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Free products amalgamating unitary subsemigroups

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Let S_i, $i \in I$, be a set of semigroups such that $S_i \cap S_j = U$, if $i \neq j$, and such that U is a unitary subsemigroup of S_i for each i in I. The semigroup amalgam $[\{S_i \mid i \in I\}; U]$ determined by this system is the partial groupoid $G = \cup S_i$ in which a product of two elements is defined if and only if they both belong to the same S_i and their product is then taken as their product in S_i. In 1962, J.M. Howie showed that the amalgam G is embeddable in the free product of the S_i, amalgamating U. To prove this result it suffices to find any semigroup in which G can be embedded. In this paper, by taking convenient representations of the S_i, adapting a method recently (1975) used by T.E. Hall for inverse semigroups, we provide a short method of constructing such a semigroup.

In a classical paper, 1962, Howie [2] showed that a family of semigroups such that each pair intersect in a subsemigroup U, almost unitary in each semigroup of the family, could be simultaneously embedded in a single semigroup. Such an embedding is possible if and only if the system

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can be embedded in its free product amalgamating U . Howie proved his theorem by proving directly that the system, or amalgam, could be embedded in this free product.

In a recent paper Hall [1] has shown that, within the category of inverse semigroups, the corresponding embedding always take place. Here, of course, U must itself be in the category; that is, U must be an inverse semigroup.

Hall, instead of working with free products, works with conveniently chosen representations of the semigroups concerned. The method he uses suggested the method used here. We consider the case when U is a unitary subsemigroup of each semigroup of the family, and construct a semigroup of mappings in which the amalgam is embedded.

To prove the general result it suffices to consider the case of two semigroups S and T , say, only, whose intersection U is a unitary subsemigroup of each. We shall restrict ourselves to this case, without further comment in what follows. We call the partial groupoid $S \cup T$ whose operation for two elements of S is that of S and for two elements of T is that of T , the *semigroup amalgam* $[S, T; U]$. The amalgam is said to be *unitary* when U is unitary in each of S and T , *almost unitary* when U is almost unitary in each of S and T .

This is the first of two papers, the second paper [3] being devoted to almost unitary amalgams. The construction used in the second paper of course applies to the special case of unitary amalgams but results then in a more complicated construction than the one given here. The elegance of Hall's approach to inverse semigroups carries over without modification to the embedding of unitary amalgams. For almost unitary amalgams an alternative procedure is used.

1. Right congruences and the basic lemma

Crucial to our construction will be certain properties of right congruences that we now discuss. First we define unitary.

Let U be a subsemigroup of T . Then U is *left unitary* (*right unitary*) in T if, for all u in U and t in T , $ut \in U$ ($tu \in U$) implies $t \in U$. If U is both left and right unitary in T then it is said to be *unitary in* T .

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If α is an equivalence relation on a set A then we denote by $a\alpha$ the equivalence class to which a belongs.

LEMMA 1. *Let* U *be a left unitary subsemigroup of* T *and let* σ *be a right congruence on* U *. Let* γ *be the right congruence on* T *generated by* σ *. Then, for all* u *in* U *,* $u\gamma = u\sigma$ *.*

Proof. Let $(u, t) \in \gamma$ and $u \in U$. Then, since γ is generated by σ , there exists a sequence of σ-transitions

$$u = u_1a_1 \to v_1a_1 = u_2a_2 \to v_2a_2 = \dots \to v_na_n = t ,$$

say, where $a_i \in T^1$ and $(u_i, v_i) \in \sigma$, for $i = 1, 2, \dots, n$. Since $(u_i, v_i) \in \sigma$, it follows that $u_i, v_i \in U$. Hence, since $u = u_1a_1 \in U$ and U is left unitary, we infer that $a_1 \in U$. Hence also $u_2a_2 = v_1a_1 \in U$ and again, by left unitariness, $a_2 \in U$. Continuing like this we finally have that $a_n \in U$ and so $t = v_na_n \in U$. The proof of the lemma is completed by observing that, since $a_i \in U$, we have $(u_ia_i, v_ia_i) \in \sigma$ for each i .

We now use this lemma to derive the properties of an equivalence relation we need for our construction.

If X is a set then we denote by T_X the semigroup of transformations of X . Assume in what follows that U has an identity element 1 , which is also the identity of T , that U is left unitary in T , and that $\phi : U \to T_X$ is a faithful representation of U for which $1\phi = 1_X$, the identical transformation of X . If $u \in U$ and $x \in X$ we shall write xu as a short-hand for $x(u\phi)$.

For each x in X , the relation

$$\sigma_x = \{(u, v) \in U \times U \mid xu = xv\}$$

is a right congruence on U . Let γ_x be the right congruence on T generated by σ_x . Define the set Γ by

$$\Gamma = \cup\{T/\gamma_x \times \{x\} \mid x \in X\} . \tag{1}$$

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By Lemma 1, the following set is a subset of Γ :

(2) $$\Gamma_U = \{(u\sigma_x, x) \mid u \in U, x \in X\} .$$

Now define δ' on Γ_U by

(3) $$(u\sigma_x, x)\delta'(v\sigma_y, y) \text{ if and only if } xu = yv .$$

LEMMA 2. *Let U be a left unitary subsemigroup of T and suppose that U and T have a common identity 1. Then $|X| = |\Gamma_U/\delta'|$ under the mapping $\mu : x \mapsto (1\sigma_x, x)\delta'$, $x \in X$.*

Proof. Since $(u\sigma_x, x)\delta'(1\sigma_{xu}, xu)$, by (3), the given mapping is onto Γ_U/δ'. Since $1\phi = 1_X$, again by (3), $(1\sigma_x, x)\delta'(1\sigma_y, y)$ implies $x = y$. Hence the mapping μ is one-to-one.

The construction we are leading up to starts with a representation of U of the kind ϕ we have considered and then obtains a representation of T that extends ϕ. We do this by representing T on a set $X \cup Y$, say. In the extension we in fact work with Γ_u/δ', identifying it with X under the mapping μ of Lemma 2.

The next result is essential if our construction is to work. We wish to extend δ' in the right fashion to Γ. As a first step, if $a = (t\gamma_x, x) \in \Gamma$ and $t' \in T$ define

(4) $$at' = ((tt')\gamma_x, x) .$$

Since γ_x is a right congruence on T this product is well-defined. Next define

$$\delta'' = 1_\Gamma \cup \{(a, b) \in \Gamma \times \Gamma \mid a = a't, b = b't, t \in T, (a', b') \in \delta'\} .$$

The relation δ'' is reflexive and symmetric. It is not necessarily transitive. We take δ to be the transitive closure of δ''.

LEMMA 3. *Let U be a left unitary subsemigroup of T. Then, for all a in Γ_U, $a\delta' = a\delta$.*

Proof. Let $(a, b) \in \delta$ and $a \in \Gamma_U$. Then there exists a sequence

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$$a = a_1 \xrightarrow{\delta''} a_2 \xrightarrow{\delta''} \cdots \xrightarrow{\delta''} a_n = b \ ,$$

say. Consider the first step in this sequence. Either $a_1 = a_2$ or there exist a_1', a_2', say, in Γ_U and t in T such that $a_1 = a_1't$, $a_2 = a_2't$, and $(a_1', a_2') \in \delta'$. Thus there exist u_1, u_2 in U and x_1, x_2 in X, say, such that $a_1' = (u_1\sigma_{x_1}, x_1)$, $a_2' = (u_2\sigma_{x_2}, x_2)$, and $x_1u_1 = x_2u_2$. Now, by assumption, $a = a_1 \in \Gamma_U$. Thus $a_1't = ((u_1t)\gamma_{x_1}, x_1) \in \Gamma_U$. By Lemma 1, therefore, $u_1t \in U$ and hence, since U is left unitary, $t \in U$. Hence $u_2t \in U$ and so $a_2 = a_2't \in \Gamma_U$. Moreover, since $x_1u_1 = x_2u_2$, we have $x_1(u_1t) = x_2(u_2t)$. Hence $(a_1, a_2) \in \delta'$. By induction it now follows that $(a, b) \in \delta'$.

The following lemma is basic for our construction and is the analogue for a unitary subsemigroup U of T of the corresponding result of Hall [1] for inverse semigroups. The final part of the proof gives the first place that right unitariness is used in the argument.

BASIC LEMMA. *Let U be a unitary subsemigroup of the semigroup T and such that U and T have a common identity element 1. Let $\phi : U \to T_X$ be a faithful representation of U for which $1\phi = 1_X$. Then there exists a set Y, disjoint from X, and a representation $\psi : T \to T_{X\cup Y}$, say, such that $1\psi = 1_{X\cup Y}$ and such that for all u in U,*

$$u\phi = (u\psi) \mid X$$

and $u\psi \mid Y : Y \to Y$.

Proof. We take $X \cup Y$ to be Γ/δ in which the subset Γ_U/δ, equal to Γ_U/δ' by Lemma 3, has been identified with X, by the mapping μ of Lemma 2. Thus $Y = (\Gamma/\delta)\backslash(\Gamma_U/\delta)$; and, for t in T, we define

$$t\psi : a\delta \mapsto (at)\delta \ , \quad a \in \Gamma \ ,$$

where at is defined by (4).

If $a\delta = b\delta$ then a and b are connected by a sequence of

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δ''-related elements. If two elements a' and b' are δ''-related then we easily obtain that $(a't)\delta''(b't)$. Hence $a\delta = b\delta$ implies $(at)\delta = (bt)\delta$. Thus $t\psi$ is well-defined. We then easily complete checking that $\psi : t \mapsto t\psi$, $t \in T$, is a representation of T in $\mathcal{T}_{X\cup Y}$ for which $1\psi = 1_{X\cup Y}$.

Let $u \in U$ and $x \in X$. Then

$$x(u\psi) = \big((1\sigma_x, x)\delta\big)(u\psi) = (u\sigma_x, x)\delta = (1\sigma_{xu}, xu)\delta = xu = x(u\phi) ,$$

under the agreed identification μ . Thus $(u\psi) \mid X = u\phi$.

Finally, if $u \in U$ and $y \in Y$, then $y = (t\gamma_x, x)\delta$, say, where $t \in T\backslash U$, and

$$y(u\psi) = \big((t\gamma_x, x)\delta\big)(u\psi) = \big((tu)\gamma_x, x\big)\delta .$$

Now suppose that $y(u\psi) \in X$. Then, $\big((tu)\gamma_x, x\big) \in \Gamma_U$, and so by Lemma 1, $tu \in U$. Since U is right unitary, $t \in U$, contradicting the assumption that $t \in T\backslash U$. Hence $y(u\psi) \in Y$ and so $(u\psi) \mid Y \in \mathcal{T}_Y$.

It will be convenient to have a notation for describing the behaviour of $\psi|U$. Denote by $(\psi|U)_X$ the set of all $(u\psi)|X$, $u \in U$, and similarly by $(\psi|U)_Y$, the set of all $(u\psi)|Y$, $u \in U$. In the basic lemma $(\psi|U)_X$ is a set of mappings in $\mathcal{T}_X$ and $(\psi|U)_Y$ is a set of mappings in $\mathcal{T}_Y$ and X and Y are disjoint. Each $u\psi$ is the union of the disjoint mappings $(u\psi)|X$ and $(u\psi)|Y$. We shall write, to indicate this situation, both

$$\psi|U = (\psi|U)_X \oplus (\psi|U)_Y$$

and

$$u\psi = (u\psi)|X \oplus (u\psi)|Y .$$

2. Embedding a unitary amalgam

We start with semigroups S and T such that $S \cap T = U$ and U is a unitary subsemigroup of each of S and T . If S, T , and U do not have an identity element in common then we adjoin one. This is done by

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adjoining an extra element e, say, to $S \cup T$, whether or not S or T has an identity element already, to form semigroups with identity $S \cup \{e\}$ and $T \cup \{e\}$ intersecting in $U \cup \{e\}$. We then easily check that $U \cup \{e\}$ is unitary in each of $S \cup \{e\}$ and $T \cup \{e\}$. Moreover if we can embed $S \cup \{e\}$ and $T \cup \{e\}$ in a semigroup P, say, in such a fashion that we preserve their intersection $U \cup \{e\}$, then we have simultaneously embedded S and T and preserved their intersection U. Thus there will be no loss of generality in assuming that S, T, and U have a common identity element 1, say; and we make this assumption in what follows.

We shall embed our amalgam in a transformation semigroup. So our problem is to find a set Z, say, and faithful representations $\tilde{\phi} : S \to \mathcal{T}_Z$ and $\tilde{\psi} : T \to \mathcal{T}_Z$, say, such that, for $s \in S$ and $t \in T$, $s\tilde{\phi} = t\tilde{\psi}$ if and only if $s = t \in U$.

We begin with a faithful representation $\phi : S \to \mathcal{T}_X$, say, for which $1\phi = 1_X$. The right regular representation of X for which $X = S$, will do. Now apply the basic lemma to obtain a representation $\psi : T \to \mathcal{T}_{X \cup Y}$ for which $\phi|U = (\psi|U)_X$, $\psi|U = (\psi|U)_X \oplus (\psi|U)_Y$, and $1\psi = 1_{X \cup Y}$. If ψ is not faithful then at this stage we can adjoin a further representation to ψ: replace Y by $Y \cup K$, where K is disjoint from $X \cup Y$, and the adjoined representation is a faithful one of T in $\mathcal{T}_K$. The new set $Y \cup K$ now becomes the Y of our discussion. Now apply the basic lemma to U as a subsemigroup of S and to the representation $(\psi|U)_Y$ of U. We obtain a set X', that may be chosen disjoint from X and Y, and a representation $\phi' : S \to \mathcal{T}_{Y \cup X'}$, say, such that $(\psi|U)_Y = (\phi'|U)_Y$, $\phi'|U = (\phi'|U)_Y \oplus (\phi'|U)_{X'}$, and $1\phi' = 1_{Y \cup X'}$. We next find a set Y', say, disjoint from X, Y, and X', and a representation $\psi' : T \to \mathcal{T}_{X' \cup Y'}$, extending $(\phi'|U)_{X'}$ in the required fashion; and so on. Set $Z = X \cup Y \cup X' \cup Y' \cup \ldots$ and

$$\tilde{\phi} = \phi \oplus \phi' \oplus \ldots,$$

$$\tilde{\psi} = \psi \oplus \psi' \oplus \ldots.$$

Then $\tilde{\phi} : S \to \mathcal{T}_Z$, $\tilde{\psi} : T \to \mathcal{T}_Z$ are faithful representations because ϕ

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and ψ are faithful.

Let $u \in U$. Then

$$\begin{aligned} u\tilde{\phi} &= u\phi \oplus u\phi' \oplus \ldots \\ &= u\phi \oplus \big((u\phi')|Y \oplus (u\phi')|X'\big) \oplus \ldots \end{aligned}$$

and

$$\begin{aligned} u\tilde{\psi} &= u\psi \oplus u\psi' \oplus \ldots \\ &= \big((u\psi)|X \oplus (u\psi)|Y\big) \oplus \big((u\psi')|X' \oplus (u\psi')|Y'\big) \oplus \ldots . \end{aligned}$$

Since, by construction, $u\phi = (u\psi)|X$, $(u\phi')|Y = (u\psi)|Y$, and so on, we have $u\tilde{\phi} = u\tilde{\psi}$.

Conversely, let $s \in S$, $t \in T$, and suppose that $s\tilde{\phi} = t\tilde{\psi}$. Then $(s\tilde{\phi})|X = s\phi$ and so $s\phi = (t\tilde{\psi})|X = (t\psi)|X$. Since $s\phi : X \to X$ therefore $(t\psi)|X : X \to X$. Returning now to the notation of the first section this means that, for all $u \in U$, $x \in X$,

$$\big((u\gamma_x, x)\delta\big)(t\psi) = \big((ut)\gamma_x, x\big)\delta \in X .$$

Thus $ut \in U$. Since U is left unitary, $t \in U$, $t = v$, say. Thus $s\phi = (v\psi)|X$. But, since $v \in U$, $(v\psi)|X = v\phi$. Hence $s\phi = v\phi$. But ϕ is faithful. Hence $s = v$. Thus $s = t \in U$.

We have proved the following theorem.

THEOREM. *Let* $[S, T; U]$ *be a unitary amalgam. Then* $[S, T; U]$ *is embeddable in a semigroup.*

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