# SHORT CONTRIBUTIONS

## DEDUCTIBLES AND THE INVERSE GAUSSIAN DISTRIBUTION

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Inverse Gaussian; Censoring; Truncation; Deductibles; Limits; Moments.

### 1. INTRODUCTION

The calculation of mean claim sizes, in the presence of a deductible, is usually achieved through numerical integration. In case of a Lognormal or Gamma distribution, the quantities of interest can easily be expressed as functions of the *cumulative* distribution function, with modified parameters. This also applies to the $F$-distribution, where the incomplete Beta function enters the scene; see for instance the appendix in HOGG and KLUGMAN (1984).

The purpose of this paper is to derive an explicit formula for the first two moments of the Inverse Gaussian distribution, in the presence of censoring. For reasons of completeness we also consider truncation of the Inverse Gaussian distribution by an upper limit.

The tractability of the derivation depends in a crucial way on two properties of the Inverse Gaussian distribution. Firstly, the cumulative distribution function of the Inverse Gaussian can be written as a simple function using the Normal probability integral. Secondly, the moment generating function of a *censorized* Inverse Gaussian distribution boils down to an expression containing the cumulative Inverse Gaussian distribution. This manifests itself most clearly in case of life insurance where the quantity of interest is the expectation of a present value. In case of non-life insurance, where the dimension of the Inverse Gaussian random variable is money instead of time, a further step is required: differentiation of the moment generating function.

So, a natural order of this paper is to address ourselves first to the derivation for the life case and afterwards tackling the more laborious derivation for the non-life case.

### 2. MATHEMATICAL PRELIMINARIES

We denote the Inverse Gaussian density with mean $\mu$ and variance $\mu^2/\phi$ by:

(2.1) $$h(x \mid \mu, \phi) = [\mu\phi/2\pi x^3]^{1/2} \exp\{-\phi(x-\mu)^2/2\mu x\}$$

and its cumulative distribution function as:

(2.2) $$H(x \mid \mu, \phi) = N[(x-\mu)\sqrt{\phi/\mu x}\,] + e^{2\phi} N[-(x+\mu)\sqrt{\phi/\mu x}\,]$$

where $N$ denotes the Normal probability integral:

$$N(z) = (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-\tfrac{1}{2}t^2)\,dt$$

which can be evaluated by means of expansions such as given in ABRAMOWITZ and STEGUN (1970). Whenever the parameters do not enter explicitely in $h$ or $H$ we will assume these are $\mu$ and $\phi$.

Observe that $e^{tx} h(x \mid \mu, \phi)$ is proportional with an Inverse Gaussian density:

(2.3) $$\exp(tx)\, h(x \mid \mu, \phi) = \exp(\phi - f)\, h(x \mid m, f)$$

where the auxiliary parameters $m$ and $f$ depend on $t$:

(2.4) $$m = \mu\phi/f$$
$$f = (\phi^2 - 2t\mu\phi)^{1/2}$$

Alternatively, we may say that the Esscher transform of (2.1) is $h(x \mid m, f)$. Integration of (2.3) over part of the positive axis is tractable using (2.2). Integrating (2.3) over the whole positive axis gives the moment generating function of (2.1) as:

$$E[e^{tX}] = \exp(\phi - f)$$

from which we easily see that the $n$-fold convolution of (2.1) is again an Inverse Gaussian density:

$$h^{n*}(x \mid \mu, \phi) = h(x \mid n\mu, n\phi)$$

a property which it has in common with the Gamma density and which formed the reason for HADWIGER (1942) to put (2.1) forward as a modelling tool in insurance and demography.

In case of deductibles or limits, this property is lost, however.

## 3. PRESENT VALUES IN LIFE INSURANCE

Consider a, not necessarily human, life duration $X$, with density (2.1). A lump sum $B$ will be paid at moment $X$. With a discount factor $\exp(-\delta)$ the present value $V$ of $B$ at moment $D < X$ is:

(3.1) $$V = B \exp[\delta(D - X)]$$

In case there is an upper limit $L$ for the moment of payment, (3.1) is valid as long as $D < X \le L$ and for $X > L$, (3.1) is replaced by:

(3.2) $$V = B \exp[\delta(D - L)]$$

The expected value of $V^\tau$, where $\tau = 1$ or 2 is of special economic interest, is then easy to derive. We have:

(3.3)
$$\begin{aligned} V^\tau &= B^\tau \exp[\delta\tau(D-X)] & D < X \leq L \\ &= B^\tau \exp[\delta\tau(D-L)] & L < X \end{aligned}$$

Using (2.1-2-3-4) with $t = -\delta\tau$ results in:

(3.4) $E[V^\tau] = B^\tau [1 - H(D)]^{-1} \{Q[H(L \mid m, f) - H(D \mid m, f)] + R[1 - H(L)]\}$

where the auxiliary variables $Q$ and $R$ are given by:

$$Q = \exp[\phi - f - tD]$$
$$R = \exp[t(L-D)]$$

Whenever $L \to \infty$, (3.4) simplifies to:

(3.5)
$$E[V^\tau] = B^\tau Q \frac{[1 - H(D \mid m, f)]}{[1 - H(D \mid \mu, \phi)]}$$

## 4. EXPECTED VALUES IN NON-LIFE INSURANCE

Now $X$ represents the size of a monetary loss, which is modified to a claim size $Y$ by a deductible $D$ and a limit $L$:

$$\begin{aligned} Y &= 0 & X \leq D \\ &= X - D & D < X \leq L \\ &= L - D & L < X \end{aligned}$$

So, the probability of a nilclaim is given by $H(D)$.

The moment generating function of $Y$ can be written as:

$$M(t) = H(D) + R[1 - H(L)] + Q\{H(L \mid m, f) - H(D \mid m, f)\}$$

In order to derive $E[Y]$ and $E[Y^2]$, we have to differentiate $M(t)$ with respect to $t$, substituting $t = 0$ afterwards.

The following auxiliary results are helpful in this task:

$$dm/dt = m^2/f$$
$$df/dt = -m$$
$$dQ/dt = Q(m - D)$$
$$dH(z \mid m, f)/dt = 2m\{N[(z-m)\sqrt{f/mz}] - H(z \mid m, f)\}$$
$$dN[(m-z)\sqrt{f/mz}]/dt = z^2 f^{-1} h(z \mid m, f)$$

where $z$ is a dummy variable, which does not depend on $t$.

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After some rewriting, we arrive then at:

$$M'(t) = R(L-D)\,[1-H(L)] + \\ + Q(m+D)\,\{H(D\,|\,m,f) - H(L\,|\,m,f)\} + \\ + 2Qm\,\{N[(m-D)\sqrt{f/mD}] - N[(m-L)\sqrt{f/mL}]\}$$

$$M''(t) = R(L-D)^2\,[1-H(L)] + \\ + Q[m^2 f^{-1} - (m+D)^2]\,\{H(D\,|\,m,f) - H(L\,|\,m,f)\} + \\ + 2Qmf^{-1}(m-2fD)\,\{N[(m-D)\sqrt{f/mD}] - N[(m-L)\sqrt{f/mL}]\} + \\ + 2Qmf^{-1}[D^2 h(D\,|\,m,f) - L^2 h(L\,|\,m,f)]$$

Now the main goal of this paper follows easily by substituting $t=0$:

$$(4.1)\quad E[Y] = M'(0) \\ = (L-D)\,[1-H(L)] + \\ + (\mu+D)\,[H(D)-H(L)] + \\ + 2\mu\,\{N[(\mu-D)\sqrt{\phi/\mu D}] - N[(\mu-L)\sqrt{\phi/\mu L}]\}$$

$$(4.2)\quad E[Y^2] = M''(0) \\ = (L-D)^2\,[1-H(L)] + \\ + [\mu^2\phi^{-1} - (\mu+D)^2]\,[H(D)-H(L)] + \\ + 2\mu\phi^{-1}(\mu-2\phi D)\,\{N[(\mu-D)\sqrt{\phi/\mu D}] - N[(\mu-L)\sqrt{\phi/\mu L}]\} + \\ + 2\mu\phi^{-1}[D^2 h(D) - L^2 h(L)]$$

If we let $L \to \infty$, (4.1) simplifies to:

$$(4.3)\quad E[Y] = 2\mu N[(\mu-D)\sqrt{\phi/\mu D}] - (\mu+D)\,[1-H(D)] \\ = (\mu-D)\,N[(\mu-D)\sqrt{\phi/\mu D}] + (\mu+D)\,e^{2\phi}N[-(\mu+D)\sqrt{\phi/\mu D}]$$

which agrees with formula (15) in CHHIKARA and FOLKS (1977) [1].

The second moment (4.2) simplifies to:

$$(4.4)\quad E[Y^2] = [(\mu+D)^2 - \mu^2\phi^{-1}]\,[1-H(D)] + \\ + 2\mu\phi^{-1}\{D^2 h(D) + (\mu - 2\phi D)\,N[(\mu-D)\sqrt{\phi/\mu D}]\}$$

Whenever interest focusses on moments, conditionally on $X > D$, the formulae (4.1-2-3-4) must be divided by the probability $[1-H(D)]$.

[1] I came across this reference after completion of this paper. It does not contain an explicit derivation of this result, however.

It is wellknown that deductibles have a loss eliminating effect. At the same time however, the coefficient of variation of the aggregate claim size distribution increases. A clear exposition of these matters can be found in chapter 5 of STERK (1979).

The availability of (4.3) and (4.4) enables a routine illustration of these findings with the Inverse Gaussian distribution.

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