SEMIMODULARITY AND BISIMPLE ω-SEMIGROUPS

by H. E. SCHEIBLICH

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1. Introduction

Let S be a completely 0-simple semigroup and let $\Lambda(S)$ be the lattice of congruences on S. G. Lallement (**2**) has described necessary and sufficient conditions on S for $\Lambda(S)$ to be modular, and has shown that $\Lambda(S)$ is always semimodular ($\lambda, \tau \succ \lambda \cap \tau$ implies $\lambda \vee \tau \succ \lambda, \tau$). This result may be stated: If S is 0-bisimple and contains a primitive idempotent, then $\Lambda(S)$ is semimodular.

Let S be a bisimple ω-semigroup. Munn (**4**) has found necessary and sufficient conditions on S for $\Lambda(S)$ to be modular. This note will describe the conditions under which $\Lambda(S)$ is semimodular. An example will show that these conditions need not be met, so that the existence of a primitive idempotent is indeed essential in Lallement's result.

2. Semimodularity

The notation will be that of (**1**), (**3**), (**4**), (**5**). Some knowledge of these papers is assumed.

Briefly, a semigroup S is an ω-semigroup if the idempotents of S form a countably infinite descending chain: $e_0 > e_1 > e_2 \ldots.$ Bisimple ω-semigroups $S = S(G, \alpha)$ have been characterized by Reilly (**5**) in terms of groups G and endomorphisms α of G.

If $S = S(G, \alpha)$ is a bisimple ω-semigroup, then $\Lambda(S) = [i, \mathscr{H}] \cup [\sigma, S \times S]$ where σ is the minimum group congruence on S. Let $\mathscr{A} = \{A : A$ is a normal α-admissible subgroup of $G\}$. For each $A \in \mathscr{A}$, let rad $A = \{g \in G\colon g\alpha^n \in A$ for some $n\}$, and let $\mathscr{A}^* = \{A \in \mathscr{A}\colon \text{rad } A = A\}$. For each $\lambda \in \Lambda(S)$, let $A_\lambda = \{g \in G\colon (0;\, g;\, 0) \in e_0\lambda\}$. Then $\lambda \to A_\lambda$ and $\tau \to A_\tau$ are, respectively, inclusion preserving bijections of $[i, \mathscr{H}]$ onto $\mathscr{A}$ and of $[\sigma, \sigma \vee \mathscr{H}]$ onto $\mathscr{A}^*$ [(**3**), (**4**)].

Lemma 2.1. *Let $\lambda \in [i, \mathscr{H}]$ and $\tau \in [\sigma, \sigma \vee \mathscr{H}]$ and $A_\lambda \subseteq A_\tau$. Then $\lambda \subseteq \tau$.*

Proof. Let $(x, y) \in \lambda$. Then $x = (m;\, g;\, n)$, $y = (m;\, h;\, n)$, and $gh^{-1} \in A_\lambda \subseteq A_\tau$ by Lemma 2.3 of (**3**). Hence $(x, y) \in \tau$ by the remarks after Lemma 5 of (**4**).

Lemma 2.2. *Let $\lambda \in [i, \mathscr{H}]$ and $\tau \in [\sigma, S \times S]$.*

(i) *Then $\tau \succ \lambda$ in $\Lambda(S)$ implies $\tau = \sigma \vee \lambda$.*
(ii) *Thus $\tau \succ \lambda$ in $\Lambda(S)$ if and only if $A_\tau = A_\lambda$.*

Proof. (i) Since λ, $\sigma \subseteq \tau$, then $\lambda \subseteq \lambda \vee \sigma \subseteq \tau$. Thus $\tau = \lambda \vee \sigma$ since $\tau \succ \lambda$.

(ii) Assume first that $\tau \succ \lambda$ in $\Lambda(S)$. Then $A_\tau = A_{\lambda \vee \sigma}$ by (i) and so

$$A_\tau = \text{rad}\,(A_\lambda)$$

by Lemma 7 of (4). Thus $A_\lambda = \text{rad}\, A_\lambda$ since otherwise there would be an idempotent separating congruence strictly between λ and τ by Lemma 2.1. Then $A_\lambda = A_\tau$ since otherwise there would be a group congruence strictly between λ and τ, again by Lemma 2.1.

Conversely, assume that $A_\lambda = A_\tau$ and $\kappa \in \Lambda(S)$ such that $\lambda \subseteq \kappa \subseteq \tau$. Then $A_\tau = A_\lambda \subseteq A_\kappa \subseteq A_\tau$ and so $\kappa = \lambda$ or $\kappa = \tau$ depending on whether $\kappa \in [i, \mathscr{H}]$ or $\kappa \in [\sigma, \sigma \vee \mathscr{H}]$.

Theorem 2.3. $\Lambda[S(G, \alpha)]$ *is semimodular if and only if* $A \in \mathscr{A}^*$ *and* $B \succ A$ *in* $\mathscr{A}$ *implies* $B \in \mathscr{A}^*$.

Proof. Assume that $\Lambda[S(G, \alpha)]$ is semimodular. Let $A \in \mathscr{A}^*$ and $B \succ A$ in $\mathscr{A}$. Choose $\lambda \in [i, \mathscr{H}]$ and $\tau \in [\sigma, \sigma \vee \mathscr{H}]$ such that $A_\tau = A$ and $A_\lambda = B$. Then $A_{\tau \cap \lambda} = A_\tau \cap A_\lambda = A \cap B = A = A_\tau$, using Lemma 6 of (4) at the first step. Thus $\tau \succ \lambda \cap \tau$ by Lemma 2.2 and $\lambda \succ \lambda \cap \tau$ since $B \succ A$ in $\mathscr{A}$. Thus $\lambda \vee \tau \succ \lambda, \tau$ since $\Lambda(S)$ is semimodular. But, again by Lemma 6 of (4) $A_{\lambda \vee \tau} = \text{rad}\,(A_\lambda A_\tau) = \text{rad}\, B$. But $\lambda \vee \tau \succ \lambda$ and so $B = \text{rad}\, B$ by Lemma 2.2.

Conversely, assume that $A \in \mathscr{A}^*$ and $B \succ A$ in $\mathscr{A}$ implies $B \in \mathscr{A}^*$. Let $\lambda \in [i, \mathscr{H}]$ and $\tau \in [\sigma, \sigma \vee \mathscr{H}]$ such that $\lambda, \tau \succ \lambda \cap \tau$. Then $A_\tau = A_{\tau \cap \lambda} = A_\tau \cap A_\lambda$ and so $A_\tau \subseteq A_\lambda$. Since $\lambda \succ \tau \cap \lambda$, then $A_\lambda \succ A_\tau$ and hence $\text{rad}\, A_\lambda = A_\lambda$. But $A_{\lambda \vee \tau} = \text{rad}\,(A_\lambda A_\tau) = A_\lambda$ and so $\lambda \vee \tau \succ \lambda$ by Lemma 2.2. Also $\lambda \vee \tau \succ \tau$ because $A_\lambda \succ A_\tau$.

Corollary 2.4. *Let* $\alpha \in \text{End}\,(G)$ *such that* $G\alpha = G$. *Then* $\Lambda[S(G, \alpha)]$ *is semimodular.*

Proof. Let $A \in \mathscr{A}^*$ and let $B \succ A$ in $\mathscr{A}$. To show that $B \in \mathscr{A}^*$, it is enough (by induction) to show that $g\alpha \in B$ implies $g \in B$.

Since $B \succ A$ in the lattice of α-admissible normal subgroups of G, then $B/A \succ 1$ in the lattice of (α/A)-admissible normal subgroups of G/A. From $B/A \lhd G/A$ it follows that $(B/A)(\alpha/A) \lhd (G/A)(\alpha/A) = G/A$ and since $(B/A)(\alpha/A)$ is (α/A)-admissible, then $(B/A)(\alpha/A) = B/A$ or $(B/A)(\alpha/A) = 1$. But α/A is injective since $A \in \mathscr{A}^*$ and so $(B/A)(\alpha/A) = B/A$.

Suppose then that $g \in G$ and $g\alpha \in B$. Then $(Ag)(\alpha/A) \in B/A$ and so

$$A(g\alpha) = A(b\alpha)$$

for some $b \in B$. Hence $(gb^{-1})\alpha \in A$ and so $gb^{-1} \in A$ since $A \in \mathscr{A}^*$. Thus $gb^{-1} \in B$ and so $g \in B$.

Corollary 2.5. *Assume* G *is abelian and* $\alpha \in \text{End}\,(G)$. *Then* $\Lambda[S(G, \alpha)]$ *is semimodular.*

Proof. The argument is essentially the same as in Corollary 2.4. The important thing there is that $(B/A)(\alpha/A) \lhd G/A$ and this is so here too because G is abelian.

Example 2.6.† Let $N = \{0, 1, 2, \ldots\}$. Let $G = \{\sigma\colon \sigma$ is a permutation of N, and $n\sigma = n$ for all but a finite number of $n \in N\}$. Let $A = \{\sigma \in G\colon \sigma$ is an even permutation$\}$. Define $\alpha\colon G \to G$ by $(2n)(\sigma\alpha) = 2(n\sigma)$ and

$$(2n+1)(\sigma\alpha) = 2(n\sigma)+1.$$

Then $\alpha \in \mathrm{End}\,(G)$, α is injective, and $G\alpha \nsubseteq A$. Hence $\mathscr{A} = \{1, A, G\}$ and $\mathscr{A}^* = \{1, G\}$. Consequently, $\Lambda[S(G, \alpha)]$ is not semimodular. In fact, $[i, \sigma \vee \mathscr{H}]$ is the non-modular five lattice and so $\Lambda[S(G, \alpha)]$ does not satisfy the Jordan-Dedekind chain condition.

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The University of South Carolina

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