being in $\mathcal{L}(p)$ is also a necessary condition for (7) to define a bounded derivation.

THEOREM 5.5. *For $1 \leq p < \infty$, a map D from $L^p([0,1],m)$ into its dual is a bounded derivation if and only if there is $\lambda \in \mathcal{L}(p)$ with*

$$D(f)(g) = \int_0^1 \lambda(x)f(x)\int_0^x g(y)\,dy\,dx$$

for all $f, g \in L^p([0,1],m)$.

PROOF. The 'if' part follows from the discussion before the statement. So we start with D, a bounded derivation from $L^p([0,1],m)$ into $L^q([0,1],m)$ and let λ be as in the statement of Lemma 5.4. Now for a sequence of polynomials (f_n) converging to f in $L^p([0,1],m)$, we have for polynomials g,

$$\left|\int_0^1 \lambda(x)G(x)\big(f_n(x)-f(x)\big)\,dx\right| \leq \|\lambda G\|_q\,\|f_n - f\|_p \longrightarrow 0$$

so that $D(f)(g) = \int_0^1 \lambda(x)f(x)G(x)\,dx$ for all $f \in L^p([0,1],m)$ and polynomials g.

We next show that $\lambda G \in L^q([0,1],m)$ for all $g \in L^p([0,1],m)$. So fix $g \in L^p([0,1],m)$ and let (g_n) be a sequence of polynomials tending to g. Now for $l \in \mathbb{N}$, $\int_{\frac{1}{l}}^1 |\lambda(x)|^q\,dx < \infty$, so that for $f \in L^p([0,1],m)$,

$$\left|\int_{\frac{1}{l}}^1 \lambda(x)f(x)\big(G(x)-G_n(x)\big)\,dx\right| \leq \int_{\frac{1}{l}}^1 |\lambda(x)f(x)|\,dx\,\|g_n - g\|_p \longrightarrow 0$$

as $n \longrightarrow \infty$. Therefore putting $f_l = f\chi_{[\frac{1}{l},1]}$ and $I_l(f) = \int_{\frac{1}{l}}^1 \lambda(x)f(x)G(x)\,dx$, we have

$$D(f_l)(g) = \lim_{n\longrightarrow\infty} D(f_l)(g_n) = \lim_{n\longrightarrow\infty}\int_{\frac{1}{l}}^1 \lambda(x)f(x)G_n(x)\,dx = I_l(f).$$

Thus, since f_l tends to f in $L^p([0,1],m)$, it follows that $I_l(f)$ tends to $D(f)(g)$ as $l \longrightarrow \infty$. Now I_l is a bounded linear function on $L^p([0,1],m)$ with norm $\left(\int_{\frac{1}{l}}^1 |\lambda(x)G(x)|^q\,dx\right)^{\frac{1}{q}}$. Thus the Banach-Steinhaus theorem together with Fatou's lemma gives that $\lambda G \in L^q([0,1],m)$. It then follows that $I_l(f)$ will converge to $\int_0^1 \lambda(x)f(x)G(x)\,dx$ for each $f \in L^p([0,1],m)$. Finally $\|\lambda G\|_q \leq \|D\|\,\|g\|_p$ and $\lambda \in \mathcal{L}(p)$. ∎

In [2] it was shown that if X is a compact totally ordered semigroup and μ a continuous measure then $L^p(X,\mu)$ is isomorphic to $L^p([0,1],m)$. Thus, for such X and μ, defining $\mathcal{L}(X,\mu,p)$ to be the set of Borel measurable functions λ for which $\lambda G \in L^q(X,\mu)$ and $\|\lambda G\|_q \leq K\|g\|_p$ for some $K \in \mathbb{R}$ and all $g \in L^p(X,\mu)$, we have an immediate generalisation of the theorem.

COROLLARY 5.6. *If X is a compact totally ordered semigroup and μ a continuous measure in $M(X)$ then for $1 \le p < \infty$ the bounded derivations from $L^p(X,\mu)$ into $L^q(X,\mu)$ are of the form*

$$D(f)(g) = \int_X \lambda(x) f(x) \int_{-\infty}^{x} g(y)\, d\mu(y)\, d\mu(x)$$

for some $\lambda \in L(X,\mu,p)$.

We finish by remarking that $L(X,\mu,1)$ can be identified with $L^\infty(X,\mu)$ and so we have another proof of the results of the last section in the case that X is compact and μ continuous. (The role that the function which is identically 1 played in this section means it is not likely that the techniques used here could be used to prove the results of the last section in their full generality.) A more intrinsic description of $L(p)$ in the cases $p > 1$ does not seem as tractable. Certainly $x \longmapsto \lambda(x)x^{\frac{1}{q}}$ being in $L^q([0,1],m)$ is a sufficient condition on λ for it to be in $L(p)$ and $x \longmapsto \lambda(x)x^{\frac{1}{q}+\epsilon}$ being in $L^q([0,1],m)$ for each $\epsilon > 0$ is a necessary condition for λ to be in $L(p)$. Whether more than this can be said we do not know.

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