MONTEL ALGEBRAS ON THE PLANE

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1. Introduction. The results of Rudin in [7] show that under certain conditions, the maximum modulus principle characterizes the algebra $A(G)$ of functions analytic on an open subset $G$ of the plane $C$ (see below). In [2], Birtel obtained a characterization of $A(C)$ in terms of the Liouville theorem; he proved that every singly generated $F$-algebra of continuous functions on $C$ which contains no non-constant bounded functions is isomorphic to $A(C)$ in the compact-open topology. In this paper we show that the Montel property of the topological algebra $A(G)$ also characterizes it. In particular, any Montel algebra $A$ of continuous complex-valued functions on $G$ which contains the polynomials and has continuous homomorphism space $M(A)$ homeomorphic to $G$ is precisely $A(G)$.

An example is given to show that this is not true if we do not require $M(A) = G$. For each $n \geq 1$, a subalgebra of continuous complex-valued functions on $G$ is constructed which contains the polynomials and is isomorphic to $P(G^n)$, the closure of polynomials in $n$ variables in the topology of uniform convergence on compact subsets of the open set $G^n$ in $C^n$. For polynomially convex open sets $G$, the algebras so constructed are Montel but cannot be isomorphic to $A(G)$ unless $n = 1$. In case $G = C$, the algebras obtained provide an answer to a question asked in [3]: Do there exist subalgebras of continuous functions on the plane which properly contain $A(C)$ but contain no non-constant bounded functions?

2. Preliminaries. We shall use the result of Rudin mentioned above in the following form. Define a uniform algebra on a topological space $X$ to be an algebra of continuous complex-valued functions on $X$ which contains the constants and is closed under uniform convergence on compact subsets of $X$. By a maximum modulus algebra on $X$ we shall mean a uniform algebra $A$ on $X$ having the property that for every compact subset $K$ of $X$, the Silov boundary of the restriction algebra $A|K$ is contained in the topological boundary of $K$. Rudin's result can be formulated as follows: if $A$ is a maximum modulus algebra on an open subset $G$ of $C$, if $A$ contains the polynomials, and if $M(A) = G$, then $A = A(G)$.

Let $A$ be a uniform algebra on $X$ and $K$ a compact subset of $X$. The $A$-convex hull of $K$, denoted $\text{hull}_A K$, is the set $\{x \in M(A) : |\hat{a}(x)| \leq ||a||_K, a \in A\}$, where $\hat{a}(x) = x(a)$ defines the Gelfand transform $\hat{a}$ of $a$. For compact subsets $K$ of $X$, $\text{hull}_A K$ is compact and the algebra $A_K$ obtained as the uniform

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closure of the restriction algebra $A|K$ has non-zero continuous homomorphism space $M(A_K) = \text{hull}_A K$ [6].

If $X$ is a $\sigma$-compact locally compact space, then any uniform algebra on $X$ is an $F$-algebra; by $\sigma$-compactness, the topology of uniform convergence on compact subsets is metrizable, and it is complete since $X$ is a $k$-space [5]. Moreover, there exist compact subsets $K_n$ of $X$ such that $K_{n+1} \supseteq K_n$, $X = \bigcup_{n=1}^{\infty} K_n$, and every compact subset of $X$ is contained in some $K_n$. Such a sequence $\{K_n\}_{n=1}^{\infty}$ is called a hemi-compact covering of $X$. If $X$ has a hemi-compact covering $\{K_n\}_{n=1}^{\infty}$ and $A$ is a uniform algebra on $X$, then $\{\text{hull}_A K_n\}_{n=1}^{\infty}$ is a hemi-compact covering of $M(A)$ [5]. It follows that when $X$ is $\sigma$-compact and locally compact, the algebra $\hat{A}$ of Gelfand transforms of elements in $A$ is a uniform algebra on $M(A)$ and $\hat{A}$ is algebraically and topologically isomorphic to $A$. If $M(A) = X$ (topologically), we shall identify the isomorphic algebras $A$ and $\hat{A}$ and say that $A$ is a uniform algebra on $M(A)$.

If $K$ is a compact subset of $C^n$, then $\text{hull}_{F^n} K$ is also a compact set in $C^n$, denoted by $\hat{K}$. Call a subset $X$ of $C^n$ polynomially convex provided $\hat{K} \subset X$ whenever $K$ is a compact subset of $X$. For arbitrary $X$, define $\hat{X}$ to be the intersection of all polynomially convex sets containing $X$.

When $A$ is a uniform algebra on $M(A)$, this concept of polynomial convexity may be generalized to one of $A$-convexity (cf. Rickart [6]). A subset $Y \subset M(A)$ is said to be $A$-convex provided $\text{hull}_A K \subset Y$ whenever $K$ is a compact subset of $Y$. For arbitrary $Y \subset M(A)$, define $\text{hull}_A Y$ to be the intersection of all $A$-convex subsets of $M(A)$ which contain $Y$. Since $M(A)$ is $A$-convex, $\text{hull}_A Y$ always exists, and is $A$-convex. If $Y$ is $\sigma$-compact and locally compact, then Lemma 1 below shows that the non-zero continuous homomorphism space $M(A_Y)$ of the uniform algebra $A_Y$ (defined as the closure of the restriction algebra $A|Y$ in the space $C(Y)$, in the topology of uniform convergence on compact subsets of $Y$) is $\text{hull}_A Y$.

Finally, suppose that $A$ is a uniform algebra on $X$ and $S \subset X$. If there is a neighbourhood $U$ of $S$ and an element $a \in A$ such that $a(x) = 1$ for $x \in S$ and $|a(x)| < 1$ for $x \in U - S$, then $S$ is said to be a local peak set in $X$, and $a$ is said to peak locally at $S$ within $U$. If $U$ can be taken to be the whole space $X$, then $S$ is a peak set of $A$. We obtain our characterization of Montel algebras by showing that, in the cases under consideration, they can have no (non-trivial) local peak sets.

3. A characterization of $A(G)$. A uniform algebra $A$ on $X$ is said to be Montel if every bounded subset (that is, every set of functions in $A$ which is uniformly bounded on compact subsets of $X$) is relatively compact in $A$.

Note that the Montel property is preserved under topological isomorphisms.

**Proposition 1.** Let $A$ be a uniform algebra on a $\sigma$-compact locally compact space $X$. If $A$ is Montel, then every local peak set of $A$ in $M(A)$ is open and closed in $M(A)$.
Proof. Suppose that $f \in A$ peaks locally on $S$ within $U$ in $M(A)$. For every positive integer $n$, the set $U_n = \{x \in U : |f(x) - 1| < 1/n\}$ is a neighbourhood of $S$ in $M(A)$, and $\{U_n\}_{n=1}^\infty$ is a fundamental sequence of neighbourhoods of $S$. Let $\{K_n\}_{n=1}^\infty$ be a hemi-compact covering of $M(A)$ by $A$-convex sets $K_n$. There is an integer $n_0$ such that $S \cap K_n \neq \emptyset$ for $n \geq n_0$. Thus for $n \geq n_0$, $S \cap K_n$ is a local peak set of $A|K_n$ in $M(A)$; hence by a well-known result (see [4, p. 62]) it is known that $S \subseteq K_n$. Let $\{f_n\}_{n=1}^\infty$ be a subsequence converging uniformly on compact subsets of $X$ to $f \in A$. Clearly $f(x) = 1$ for $x \in S$ and if $y \in M(A) - S$, then $f(y) = 0$. Since $f \in C(M(A))$, it must be that $S$ is open and closed in $M(A)$.

Corollary 1. Let $A$ be a uniform algebra on a connected $\sigma$-compact, locally compact space $X$. If $A$ is Montel, then $A$ is a maximum modulus algebra on $M(A)$.

Proof. Suppose that there is a compact subset $K$ of $M(A)$ and a function $f \in A$ such that $\{x \in M(A) : |f(x)| = 1\}$ does not meet the boundary of $K$ in $M(A)$. If $x$ is chosen to be any element of this set, then the function $g \in A$ defined by $g = ((f/f(x)) + 1)/2$ peaks in $K$ on $y \in K$: $f(y) = f(x) = S$, which is in the interior of $K$. Thus $S$ is a local peak set of $A$ in $M(A)$, hence $S$ is open and closed in $M(A)$, whence $S = M(A)$, which is impossible.

Applying the result of Rudin in the form stated above, we obtain the following result.

Corollary 2. Let $A$ be a uniform algebra on an open subset $G$ of $C$ and suppose that $A$ contains the polynomials and $M(A) = G$. Then $A$ is Montel if and only if $A = A(G)$.

4. Montel algebras of non-analytic functions. In this section we show that if $G$ is a polynomially convex open connected subset of $C$ and $n \geq 1$, there is a uniform algebra $A$ on $G$ which is algebraically and topologically isomorphic to the algebra of all analytic functions on an open subset of $C^n$, in the compact-open topology. Since the Montel property is preserved under isomorphisms, the algebra is Montel. However, if $n > 1$, then $A \neq A(G)$ since the continuous homomorphism space of $A$ is an open subset of $C^n$ while that of $A(G)$ is $G$ (cf. [4, p. 58]).

In the construction, the following standard fact is used.

Lemma. If $K$ is a compact connected subset of $C$ and $\epsilon$ is any positive real number, then there exists a simple closed curve $J$ such that $K$ is contained in the relatively compact component of $C - J$ and every point of $J$ is at a distance less than $\epsilon$ from some point of $K$; cf. [8, p. 35].

Proposition 2. If $G$ is an open connected subset of $C$ and $n \geq 1$, then $P(G^n)$ is algebraically and topologically isomorphic to a subalgebra of $C(G)$.

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Proof. $G$ is $\sigma$-compact and locally compact, thus there exists a hemi-compact covering $\{K_j\}_{j=1}^\nu$ of $G$. Since $G$ is connected, locally connected, and locally compact, every compact subset of $G$ is contained in a compact connected subset, thus we may assume that $\{K_j\}_{j=1}^\nu$ is chosen so that each $K_j$ is connected.

Choose a sequence of simple closed curves $J_j$ in $G$ as follows. $K_1$ is a compact connected subset of the open set $G$ in $C$, thus there exists a simple closed curve $J_1$ in $G$ such that the relatively compact component $i(J_1)$ of $C - J_1$ contains $K_1$. Applying the lemma now to the compact connected set $J_1$, a simple closed curve $J_2$ in $G$ may be chosen so that the closure $c(J_1)$ of $i(J_1)$ in $C$ lies in $i(J_2)$ and $c(J_2) - i(J_1) \subset G$. Suppose by way of induction that $J_j$ have been chosen, $1 \leq j \leq 2k$, such that

1. $J_j \subset G$, $1 \leq j \leq 2k$,
2. $K_i \cup c(J_{2i-2}) \subset i(J_{2i-1}) \subset c(J_{2i}) \subset i(J_{2i+1})$, $1 \leq i \leq k$, and
3. $c(J_{2i}) - i(J_{2i-1}) \subset G$, $1 \leq i \leq k$.

Now $K_{k+1} \cup J_{2k}$ is a compact subset of $G$, hence is contained in a compact connected subset $L_{k+1}$. By the lemma, there exists a simple closed curve $J_{2k+1}$ in $G$ such that $L_{k+1} \subset i(J_{2k+1})$ and another curve $J_{2k+2}$ such that $J_{2k+1} \subset i(J_{2k+2})$ and $c(J_{2k+2}) - i(J_{2k+1}) \subset G$. Thus

1. $J_{2k+1}, J_{2k+2} \subset G$,
2. $K_{k+1} \cup c(J_{2k}) \subset i(J_{2k+1}) \subset c(J_{2k+1}) \subset i(J_{2k+2})$,
3. $c(J_{2k+2}) - i(J_{2k+1}) \subset G$,

and by induction, (1), (2), and (3) hold for all positive integers. Define $R_j = c(J_{2j}) - i(J_{2j-1}), j \geq 1$. Note that $\bigcup_{j=1}^\nu R_j$ is closed in $G$.

Since $\{K_j\}_{j=1}^\nu$ is a hemi-compact covering of $G$, it follows from the definition of $\hat{G}$ that $\hat{G} = \bigcup_{j=1}^\nu \hat{K}_j$. Furthermore, $\hat{J}_j = (c(J_j))^\Lambda = c(J_j)$ for all $j$, thus by (2) above, $\hat{G} = \bigcup_{j=1}^\nu \hat{K}_j \subset \bigcup_{j=1}^\nu (c(J_j))^\Lambda = \bigcup_{j=1}^\nu c(J_j)$. Moreover, by (1), we have $\bigcup_{j=1}^\nu c(J_j) = \bigcup_{j=1}^\nu \hat{J}_j \subset \hat{G}$, hence

$$\hat{G} = \bigcup_{j=1}^\infty c(J_j).$$

Now let $\{T_j\}_{j=1}^\nu$ be a sequence of disjoint closed annuli $T_j = \{t \in C: r_j' \leq |t| \leq r_j\}$ whose outer radii $r_j$ increase to infinity. Let $I_j$ be the closed interval $I_j = \{t \in C: \arg(t) = 0$ and $r_j' \leq |t| \leq r_j\}$. By the representation (4) of $\hat{G}$ and the fact that $c(J_j) \subset i(J_{j+1})$, $j \geq 1$, there is a homeomorphism $\varphi: \hat{G} \to C$ such that $\varphi(R_j) = T_j$, $j \geq 1$. To show the existence of $\varphi$, it is enough to note that if $J'$ and $J$ are simple closed curves with $J' \subset i(J)$ and if $r'$ and $r$ are real numbers with $r' < r$, then any onto homeomorphism

$$\varphi: c(J') \to \{t \in C: |t| \leq r'\}$$

can be extended to a homeomorphism $\hat{\varphi}: c(J) \to \{t \in C: |t| \leq r\}$.

For each positive integer $j$, take a space-filling continuous function $g_j: I_j \to C^{n-1}$ with $g_j(I_j) = D_{r_j^{n-1}}$, $\ldots$
where $D_{r,j}^{-1}$ is by definition the polydisc

\[ s = (s_1, \ldots, s_{n-1}) \in C^{n-1}: |s_k| \leq r, 1 \leq k \leq n - 1. \]

For $1 \leq k \leq n - 1$, let $\pi_k$ denote the projection map to the $k$th coordinate and define functions $h_k$ on $\bigcup_{j=1}^{\infty} R_j$ by

\[ h_k(t) = \pi_k(g_j(|\varphi(t)|)), \quad t \in R_j. \]

Then $h_k$ is continuous on the closed subset $\bigcup_{j=1}^{\infty} R_j$ of the normal space $G$ and by the Tietze extension theorem has a continuous extension to a function, which we shall also call $h_k$, on $G$ to $C$. Let $f_k = \varphi^{-1} \circ h_k$, $1 \leq k \leq n - 1$.

Define a map $F: G \to C^n$ by $F(t) = (f_1(t), \ldots, f_{n-1}(t), t)$ for $t \in G$. Clearly $F(G) \subset (\hat{G})^n$ by definition of the functions $f_1, \ldots, f_{n-1}$. On the other hand, if $s = (s_1, \ldots, s_n)$ is an element of $(\hat{G})^n$, then by the representation (4) there is a positive integer $j$ such that $s_1, \ldots, s_n \in c(J_{2j-2})$, thus

\[ \varphi(s_1), \ldots, \varphi(s_n) \in D_{r,j}^{-1} \subset D_{r,j}. \]

To see this, observe that the image under the homeomorphism $\varphi$ of the connected set $i(J_{2j-2}) - R_{j-1} = i(J_{2j-3})$ must lie in a single component of $C - \varphi(R_{j-1})$. However, $\varphi(c(J_{2j-3}))$ is compact and equal to the closure of $\varphi(i(J_{2j-3}))$; therefore $\varphi(i(J_{2j-3}))$ must be contained in the bounded component of the complement of $\varphi(R_{j-1}) = T_{j-1}$. Thus $|\varphi(t)| \leq r_{j-1}$ for all $t \in c(J_{2j-3})$.

It now follows that there exists $r \in I_j$ such that $g_j(r) = (\varphi(s_1), \ldots, \varphi(s_{n-1}))$, and $|\varphi(s_n)| \leq r_{j-1} < r$. Let $p$ be any polynomial on $C^n$ (in fact, any entire function on $C^n$). Then

\begin{equation}
|p(s)| \leq \sup \{|p(s_1, \ldots, s_{n-1}, t)|: t \in \hat{G}, |\varphi(t)| \leq r\}
= \sup \{|p(s_1, \ldots, s_{n-1}, t)|: t \in \hat{G}, |\varphi(t)| = r\}
= \sup \{|p(s_1, \ldots, s_{n-1}, t)|: t \in G, |\varphi(t)| = r\}
= \sup \{|p(F(t))|: t \in G, |\varphi(t)| = r\}.
\end{equation}

However, $\{F(t): t \in G \text{ and } |\varphi(t)| = r\}$ is a compact subset of $F(G)$, thus $s \in (F(G))^A$. That $\varphi$ is a homeomorphism is used to conclude that $\{t \in \hat{G}: |\varphi(t)| \leq r\}$ is compact and that $|\varphi(t)| = r$ implies $t \in G$.

We have shown that $F(G) \subset (\hat{G})^n \subset F(G)^A$ and thus $F(G) \subset (G^n)^A \subset F(G)^A$ since $(\hat{G})^n = (G^n)^A$ is immediate.

If $L$ is a compact subset of $F(G)$, then $\pi_n(L)$ is compact in $G$ and $L = F(\pi_n(L))$. However, $G$, and hence $F(G)$, is hemi-compact; since $F(G)$ is also first countable, it is $\sigma$-compact and locally compact [1]. It follows from Lemma 1 below that $P(F(G)) = P((G^n)^A) = P(G^n)$. Finally, we use Lemma 2 to conclude that $P(G^n)$ is algebraically and topologically isomorphic to the subalgebra $A$ of $C(G)$ generated by the functions $f_1, \ldots, f_{n-1}$, and $z$.

**Lemma 1.** Suppose that $A$ is a uniform algebra on $M(A)$ and $Y$ is a $\sigma$-compact locally compact subset of $M(A)$. If $Y \subset X \subset \text{hull}_A Y$, then $A_X = A_Y$ and
Montel algebras

$M(A_X) = \text{hull}_A Y$; in particular, the restriction map $f \rightarrow f|Y$ is an algebraic and topological isomorphism.

Proof. Let $\{K_n\}_{n=1}^\infty$ be a hemi-compact covering of $Y$. Then $\{\text{hull}_A K_n\}_{n=1}^\infty$ is a hemi-compact covering of $M(A_Y)$, thus

$$M(A_Y) = \bigcup_{n=1}^\infty \text{hull}_A K_n \subset \text{hull}_A Y.$$  

However, $M(A_Y)$ is easily seen to be $A$-convex, whence $\text{hull}_A Y = M(A_Y)$ is a hemi-compact union of the $\text{hull}_A K_n$. Clearly, $f \rightarrow f|Y$ is an algebraic and topological isomorphism.

Let $\{K_n\}_{n=1}^\infty$ be a hemi-compact covering of $Y$. Then $\{\text{hull}_A K_n\}_{n=1}^\infty$ is a hemi-compact covering of $M(A_Y)$, thus $M(A_Y) = \bigcup_{n=1}^\infty \text{hull}_A K_n \subset \text{hull}_A Y$. However, $\text{hull}_A Y$ is easily seen to be $A$-convex, whence $\text{hull}_A Y = M(A_Y)$ is a hemi-compact union of the $\text{hull}_A K_n$. Clearly, $f \rightarrow f|F$ is an algebraic homomorphism of $A_X$ into $A_Y$. If $g \in A_Y$, the Gelfand transform $\hat{g} \in \hat{A}_Y$ is such that $\hat{g}|F = g$. However, $\hat{g} \in \text{hull}_A Y$; for if $L$ is a compact subset of $\text{hull}_A Y$ and $\epsilon > 0$, then taking a compact subset $K$ of $Y$ such that $L \subset \text{hull}_A K$ and an element $p \in A$ such that $||g - p||K < \epsilon$, it follows that $||\hat{g} - p||L \leq ||\hat{g} - p||\text{hull}_A K = ||g - p||K < \epsilon$,

or $\hat{g} \in \text{hull}_A Y$. Let $f = \hat{g}|X$. Then $f \in A_X$ and $f \rightarrow f|Y = g$, thus the homomorphism is onto. This also shows that the map is one-to-one. Thus $f \rightarrow f|Y$ is an algebraic isomorphism. The inequalities $||f|Y||K \leq ||f|\text{hull}_A Y||$ and $||f||\text{hull}_A Y$ which hold for $K$ and $L$ as above show that the map is in fact topological, whence $A_X = A_Y$ and $M(A_X) = M(A_Y)$.

Lemma 2. Let $X$ be $\sigma$-compact and locally compact and let $f_1, \ldots, f_n$ be functions in $C(X)$. Suppose that the map $F: X \rightarrow C^n$ defined by $F(x) = (f_1(x), \ldots, f_n(x))$, $x \in X$, has the property that if $L$ is a compact subset of $F(X)$, then there exists $K$ compact in $X$ such that $L \subset F(K)$. Then the uniform algebra $A$ on $X$ generated by $f_1, \ldots, f_n$ is algebraically and topologically isomorphic to $P(F(X))$.

Proof. $X$ is $\sigma$-compact and locally compact, and the property of $F$ assumed in the hypothesis guarantees that $F(X)$ is also $\sigma$-compact and locally compact, since it is semi-compact and first countable. Thus the uniform algebras $A$ and $P(F(X))$ are $F$-algebras. Define a mapping $\varphi$ on $P(F(X))$ by $\varphi(g) = g \circ F$. Note that the image under $\varphi$ of a dense subset of $P(F(X))$ is dense in $A$. Furthermore, if $K$ is a compact subset of $X$, then $F(K)$ is compact in $F(X)$ and if $p$ is any polynomial on $C^n$, then $||p||F(K) = ||p \circ F||K = ||\varphi(p)||K$. Thus $\varphi$ is continuous and, since $A$ is complete, into $A$. It is clear that $\varphi$ is one-to-one. We show that $\varphi$ is onto. Suppose that $f \in A$ and $K$ is a compact subset of $X$, $\delta$ a real number with $\delta > 0$. Choose a polynomial $p(K, \delta)$ such that $||p(K, \delta) \circ F - f||K < \delta$.

If the indices $(K, \delta)$ are ordered by $(K_1, \delta_1) < (K_2, \delta_2)$ if and only if $K_1 \subseteq K_2$ and $\delta_2 \leq \delta_1$, then $\{p(K, \delta)\}$ may be shown to be a Cauchy net as follows. Let $L$ be an arbitrary compact set in $F(X)$ and let $\epsilon > 0$. Choose a compact set $K_L \subset X$ such that $F(K_L) \supseteq L$. Then, if $(K_L, \delta) < (K_i, \delta_i)$ ($i = 1, 2$), we have $||p(K_1, \delta_1) - p(K_2, \delta_2)||_L < \epsilon$, thus $\{p(K, \delta)\}$ is a Cauchy net. By completeness of $P(F(X))$, $\{p(L, \delta)\}$ has a limit $g \in P(F(X))$. By continuity, $\varphi(g) = f$.

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We have established that $\varphi$ is a continuous algebraic isomorphism of an $F$-algebra onto another, and the interior mapping principle enables us to conclude that $\varphi$ is topological.

For polynomially convex open sets $G$ in $C$, $P(G^n) = A(G^n)$ by Runge's theorem, hence we have the following corollary to Proposition 2.

**Corollary 3.** If $G$ is a polynomially convex open connected subset of $C$, then for each positive integer $n$ there is a uniform algebra $A_n$ on $G$ containing the polynomials and such that $A_n = A(G^n)$ (algebraically and topologically).

By an earlier remark, each of the algebras $A_n$ is Montel since $A(G^n)$ is, thus we have found infinitely many non-isomorphic Montel algebras $A_n$ on $G$. Of course, for $n > 1$, $M(A_n) = M(A(G_n)) = G^n \neq G$, thus $A(G) \subset A_n (A(G) \neq A_n)$.

In the case $G = C$, the algebras $A_n$ constructed above contain no non-constant bounded functions. For suppose that $f \in A_n$ is bounded. By Lemmas 1 and 2, $f = g \circ \varphi$, where $g$ can be taken in the algebra $P(F(C)) = P(C^n) = A(C^n)$. However, $g$ is bounded on $F(C)$; thus by (5), $g$ is bounded on $C^n$. It follows that $g$, and hence $f$, is constant. We have therefore also found (infinitely many non-isomorphic) uniform algebras on $C$ having no non-constant bounded functions and properly containing $A(C)$, answering a question about the existence of such algebras raised by Birtel and Lindberg [3].

**References**


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