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RELATIVE RELATION MODULES OF FINITE GROUPS

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Let E be a free product of a finite number of cyclic groups, and S a normal subgroup of E such that $E/S \cong G$ is finite. For a prime p, $\hat{S} = S/S'S^p$ may be regarded as an $\mathbb{F}_pG$-module. Whenever E is a free group, $\hat{S}$ is called a relation module (modulo p); in general we call $\hat{S}$ a relative relation module (modulo p). Gaschütz, Gruenberg and others have studied relation modules; the aim of this paper is to study relative relation modules.

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1. Introduction

Throughout the paper, let p be a (fixed) prime, $\mathbb{F}_p$ the field of p elements, G a finite group of order n generated by $X = \{g_i, 1 \leqq i \leqq d\}$, G_i the cyclic subgroup of G of order n_i generated by g_i and E_i a cyclic group of order m_i, where $m_i = k_i n_i$, $1 \leqq k_i \leqq \infty$. Suppose that $k_i < \infty$ and $p \not| k_i$ if $i \leqq \delta$, $\delta \leqq d$, and $k_i = \infty$ or $p | k_i$ if $\delta + 1 \leqq i \leqq d$. Let E be the free product of the E_i, $1 \leqq i \leqq d$; and S the kernel of the natural epimorphism of E onto G.

$\hat{S} = S/S'S^p$, regarded as an $\mathbb{F}_pG$-module via conjugation in E, is called a relation module (modulo p) when E is a free group. In general $\hat{S}$ will be called a relative relation module (For convenience, reference to p is being dropped). We remark that our notion of relative relation module is different from that discussed in [**5**]. Gaschütz [**2**]; Gruenberg [**3**] and [**4**], and others have studied relation modules. In this paper, along the lines of Gruenberg's theory of relation modules, we study exact sequences, decomposition and comparison of relative relation modules.

Crucial to our study is the relationship between relation and relative relation modules, which enables us to establish an exact sequence: $\hat{S}$ is embedded in the direct sum of the augmentation ideals of the $\mathbb{F}_pG_i$ induced to G, $1 \leqq i \leqq \delta$, and a free module of rank $d - \delta$; the resulting factor module is isomorphic to the augmentation ideal of $\mathbb{F}_pG$. Another exact sequence is established in which $\hat{S}$, when $\delta \geqq 1$, is embedded in a free module of rank $d - 1$. These exact sequences are established in Section 2.

In Section 3 we discuss the structure of $\hat{S}$. When p is coprime to $|G|$, relative relation modules are easily described; our attempts in the non-coprime case have been only partially successful however. Whenever $p \,|\, |G|$ but $p \not| |G_i|$, $1 \leqq i \leqq \delta$; we say that p is semi-coprime to $|G|$ (with respect to X). In the coprime and semi-coprime cases, a characterisation, including a criterion for counting projective summands, of $\hat{S}$ is given.

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Given a decomposition $\hat{S} \cong P(\hat{S}) \oplus N(\hat{S})$, where $P(\hat{S})$ is $\mathbb{F}_pG$-projective and $N(\hat{S})$ contains no projective summand, $P(\hat{S})$ is called the projective part and $N(\hat{S})$ the non-projective part of $\hat{S}$. In the semi-coprime case, $N(\hat{S})$ is nonzero and indecomposable (and also isomorphic to $N(\hat{S})$ of the case when E is a free group). Nothing is predictable in the non semi-coprime case, $N(\hat{S})$ may be zero or decomposable (and may not even be a homomorphic image of $N(\hat{S})$ of the case when E is a free group). The author [**9**] has shown that $\hat{S}$ is non-projective and indecomposable in the case when G is a p-group, X is a minimal generating set of G and $\delta = d$.

In Section 4 we compare various relative relation modules. Two relative relation modules, isomorphic as $\mathbb{F}_p$-spaces, are rarely isomorphic as G-modules; that is, $\hat{S}$ not only depends on G, p and d but also on the mapping of E to G. In cases when one relative relation module is embedded into another it does not necessarily mean that the bigger module splits over the smaller; however there are some known cases when it does happen.

2. Relative relation modules and associated exact sequences

Let F be a free group freely generated by $\{f_i, 1 \leqq i \leqq d\}$ and consider

$$1 \to R \to F \xrightarrow{\pi} G \to 1 \tag{2.1}$$

and

$$1 \to S \to E \xrightarrow{\psi} G \to 1 \tag{2.2}$$

where $f_i\pi = g_i = e_i\psi$, $1 \leqq i \leqq d$. Let $\hat{R}$ and $\hat{S}$ be the corresponding relation and relative relation modules of G determined by π and ψ.

The natural epimorphism $\phi: F \to E$ defined by $f_i\phi = e_i$, $1 \leqq i \leqq d$, induces an epimorphism $\hat{\phi}: F/R'R^p \to E/S'S^p$ such that $(f_iR'R^p)\hat{\phi} = e_iS'S^p$, $1 \leqq i \leqq d$. If $\hat{Q}$ denotes the kernel of $\hat{\phi}$, and $\hat{\theta}$ the restriction of $\hat{\phi}$ to $\hat{R}$, then it may be checked that

$$0 \to \hat{Q} \to \hat{R} \xrightarrow{\hat{\theta}} \hat{S} \to 0 \tag{2.3}$$

is $\mathbb{F}_pG$-exact, and that

$$\hat{Q} = \langle (f^{-1} f_i^{m_i} f)R'R^p; f \in F, 1 \leqq i \leqq \delta \rangle.$$

Let $B = \bigoplus_{i=1}^{d} b_i\mathbb{F}_pG$, a free module with $\{b_i, 1 \leqq i \leqq d\}$ as an $\mathbb{F}_pG$ basis. By the embedding

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theorem of Magnus [5], $F/R'R^p$ may be embedded into the semidirect product of B and G such that $f_iR'R^p \to (g_i, b_i)$, $1 \leqq i \leqq d$. This together with (2.3) gives:

Lemma 2.4. *Let T_i be the trivial irreducible submodule of $b_i\mathbb{F}_pG_i$ generated by $t_i = b_i(1 + g_i + g_i^2 + \cdots + g_i^{n_i - 1})$. Then*

$$0 \to \bigoplus_{i=1}^{\delta} T_i^G \to \hat{R} \xrightarrow{\hat{\theta}} \hat{S} \to 0, \tag{2.5}$$

where $\hat{\theta}$ is as defined above, is exact, and $\hat{Q} = \bigoplus_{i=1}^{\delta} T_i^G$.

Corollary 2.6.

$$\dim_{\mathbb{F}_p} \hat{S} = n(d-1) - \left(\sum_{i=1}^{\delta} \frac{n}{n_i}\right) + 1.$$

Proof. By O. Schreier the rank of the free group R is equal to $n(d-1)+1$, which is also equal to the dimension of $\hat{R}$. As $\dim T_i^G = (n/n_i)$, the result follows from (2.5).

Note 2.7. If $m_i = n_i$ for $1 \leqq i \leqq \delta$, and $m_i = \infty$ for $\delta + 1 \leqq i \leqq d$, then by the Kurosh subgroup theorem ([7, Cor. 4.9.1, p. 243]) S is a free group. In this case the rank of S equals the dimension of $\hat{S}$ given by Corollary 2.6.

Proposition 2.8. (Gaschütz [2]). *Given the free presentation* (2.1), *then*

$$0 \to \hat{R} \to \bigoplus_{i=1}^{d} b_i\mathbb{F}_pG \xrightarrow{\hat{\pi}} \mathbf{g} \to 0 \tag{2.9}$$

is $\mathbb{F}_pG$-exact, where $\mathbf{g}$ *denotes the augmentation ideal of $\mathbb{F}_pG$, and $\hat{\pi}$ is determined by $b_i \to g_i - 1$, $1 \leqq i \leqq d$.*

The sequence (2.9) is called the relation sequence (modulo p) of G determined by (2.1). We have:

Proposition 2.10. *Given* (2.2) *then the sequence, which we will call the relative relation sequence,*

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$$0 \to \hat{S} \to \bigoplus_{i=1}^{\delta} \mathbf{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i \mathbb{F}_p G \xrightarrow{\hat{\psi}} \mathbf{g} \to 0 \tag{2.11}$$

is $\mathbb{F}_pG$-exact, where $\mathbf{g}_i$ denotes the augmentation ideal of $b_i\mathbb{F}_pG_i$, and $\hat{\psi}$ is determined by

$$b_i(g_i - 1) \to g_i - 1, \quad 1 \leqq i \leqq \delta;$$

$$b_i \to g_i - 1, \quad \delta + 1 \leqq i \leqq d.$$

Proof. The result follows by an application of the 3×3 lemma to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{\delta} T_i^G & \longrightarrow & \bigoplus_{i=1}^{\delta} T_i^G & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{R} & \longrightarrow & \bigoplus_{i=1}^{d} b_i\mathbb{F}_pG & \xrightarrow{\hat{\pi}} & \mathbf{g} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\theta}} & & \downarrow{\scriptstyle \xi} & & \downarrow & & \\
0 & \longrightarrow & \hat{S} & \xrightarrow{\eta} & \bigoplus_{i=1}^{\delta} \mathbf{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i\mathbb{F}_pG & \xrightarrow{\hat{\psi}} & \mathbf{g} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where

$$\left(\sum_{i=1}^{d} b_i x_i\right)\xi = \sum_{i=1}^{\delta} b_i(g_i - 1)x_i + \sum_{i=\delta+1}^{d} b_i x_i, \qquad x_i \in \mathbb{F}_pG.$$

Next suppose that $\delta \geqq 1$, consider the diagram

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$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{S} & \longrightarrow & \bigoplus_{i=1}^{\delta} \mathfrak{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i \mathbb{F}_p G & \xrightarrow{\hat{\psi}} & \mathfrak{g} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \bigoplus_{i=1}^{d} b_i \mathbb{F}_p G & \xrightarrow{\lambda} & \mathbb{F}_p G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & M & \longrightarrow & \bigoplus_{i=1}^{\delta} T_i^G & \xrightarrow{\beta} & \mathbb{F}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where

$$\left(\sum_{i=1}^{d} b_i x_i\right)\mu = \sum_{i=1}^{\delta} t_i x_i, \quad x_i \in \mathbb{F}_p G,$$

$$\left(\sum_{i=1}^{d} b_i x_i\right)\lambda = \sum_{i=1}^{\delta} x_i + \sum_{i=\delta+1}^{d} (g_i - 1)x_i, \quad x_i \in \mathbb{F}_p G.$$

Let L be the kernel of λ, ρ the augmentation, M the kernel of β and α the restriction of μ to L. The 3×3 lemma yields:

Proposition 2.12. *Suppose that $\delta \geqq 1$, and let L be a free module of rank $d-1$. Then*

$$0 \to \hat{S} \to L \xrightarrow{\alpha} \bigoplus_{i=1}^{\delta} T_i^G \xrightarrow{\beta} \mathbb{F}_p \to 0 \tag{2.13}$$

is $\mathbb{F}_p G$-exact, where α and β are as described above.

It may be noted that, if $\delta = 0$, $\hat{S}$ being a relation module cannot be embedded into a free module of rank $d-1$.

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3. Structure of relative relation modules

An application of Maschke's Theorem to the relative relation sequence (2.11) yields:

Proposition 3.1. *Suppose that $p \nmid n$. Then*

$$\hat{S} \oplus \mathbf{g} \cong \bigoplus_{i=1}^{\delta} \mathbf{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i \mathbb{F}_p G.$$

Since $\mathbf{g}$ and $\mathbf{g}_i^G$ contain no trivial submodules in the coprime case, $\hat{S}$ contains exactly $d-\delta$ copies of the trivial module $\mathbb{F}_p$.

Both T_i^G and $\mathbf{g}_i^G$ are projective, precisely when $p \nmid n_i$. Finiteness of G allows us to treat a projective module as injective, and vice versa (see, [**1**, Theorem (52.3), p. 421]). From (2.5), we have

Lemma 3.2. *Suppose that $p \nmid n_i$, $1 \leqq i \leqq \delta$. Then*

$$\hat{S} \bigoplus_{i=1}^{\delta} T_i^G \cong \hat{R}.$$

For a module V, we define ϕV to be the smallest submodule of V such that $V/\phi V$ is completely reducible. Equivalently, ϕV is the intersection of all maximal submodules of V. We say that p is semi-coprime to n (with respect to X) if $p \mid n$ but $p \nmid n_i$, $1 \leqq i \leqq \delta$. Gaschütz's theorem ([**4**, Theorem 2.9, p. 9]) and Lemma (3.2) yield:

Proposition 3.3. *Suppose that p is semi-coprime to n. Let*

$$0 \to J \to V \xrightarrow{f} W \xrightarrow{g} \mathbb{F}_p \to 0$$

be an $\mathbb{F}_p G$-exact sequence, where V and W are projective modules, with $\operatorname{Ker} g = \phi W$ *and $J \leqq \phi V$. Then*

$$\hat{S} \oplus V \oplus \mathbb{F}_p G \cong J \bigoplus_{i=1}^{\delta} \mathbf{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i \mathbb{F}_p G \oplus W,$$

and J is a non-zero, non-projective and indecomposable module.

The Krull–Schmidt theorem may be applied to simplify the isomorphism of (3.1) and (3.3) in various cases. For example, when $\delta = 0$, Proposition (3.1) and (3.3) reduce to Gaschütz's theorem ([**4**, Theorem 2.7, p. 8, Theorem 2.9, p. 9]).

Of course, the non-projective part $N(\hat{S})$ of $\hat{S}$ in the semi-coprime case is non-zero and indecomposable, as well as isomorphic to $N(\hat{R})$. None of these properties of $N(\hat{S})$ hold for the non semi-coprime case.

(It should be noted that, in all the examples of this and the next section, we consider

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only those relative relation modules, the middle term of whose relative relation sequences do not contain any free direct summand (that is when $\delta = d$)).

Example 3.4. Let $G = S_3$, $p = 2$ and $X = \{g_1, g_2\}$, where $g_1^2 = g_2^3 = 1$. We know that $\mathbb{F}_2 S_3$ has only two distinct principle indecomposable modules, one of them being irreducible. Then it follows that $\hat{S}$ is isomorphic to the irreducible two dimensional module and so $N(\hat{S}) = \{0\}$.

Example 3.5. Let $G = S_4$, $p = 3$, $X = \{g_1, g_2\}$, where $g_1^2 = g_2^3 = 1$. We know (see [**8**, Ex. 18.5, p. 153]) that $\mathbb{F}_3 S_4$ has precisely four principal indecomposable modules each of dimension three; two of them are irreducible. The other two, using (2.9), in applying (3.3) to obtain $N(\hat{R})$, give that $\dim N(\hat{R}) = 1$. Since the dimension of $\hat{S}$ is five (see Corollary (2.6)), therefore the dimension of $N(\hat{S})$ is either two or five. In any case, $N(\hat{S})$ cannot be a homomorphic image of $N(\hat{R})$.

In view of remark (4.9) below, $N(\hat{S})$ in the non semi-coprime case can be decomposable.

A suitable characterisation of $N(\hat{S})$, and so one of $\hat{S}$ in general, remains unknown, however when G is a p-group and X is a minimal generating set for G, $\hat{S}$ turns out to be non-projective and indecomposable, which is discussed by the author in [**9**].

4. Comparison of relative relation modules

Let $X_1 = \{g_i,\ 1 \leqq i \leqq d_1\}$ and $X_2 = \{h_i,\ 1 \leqq i \leqq d_2)$ be two generating sets of G, and consider the resulting relative relation sequences

$$0 \to \hat{S}_1 \to \bigoplus_{i=1}^{\delta_1} \mathbf{g}_i^G \bigoplus_{i=\delta_1+1}^{d_1} b_i \mathbb{F}_p G \overset{\psi_1}{\to} \mathbf{g} \to 0. \tag{4.1}$$

and

$$0 \to \hat{S}_2 \to \bigoplus_{i=1}^{\delta_2} \mathbf{h}_i^G \bigoplus_{i=\delta_2+1}^{d_2} b_i \mathbb{F}_p G \overset{\psi_2}{\to} \mathbf{g} \to 0 \tag{4.2}$$

where $\hat{S}_i = S_i / S_i' S_i^p$, $i = 1, 2$. Whenever $\delta_i = 0$, we shall identify $\hat{S}_i$ with $\hat{R}_i$.

An application of Schanuel's Lemma ([**3**, Lemma 11, p. 162]) to (4.1) and (4.2), together with the Krull–Schmidt theorem on putting $\delta_1 = \delta_2 = 0$ gives:

Proposition 4.3 (Gaschütz [**3**]). *If $d_1 \leqq d_2$, then $\hat{R}_2 \cong \hat{R}_1 \oplus L$, where L is a free module of* rank $d_2 - d_1$.

The analogous result for relative relation modules is far from being true, which we analyse in this section. The following example shows that $\hat{S}_1$ and $\hat{S}_2$ may not be isomorphic even if there is an automorphism of G mapping X_1 onto X_2. (Recall that in all examples we suppose that $\delta = d$).

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Example 4.4. Let $G=\mathbb{Z}_q\times\mathbb{Z}_q$, where q is an integer greater than one,

$$X_1=\{g_1,g_2\}, X_2=\{g_0,g_2\}, \text{ where } g_0=g_1g_2.$$

Case (i). $p\nmid q$. By Lemma (3.2) and Proposition (4.3)

$$\hat{S}_1\oplus T_1^G\oplus T_2^G\cong\hat{S}_2\oplus T_0^G\oplus T_2^G.$$

Since the action of $\langle g_1\rangle$ is trivial on T_1^G but non-trivial on T_0^G, $\hat{S}_1$ and $\hat{S}_2$ cannot be isomorphic as G-modules. As both $\hat{S}_1$ and $\hat{S}_2$ are projective, we also conclude that under the hypotheses the projective parts of two relative relation modules may not be isomorphic.

Case (ii) $p=q>2$. Let $x=(b_1-b_2)(g_1-1)(g_2-1)$ and $y=(b_0-b_2)(g_0-1)(g_2-1)$. It can be shown that $\hat{S}_1$ as an $\mathbb{F}_pG$-module is generated by x, and $\hat{S}_2$ by y. If there exists any $\mathbb{F}_pG$ isomorphism, θ say, of $\hat{S}_1$ to $\hat{S}_2$, then $x\theta=ya$, for some $a\in\mathbb{F}_pG$. As a can be expressed as a sum of an element of $\mathbf{g}$ (which is generated by $\{(g_1-1),(g_2-1)\}$) and an element of $\mathbb{F}_p$, we have

$$ya=\alpha y+y(g_1-1)a_1+y(g_2-1)a_2,\quad \alpha\in\mathbb{F}_p,\ a_1,a_2\in\mathbb{F}_pG.$$

Since ya must be a generator of $\hat{S}_2$, we must have $\alpha\neq 0$. If $b=(g_1-1)^{p-1}(g_2-1)^{p-3}$, then using the commutativity of $\mathbb{F}_pG$,

$$\begin{aligned}(xb)\theta&=\alpha yb\qquad(\text{as } (g_1-1)^p=0)\\&=\alpha(b_0-b_2)(g_1-1)^{p-1}(g_2-1)^{p-1},\end{aligned}$$

which is a non-zero element. But $xb=0$, and so θ cannot be an isomorphism. It may also be noted that both $\hat{S}_1$ and $\hat{S}_2$ are non-projective (and indecomposable), as (by (2.13)) both are embedded in the unique principal indecomposable module $\mathbb{F}_p(\mathbb{Z}_p\times\mathbb{Z}_p)$. Thus the non-projective parts of two relative relation modules may not be isomorphic as well.

We have:

Proposition 4.5. *Suppose that $d_1=d_2=d$ and $\delta_1=\delta_2=\delta$. Let x be a fixed element of G such that $\langle h_i\rangle=\langle g_i\rangle^x$, $1\leqq i\leqq\delta$. Then $\hat{S}_1\cong\hat{S}_2$.*

Proof. Clearly $h_i=(g_i^{\alpha_i})^x$, for some integer α_i such that $\langle g_i^{\alpha_i}\rangle=G_i$, $1\leqq i\leqq\delta$. Therefore, it suffices to prove the result in two cases, namely

(i) When $h_i=g_i^{\alpha_i}$, $1\leqq i\leqq\delta$, and (ii) when $h_i=g_i^x$, $1\leqq i\leqq\delta$.

Case (i). Clearly $\mathbf{h}_i^G=\mathbf{g}_i^G$, $1\leqq i\leqq\delta$; and so it is enough to show that $\hat{\psi}_1$ and $\hat{\psi}_2$ coincide on $\mathbf{h}_i^G$. For, if $a_i\in\mathbb{F}_pG$,

$$(b_i(h_i-1))\hat{\psi}_2=\sum_{\gamma_i=0}^{\alpha_i-1}(b_i(g_i-1)g_i^{\gamma_i})a_i=(b_i(h_i-1)a_i)\hat{\psi}_1;$$

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and so, $\hat{S}_1 \cong \hat{S}_2$.
Case (ii). Let Λ be the $\mathbb{F}_pG$-isomorphism of $\mathbb{F}_pG$ given by $a \to xa$, and consider

$$\begin{array}{ccc} \bigoplus_{i=1}^{\delta} \mathbf{h}_i^G \bigoplus_{i=\delta+1}^{d} b_i\mathbb{F}_pG & \xrightarrow{\hat{\psi}_2} & \mathbf{g} \\ \Big\downarrow \xi & & \Big\downarrow \kappa \\ \bigoplus_{i=1}^{\delta} \mathbf{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i\mathbb{F}_pG & \xrightarrow{\hat{\psi}_1} & \mathbf{g} \end{array}$$

where κ is the restriction of Λ given by $(h_i - 1) \to (g_i - 1)x$, $1 \leqq i \leqq d$; and ξ is so defined that its restriction to $\mathbf{h}_i^G$ and $b_i\mathbb{F}_pG$ coincide with the restriction of Λ given by $b_i(h_i - 1) \to b_i(g_i - 1)x$ and $b_i \to b_ix$, respectively. Then $\xi\hat{\psi}_1 = \hat{\psi}_2\kappa$, and so the existence of an isomorphism of $\hat{S}_2$ to $\hat{S}_1$, namely the restriction of ξ, is immediate.

It is not known whether there is an isomorphism between $\hat{S}_1$ and $\hat{S}_2$ if we only suppose that the elements of X_1 and X_2 are pairwise conjugate to each other. However, this is true in the semi-coprime case and follows from Schanuel's Lemma because the middle terms of (4.1) and (4.2) become projective.

For the rest of the section, suppose that $X_2 = X_1 \cup \{g_0\}$, $\delta_2 = \delta_1 + 1$. We prove:

Lemma 4.6. *$\hat{S}_1$ is a submodule of $\hat{S}_2$, and $\hat{S}_2/\hat{S}_1 \cong \mathbf{g}_0^G$.*

Proof. Let $V = \bigoplus_{i=1}^{\delta_1} \mathbf{g}_i^G \bigoplus_{i=\delta_1+1}^{d_1} b_i\mathbb{F}_pG$, and $W = \mathbf{g}_i^G$. Clearly, $\hat{\psi}_1$ and the restriction of $\hat{\psi}_2$ to V coincide; and so, $\hat{S}_2 \cap V = \hat{S}_1$. Moreover,

$$(\hat{S}_2/\hat{S}_1) \cap (V/\hat{S}_1) = (\hat{S}_2 \cap V)/\hat{S}_1 = \{0\};$$

therefore the sum of $\hat{S}_2/\hat{S}_1$ and $V/\hat{S}_1$ is a direct sum. Since $\dim(\hat{S}_2/\hat{S}_1) + \dim(V/\hat{S}_1) = \dim((V \oplus W)/\hat{S}_1)$; therefore

$$\hat{S}_2/\hat{S}_1 \oplus V/\hat{S}_1 = (V \oplus W)/\hat{S}_1 \cong V/\hat{S}_1 \oplus W.$$

The Krull–Schmidt theorem now gives the required result.
The following example shows that $\hat{S}_2$ need not split over $\hat{S}_1$.

Example 4.7. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $p = 2$, $X_1 = \{g_1, g_2\}$ and $X_2 = \{g_0, g_1, g_2\}$ (where $g_0 = g_1g_2$).

Note that $b_0(g_0 + 1)g_1 = b_0(g_0 + 1)g_2 = b_0(g_1 + g_2)$, is an element of $\mathbf{g}_0^G$. Let $y_1 = b_1(g_1 + g_2) + b_1(g_1 + 1) + b_2(g_2 + 1)$; $y_2 = (b_1 + b_2)(g_1 + 1)(g_2 + 1)$ and $y_3 = b_0(g_1 + g_2)(g_1 + 1) + b_1(g_1 + 1)(g_2 + 1)$. It may be verified that y_1, y_2, y_3 is an $\mathbb{F}_p$-linearly independent subset of $\hat{S}_2$, and so must be a basis, because $\dim \hat{S}_2 = 3$. It is easy to check that the space spanned by y_2, y_3 is the (unique) maximal submodule of $\hat{S}_2$, and so $\hat{S}_2$ must be indecomposable.

Finally we prove:

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Proposition 4.8. *Suppose that either $g_0 \in X_1$ or else p does not divide the order of $\langle g_0 \rangle$. Then $\hat{S}_2 \cong \hat{S}_1 \oplus \mathbf{g}_0^G$.*

Proof. Suppose that $g_0 \in X_1$. We suppose without loss of generality that $g_0 = g_1$. Let $V = \bigoplus_{i=1}^{\delta} \mathbf{g}_i^G \bigoplus_{i=\delta+1}^{d} b_i \mathbb{F}_p G$. Let η_1 be the identity isomorphism of V, and η_2 the epimorphism of $V \oplus \mathbf{g}_0^G$ to V such that the restriction of η_2 to V and $\mathbf{g}_0^G$ is the identity isomorphism. Then $\hat{\psi}_1 = \eta_1 \hat{\psi}_2$ and $\hat{\psi}_2 = \eta_2 \hat{\psi}_1$. Now as in the proof of Schanuel's Lemma ([3, Lemma 11, p. 162]), we may deduce the result by applying Krull–Schmidt theorem.
When $p \nmid |\langle g_0 \rangle|$, the result is obvious because $\mathbf{g}_0^G$ is projective.

It is not known whether $\hat{S}_2$ splits over $\hat{S}_1$ and $\mathbf{g}_0^G$ if we only suppose that g_0 is a conjugate of an element of X_1 in G.

Remark 4.9. If in Proposition 4.8 we suppose that G is a p-group and $g_0 \in X_1$, then $N(\hat{S}_2) \cong N(\hat{S}_1) \oplus \mathbf{g}_0^G$, because $\mathbf{g}_0^G$ being embedded in $\mathbb{F}_p G$, is non-projective. This substantiates our earlier claim that the non-projective part of a relative relation module can be decomposable.

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