A CRITERION FOR THE
HALL-CLOSURE OF FITTING CLASSES

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In a recent paper, Cusack has given a criterion, in terms of the Fitting class "join" operation, for a normal Fitting class to be closed under the taking of Hall $\pi$-subgroups. Here we show that Cusack's result can be slightly modified so as to give a criterion for any Fitting class of finite soluble groups to be closed under taking Hall $\pi$-subgroups.

1. Introduction

We will take our groups and classes of groups from the universe $\mathcal{S}$ of all finite soluble groups. Let $F$ be a Fitting class and $\pi$ be a set of primes, and let $\mathcal{S}_\pi$ denote the class of all (finite, soluble) $\pi$-groups. Then $F$ is said to be Hall $\pi$-closed if whenever $G$ belongs to $F$, then the Hall $\pi$-subgroups of $G$ also belong to $F$. If we define $Y(\mathcal{S}_\pi, F)$ to be the class of all those groups whose Hall $\pi$-subgroups belong to $F$, then it is clear that $F$ is Hall $\pi$-closed if and only if $F \subseteq Y(\mathcal{S}_\pi, F)$. It is not hard to see that $Y(\mathcal{S}_\pi, F)$ is itself a Fitting class. If $G$ is a further Fitting class, then the join, $F \vee G$, is the smallest Fitting class to contain both $F$ and $G$. In [6], Lockett associates with each Fitting class $F$ the "new" Fitting classes $F^*$ and $F^\#$, and shows that $S_4$ is the so-called smallest normal Fitting class introduced in [2]. Then the result of Cusack in which we are interested is the following.

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THEOREM [5, Theorem 5]. Let $\mathcal{F}$ be a normal Fitting class and $\pi$ be a set of primes. Then $\mathcal{F}$ is Hall $\pi$-closed if and only if

$$
\mathcal{F} = (\mathcal{S}_\pi \cap \mathcal{F}) \cup (Y(\mathcal{S}_\pi, \mathcal{S}_\pi) \cap \mathcal{F})
$$

2. Preliminaries

If $\mathcal{F}$ and $\mathcal{G}$ are Fitting classes, then $\mathcal{F}\mathcal{G}$ denotes the class

$$
\mathcal{F}\mathcal{G} = \{ X \in \mathcal{F} : X/\mathcal{F}_X \in \mathcal{G} \},
$$

where $\mathcal{F}_X$ denotes the $\mathcal{F}$-radical of $X$. It is well-known that $\mathcal{F}\mathcal{G}$ is again a Fitting class.

We refer to [6] for the definitions of the classes $\mathcal{F}^*$ and $\mathcal{F}_*$; the following result, which is due to Lockett, collects the properties we need of these classes.

THEOREM 2.1 [6]. Let $\mathcal{F}$ be a Fitting class and let $H \in \mathcal{F}$. Then

(a) $\mathcal{F}^*$ and $\mathcal{F}_*$ are Fitting classes with $\mathcal{F}_* \subseteq \mathcal{F} \subseteq \mathcal{F}^*$;

(b) $H' \leq H_{\mathcal{F}^*}$;

(c) $(H \times H)_{\mathcal{F}_*} = H_{\mathcal{F}_*} \times H_{\mathcal{F}_*} \langle (h^{-1}, h) : h \in H \rangle$; and

(d) if $\mathcal{G}$ is a further Fitting class then $(\mathcal{F} \cap \mathcal{G})^* = \mathcal{F}^* \cap \mathcal{G}^*$.

Recall that if $G$ and $H$ are groups and $N \leq G \times H$, then $N$ is said to be subdirect in $G \times H$ if $N(1 \times H) = G \times H = (G \times 1)N$. It is clear that any subgroup of $H \times H$ which contains $\langle (h^{-1}, h) : h \in H \rangle$ is subdirect in $H \times H$. We need the following result of Cusack.

THEOREM 2.2 [4, Corollary 2.6]. Let $\mathcal{U}$ and $\mathcal{V}$ be Fitting classes such that $\mathcal{U} \subseteq \mathcal{V}^*$. Then a group $G$ lies in $\mathcal{U} \vee \mathcal{V}$ if and only if there exists a group $H \in \mathcal{U}$ such that $(G \times H)_{\mathcal{V}_H}$ is subdirect in $G \times H$.

The following facts about $Y(\mathcal{S}_\pi, \mathcal{F})$ can be found in [3] (where $Y(\mathcal{S}_\pi, \mathcal{F})$ is called $K_\pi(\mathcal{F})$). Note that part (b) has also appeared in [1].

THEOREM 2.3. Let $\mathcal{F}$ be a Fitting class, $\pi$ be a set of primes and $G$ be a group. Then
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(a) \( G_y(S_n, F) \cap H = H_{F_y} \) for any Hall \( \pi \)-subgroup \( H \) of \( G \);

(b) \( y(S_n, F^a) = (y(S_n, F))^a \);

(c) \( y(S_n, F) = y(S_n, F)S_{n'} \); and

(d) if \( F \) is Hall \( \pi \)-closed, then so also are \( F^a \) and \( F^a \).

3. The theorem

We model our proof on Cusack's; in particular, the three results below correspond, in order, to Lemma 3, Theorem 4 and Theorem 5 of [5]. The main difference is that here we use Theorem 2.3.

**Lemma 3.1.** Let \( \pi \) be a set of primes and \( F \) be a Hall \( \pi \)-closed Fitting class. Suppose that \( G \in WS_n \) and that \( H \) is a Hall \( \pi \)-subgroup of \( G \). Then \( G_{W_{n}, \pi} \) is the \( Y(S_n, W) \)-radical of \( G \).

**Proof.** Let \( Y \) denote \( Y(S_n, W) \); then \( G_{W_{n}, \pi} \leq G_{\pi} \) since \( W \) is Hall \( \pi \)-closed. Now \( G = G_{W_{n}, \pi} \) by hypothesis, while \( G_{\pi} \cap H = H_{W_{n}, \pi} \) by Theorem 2.3 (a). Applying Dedekind's law, we find that \( G_{\pi} = G_{W_{n}, \pi}(H \cap G_{\pi}) = G_{W_{n}, \pi}H_{W_{n}, \pi} \), as claimed.

**Proposition 3.2.** Let \( \pi \) be a set of primes and \( F \) be a Hall \( \pi \)-closed Fitting class. Then \( F = (S_n \cap F) \vee (y(S_n, F^a) \cap F) \).

**Proof.** Let \( Y \) denote \( Y(S_n, W) \). It follows from Theorem 2.3 (c), (d) that \( F_{S_n, n} \subseteq \overline{Y_{S_n}} = Y \), and so

\[(3.3) \quad F_{S_n, n} \setminus F \subseteq Y \setminus F.\]

Now let \( G \in F_{S_n, n} \cap F \), and let \( H \) be a Hall \( \pi \)-subgroup of \( G \); then \( H \in F \). Form \( G \times H \in F \). Applying Lemma 3.1 with \( W = F^a \), and Theorem 2.1 (a) with \( H \in F \), we find that

\[
(G \times H)_{F^a} = (G \times H)_{F^a}(H \times H)_{F^a} \\
\geq (G_{F^a}H_{F^a} \times H_{F^a})(h^{-1}, h) : h \in H).
\]

But clearly \( G = G_{F^a}H \), and it follows that \( (G \times H)_{F^a} \) is subdirect in
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Since $G \times H \in \mathbb{F}$, this says that $(G \times H)_{Y \mathbb{F}}$ is subdirect in $G \times H$.

We now wish to apply Theorem 2.2 with $U = S_{\pi} \cap \mathbb{F}$ and $V = Y \cap \mathbb{F}$.

Note that by Theorem 2.1 (d) and Theorem 2.3 (b), (d), we have

$$Y^{*} = Y_{\mathbb{F}} = Y_{\mathbb{F}} \cap \mathbb{F}^{*} = \mathbb{F}^{*}.$$

Thus $U \subseteq V^{*}$, and Theorem 2.2 implies that

$$(3.4) \quad F_{\mathbb{F}} \cap \mathbb{F} \subseteq (S_{\pi} \cap \mathbb{F}) \cup (Y \cap \mathbb{F}).$$

But it follows from Theorem 2.1 (b) that

$$F = (F_{\mathbb{F}} \cap \mathbb{F}) \cup (F_{\mathbb{F}} \cap \mathbb{F}),$$

and so, combining (3.3) and (3.4), we conclude that

$$F = (S_{\pi} \cap \mathbb{F}) \cup (Y \cap \mathbb{F}),$$

as required.

THEOREM 3.5. Let $\pi$ be a set of primes and $\mathbb{F}$ be a Fitting class. Then $\mathbb{F}$ is Hall $\pi$-closed if and only if

$$\mathbb{F} = (S_{\pi} \cap \mathbb{F}) \cup (Y(S_{\pi}, \mathbb{F}) \cap \mathbb{F}).$$

Proof. The "only if" assertion has been proved above. Thus suppose that $\mathbb{F} = (S_{\pi} \cap \mathbb{F}) \cup (Y(S_{\pi}, \mathbb{F}) \cap \mathbb{F})$. Since $S_{\pi} \cap \mathbb{F} \subseteq Y(S_{\pi}, \mathbb{F})$, and since the operator $Y(S_{\pi}, \mathbb{F})$ clearly respects inclusions, then Theorem 2.1 (a) implies that

$$\mathbb{F} \subseteq Y(S_{\pi}, \mathbb{F}) \cup (Y(S_{\pi}, \mathbb{F}) \cap \mathbb{F}) = Y(S_{\pi}, \mathbb{F}).$$

Thus $\mathbb{F}$ is Hall $\pi$-closed, and the proof is complete.

References


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