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OPERATOR SEMISTABLE PROBABILITY MEASURES ON A HILBERT SPACE

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A characterization of the class of operator semistable probability measures on a real separable Hilbert space is given.

1. Notation and preliminaries

Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let B denote the σ-field generated by the class of open subsets of H. Let $L = L(H)$ denote the algebra of all bounded linear operators on H with the usual norm topology. For any subset F of L, let $\text{Sem}(F)$ denote the closed multiplicative semigroup of operators generated by F. Denote by I the identity operator and by 0, the zero operator. Let G be the multiplicative group of all invertible operators in L. Let $P = P(H)$ be the class of all probability measures on B.

The convolution of two measures μ and ν is denoted by $\mu*\nu$ and the power μ^n is taken in the sense of convolution. Let δ_x denote the probability measure degenerate at $x \in H$. For any $P \in P$ let $\hat{P}$ denote the Fourier transform of P. Thus

$$\hat{P}(y) = \int_H \exp(i\langle x, y\rangle)dP(x) , \quad y \in H .$$

Let $\{P_n\}$ be a sequence of probability measures in P and let

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$P \in \mathcal{P}$. We write $P_n \Rightarrow P$ to say that $\{P_n\}$ converges weakly to P . If $P_n \Rightarrow P$ we write $P = \lim_{n\to\infty} P_n$.

Let $P \in \mathcal{P}$. A closed subset $S(P)$ of H is said to be the *support* of P if the complement of $S(P)$ has P-measure zero and for any $x \in S(P)$ and any neighborhood V_x of x , $PV_x > 0$. We call $P \in \mathcal{P}$ *full* if its support is not contained in any proper hyperplane of H .

Let $A \in G$, and $P \in \mathcal{P}$. Define AP on $\mathcal{B}$ by setting

$$AP(E) = P\left(A^{-1}E\right) , \quad E \in \mathcal{B} .$$

Then $AP \in \mathcal{P}$ and

$$\widehat{AP}(y) = \hat{P}(A^*y) , \quad y \in H ,$$

where A^* is the adjoint of $A \in L$. Moreover, for $P_1, P_2 \in \mathcal{P}$ and $A \in G$ we have

$$A\left(P_1 * P_2\right) = AP_1 * AP_2 .$$

Let $A_n \in L$, $A \in L$, P_n and $P \in \mathcal{P}$ such that $A_n \to A$ and $P_n \Rightarrow P$. Then $A_nP_n \Rightarrow AP$.

Given $P \in \mathcal{P}$, we define $P^- \in \mathcal{P}$ by setting $P^-(E) = P(-E)$. A measure $P \in \mathcal{P}$ is said to be *symmetric* if $P = P^-$. For any $P \in \mathcal{P}$, the measure $\mathring{P} = P * P^-$ is symmetric, and is called the *symmetrization* of P .

In [4] Sharpe introduced and investigated the class of operator stable measures in $\mathbb{R}_n$. In particular, he characterized the class of all full operator stable measures. In [2] Kruglov considered the class of semi-stable measures on $\mathbb{R}_1$ and gave a characterization of this class. In [1] Jajte described the class of all full measures on $\mathbb{R}_n$ which are operator semistable. In this paper we consider the class of operator-semistable measures on $(H, \mathcal{B})$.

DEFINITION 1. A probability measure $P \in \mathcal{P}$ is said to be operator stable, if there exist $A_n \in G$, $x_n \in H$, and $Q \in \mathcal{P}$ such that the relation

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(1) $$P = \lim_{n\to\infty} A_n Q^n * \delta_{x_n}$$

holds.

DEFINITION 2. Let $P \in \mathcal{P}$. We say that P is operator semistable if there exist $A_n \in G$, $x_n \in H$, $Q \in \mathcal{P}$, and a sequence of positive integers k_n , $k_n \to \infty$ as $n \to \infty$, with $k_{n+1}/k_n \to \gamma$ for some $\gamma \geq 1$ such that

(2) $$P = \lim_{n\to\infty} A_n Q^{k_n} * \delta_{x_n} .$$

In the following we will assume that the sequence $\{A_n\}$, $A_n \in G$, satisfies the condition:

(3) $\operatorname{Sem}\left(\left\{A_m A_n^{-1}, n = 1, 2, \ldots, m; m = 1, 2, \ldots\right\}\right)$ is compact in the norm topology of L .

We note that for full probability measures on finite-dimensional spaces the compactness condition (3) can be omitted (see, for example, [*1*]).

We will prove the following theorem which characterizes the class of operator semistable measures on H .

THEOREM 1. *Let P be a full probability measure on $(H, \mathcal{B})$. Then P is operator semistable if and only if P is infinitely divisible and there exist a c , $0 < c < 1$, an element $b \in H$, and an operator $A \in G$, such that the relation*

(4) $$P^c = AP * \delta_b$$

holds. (*Here P^c is the probability measure corresponding to* $[\hat{P}]^c$.)

The proof of Theorem 1 is carried out with the help of several lemmas.

2. Norming sequences

We will prove two lemmas concerning the norming sequences $\{A_n\}$ of operators from G satisfying condition (3).

LEMMA 1. *Let $P \in \mathcal{P}$ be a full operator semistable measure. Then*

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$A_n \to 0$ *as* $n \to \infty$ *where* 0 *is the zero operator in* H .

Proof. Let $P \in \mathcal{P}$ be a full operator semistable probability measure. Suppose that $A_n Q^{k_n} * \delta_{x_n} \Rightarrow P$. By symmetrization we see that

$$\mathring{P} = \lim_{n\to\infty} A_n \mathring{Q}^{k_n}$$

so that

$$\hat{\mathring{P}}(y) = \lim_{n\to\infty} [\hat{\mathring{Q}}(A_n^* y)]^{k_n} , \quad y \in H .$$

It follows that $\hat{\mathring{Q}}(A_n^* y) \to 1$ as $n \to \infty$ uniformly in $y \in H$. We show that $A_n \mathring{Q} \Rightarrow \delta_0$. Let A be an arbitrary limit point of $\{A_n\}$. In view of (3), $\mathrm{Sem}(\{A_n, n \geq 1\})$ is compact so that there exists a subsequence $\{A_{n_j}\} \subset \{A_n\}$ such that $A_{n_j} \to A$ as $n_j \to \infty$. It follows that $A_{n_j} \mathring{Q} \Rightarrow A\mathring{Q}$ so that $\{A_n \mathring{Q}\}$ is relatively compact and hence tight (Theorem 3.9.2, p. 208 of [3]). From Proposition 7.4.2 on page 463 of Laha and Rohatgi [3], we conclude that $A_n \mathring{Q} \Rightarrow \delta_0$.

Let $n \leq n_j$. We write

(5)
$$A_{n_j} \mathring{Q}^{k_{n_j}} = A_{n_j} \mathring{Q}^{k_n} * A_{n_j} \mathring{Q}^{k_{n_j} - k_n} .$$

Letting $j \to \infty$ and noting that $A_n \mathring{Q} \Rightarrow \delta_0$ we conclude from (5) that

(6)
$$\mathring{P} = A\mathring{Q}^{k_n} * \mathring{P}$$

holds for every n .

Next note that $\mathrm{Sem}\left(\left\{AA_{n_j}^{-1}, j \geq 1\right\}\right)$ is compact. Let B be a limit point of $\left\{AA_{n_j}^{-1}\right\}$. Then without loss of generality, passing to a subsequence if necessary we may assume that $AA_{n_j}^{-1} \to B$ as $j \to \infty$. Consequently,

$$A = AA_{n_j}^{-1}A_{n_j} \to BA \quad \text{as} \quad j \to \infty$$

so that

$$A = BA . \tag{7}$$

From (6) we have the equation

$$\overset{\circ}{P} = AA_{n_j}^{-1}A_{n_j}\overset{\circ}{Q}^{k_{n_j}} * \overset{\circ}{P} .$$

Letting $j \to \infty$ we obtain

$$\overset{\circ}{P} = B\overset{\circ}{P} * \overset{\circ}{P}$$

and hence, for $y \in H$,

$$\hat{\overset{\circ}{P}}(y) = \hat{\overset{\circ}{P}}(B^*y)\hat{\overset{\circ}{P}}(y) .$$

Thus

$$|\hat{P}(y)|^2 = |\hat{P}(B^*y)|^2|\hat{P}(y)|^2 , \quad y \in H .$$

We note that $\hat{P}(y) \neq 0$ in some neighborhood of $0 \in H$. It follows that in some neighborhood of $0 \in H$,

$$|\hat{P}(B^*y)| = |\widehat{BP}(y)| = 1$$

and it follows ([3], page 463) that $BP = \delta_x$ for some $x \in H$. Since, however, P is a full measure on $\mathcal{B}$ this is possible if and only if $B = 0$ and $x = 0$. From (7) we conclude that $A = 0$. Since A is an arbitrary limit point of $\{A_n\}$ we conclude that $A_n \to 0$ as $n \to \infty$.

LEMMA 2. *Let C be a limit point of $\left\{A_{n+1}A_n^{-1}, n = 1, 2, \ldots\right\}$. Then the relation*

$$\hat{P}(y) = [\widehat{CP}(y)]^{\gamma}\exp(i\langle b, y\rangle)$$

holds for all $y \in H$, where $b \in H$ and $\gamma = \lim_{n\to\infty} \left(k_{n+1}/k_n\right) \geq 1$.

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Proof. Let $\{A_{n_j}\} \subset \{A_n\}$ be a subsequence such that $A_{n_{j+1}} A_{n_j}^{-1} \to C$ as $j \to \infty$. Then

$$\hat{P}(y) = \lim_{j\to\infty} \left\{ \left[\widehat{A_{n_{j+1}} A_{n_j}^{-1} A_{n_j} Q}(y) \right]^{k_{n_j} (k_{n_{j+1}}/k_{n_j})} \cdot \exp\left[i (k_{n_{j+1}}/k_{n_j}) \left\langle A_{n_{j+1}} A_{n_j}^{-1} x_{n_j}, y \right\rangle \right] \cdot \exp\left[i \left\langle x_{n_{j+1}} - (k_{n_{j+1}}/k_{n_j}) A_{n_{j+1}} A_{n_j}^{-1} x_{n_j}, y \right\rangle \right] \right\}$$

$$= \lim_{j\to\infty} \left\{ \left[\left(\widehat{A_{n_{j+1}} A_{n_j}^{-1} A_{n_j} Q}(y) \right)^{k_{n_j}} \cdot \exp\left(i \left\langle A_{n_{j+1}} A_{n_j}^{-1} x_{n_j}, y \right\rangle \right) \right]^{k_{n_{j+1}}/k_{n_j}} \cdot \exp\left(i \langle b_{n_j}, y \rangle \right) \right\}$$

where $b_{n_j} = x_{n_{j+1}} - (k_{n_{j+1}}/k_{n_j}) A_{n_{j+1}} A_{n_j}^{-1} x_{n_j} \in H$. It follows that (8) holds for some $b \in H$ where $\langle b_{n_j}, y \rangle \to \langle b, y \rangle$, $y \in H$ as $j \to \infty$.

3. Proof of Theorem 1

Suppose that P is infinitely divisible and (4) holds. Set $\gamma = c^{-1}$. Then $\gamma > 1$ and we can write (4) as

$$P = (AP)^{\gamma} * \delta_{\gamma b} .$$

By iteration n times we obtain

$$P = \left(A^n P\right)^{\gamma^n} * \delta_{b_n}$$

for some $b_n \in H$. Set $k_n = [\gamma^n]$. Then $k_{n+1}/k_n \to \gamma$ as $n \to \infty$. Moreover

$$P = \lim_{n\to\infty} \left(A^n P\right)^{k_n} * \delta_{b_n} = \lim_{n\to\infty} A^n P^{k_n} * \delta_{b_n}$$

and it follows that P is operator semistable.

Conversely, suppose that P is a full operator semistable probability measure on $\mathcal{B}$. Then P is infinitely divisible in view of Lemma 1. If $\gamma > 1$, (4) follows easily from Lemma 2. It remains to prove (4) for the $\gamma = 1$ case. This is done in Lemma 4 below.

LEMMA 3. *Let $P \in \mathcal{P}$ be a full measure on $\mathcal{B}$. Then P is operator stable if and only if for every $n \geq 1$,*

$$P = C_n P^n * \delta_{c_n} \tag{9}$$

where $C_n \in G$ and $c_n \in H$.

Proof. Clearly if P satisfies (9) for every $n \geq 1$ then

$$P = \lim_{n\to\infty} C_n P^n * \delta_{c_n}$$

so that P is operator stable.

Conversely, suppose that P is a full operator stable measure on $\mathcal{B}$. Then there exist $A_n \in G$, $x_n \in H$ and $Q \in \mathcal{P}$ such that

$$P = \lim_{n\to\infty} A_n Q^n * \delta_{x_n} . \tag{10}$$

Then, for every $m \geq 1$,

$$P^m = \lim_{n\to\infty} \left[A_n Q^n * \delta_{x_n}\right]^m$$

$$= \lim_{n\to\infty} A_n Q^{mn} * \delta_{mx_n} .$$

In view of (10),

$$P = \lim_{n\to\infty} A_{mn} Q^{mn} * \delta_{x_{mn}} .$$

Therefore

$$P^m = \lim_{n\to\infty} A_n Q^{mn} * \delta_{mx_n}$$

$$= \lim_{n\to\infty} A_n A_{mn}^{-1} A_{mn} Q^{mn} * \delta_{x_{mn}} * \delta_{mx_n - x_{mn}} .$$

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In view of condition (3), $\left\{A_n A_{mn}^{-1}\right\}$ is compact. Let $C_m \in G$ be a limit point of this sequence. Passing to a subsequence if necessary we obtain

$$P^m = C_m P * \delta_{a_m} ,$$

for some $a_m \in H$, and every $m \geq 1$. This completes the proof of Lemma 3.

LEMMA 4. *Let $P \in \mathcal{P}$ be full such that*

$$P = \lim_{n\to\infty} A_n Q^{k_n} * \delta_{x_n}$$

where $A_n \in G$, $x_n \in H$ and $k_{n+1}/k_n \to \infty$ as $n \to \infty$. Then P is operator stable and moreover, the relation (4) *holds.*

Proof. In view of Lemma 1, P is infinitely divisible so that its powers with any positive exponent exist and are infinitely divisible. Let $0 < \alpha < 1$ be arbitrary. Select, and fix, a sequence $l(n)$ of integers such that $k_{l(n)}/k_n \to \alpha$ as $n \to \infty$. Then, for $y \in H$,

$$\begin{aligned}
[\hat{P}(y)]^{\alpha} &= \lim_{n\to\infty} [\widehat{A_n Q}(y)]^{\alpha k_n} \cdot \exp(i\langle \alpha x_n, y\rangle) \\
&= \lim_{n\to\infty} \left[\widehat{A_n A_{l(n)}^{-1} A_{l(n)} Q}(y)\right]^{k_n (k_{l(n)}/k_n)} \cdot \exp(i\langle \alpha x_n, y\rangle) \\
&= \lim_{n\to\infty} \left\{\left[\widehat{A_n A_{l(n)}^{-1} A_{l(n)} Q}(y)\right]^{k_{l(n)}} \right. \\
&\qquad \left. \cdot \exp\left[i\left\langle A_n A_{l(n)}^{-1} x_{l(n)}, y\right\rangle\right] \cdot \exp\left[i\left\langle \alpha x_n - A_n A_{l(n)}^{-1} x_{l(n)}, y\right\rangle\right]\right\} \\
&= \lim_{n\to\infty} [\widehat{C_n A_{l(n)} Q}(y)]^{k_{l(n)}} \cdot \exp(i\langle c_n x_{l(n)}, y\rangle) \cdot \exp(i\langle a_n, y\rangle)
\end{aligned}$$

for some $a_n \in H$, where $C_n = A_n A_{l(n)}^{-1} \in G$. Write

$P_n = A_{l(n)} Q^{k_{l(n)}} * \delta_{x_{l(n)}}$ and note that $P_n \Rightarrow P$ where P is a full probability measure. Also, for $y \in H$,

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(11) $$[\hat{P}(y)]^{\alpha} = \lim_{n\to\infty} \{\widehat{C_n P_n}(y)\cdot\exp(i\langle a_n, y\rangle)\} .$$

Note that in view of (3) the sequence $\{C_n\}$ is compact. Let $C_{1/\alpha} \in G$ be a limit point of $\{C_n\}$ and suppose that $C_{n_j} \to C_{1/\alpha}$. Then $C_{n_j}P_{n_j} \Rightarrow CP$ and, moreover,

$$[\hat{P}(y)]^{\alpha} = \lim_{j\to\infty} \{\widehat{C_{n_j} P_{n_j}}(y)\cdot\exp(i\langle a_{n_j}, y\rangle)\} .$$

It follows that there exists an element $a_\alpha \in H$ such that as $n \to \infty$,

$$\langle a_{n_j}, y\rangle \to \langle a_\alpha, y\rangle , \quad y \in H .$$

Hence

$$\widehat{C_{n_j} P_{n_j}}(y)\cdot\exp(i\langle a_{n_j}, y\rangle) \to \widehat{C_{1/\alpha}P}(y)\cdot\exp(i\langle a_\alpha, y\rangle) , \quad y \in H ,$$

so that

$$[\hat{P}(y)]^{\alpha} = \widehat{C_{1/\alpha}P}(y)\cdot\exp(i\langle a_\alpha, y\rangle) .$$

By uniqueness it follows that the limiting measure is given by

(12) $$P^{\alpha} = C_{1/\alpha}P*\delta_{a_\alpha}$$

which is also full. Setting $\alpha = 1/n$ we obtain

$$P = C_n P^n * \delta_{c_n}$$

for some $c_n \in H$ and every $n \geq 1$. It follows at once from Lemma 3 that P is operator stable. Since $0 < \alpha < 1$ we see from (12) that (4) also holds. This completes the proof of Lemma 4 as well as that of Theorem 1.

We remark that in view of Lemma 4 the class of operator stable measures on H is a subclass of the class of operator semistable measures on H.

Added in proof, 30 October 1980. See the corrigenda, *Bull. Austral. Math. Soc.* 22 (1980), 479-480.

References

[1] R. Jajte, "Semi-stable probability measures on R^N ", *Studia Math.* 61 (1977), 29-39.

[2] V.M. Kruglov, "On an extension of the class of stable distributions", *Theory Probab. Appl.* 17 (1972), 685-694.

[3] R.G. Laha and V.K. Rohatgi, *Probability theory* (John Wiley & Sons, New York, Brisbane, Toronto, 1979).

[4] Michael Sharpe, "Operator-stable probability distributions on vector groups", *Trans. Amer. Math. Soc.* 136 (1969), 51-65.

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