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ON BAR LENGTHS IN PARTITIONS

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Abstract We present, given an odd integer d, a decomposition of the multiset of bar lengths of a bar partition λ as the union of two multisets, one consisting of the bar lengths in its $\bar{d}$-core partition $\bar{c}_d(\lambda)$ and the other consisting of modified bar lengths in its $\bar{d}$-quotient partition. In particular, we obtain that the multiset of bar lengths in $\bar{c}_d(\lambda)$ is a sub-multiset of the multiset of bar lengths in λ. Also, we obtain a relative bar formula for the degrees of spin characters of the Schur extensions of $\mathfrak{S}_n$. The proof involves a recent similar result for partitions, proved by Bessenrodt and the authors.

Keywords: partitions; bar partitions; bar lengths; symmetric group; covering groups

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Secondary 20C15; 20C20

1. Introduction

For any positive integer n, we call any partition λ of n into distinct parts a *bar partition* of n. It was proved by Schur [**7**] that the bar partitions of n canonically label the associate classes of irreducible projective representations of the symmetric group $\mathfrak{S}_n$, or the associate classes of faithful irreducible characters (*spin characters*) of a two-fold covering group $\hat{\mathfrak{S}}_n$ of $\mathfrak{S}_n$.

In [**4**, Theorem 1], Morris proved a formula (the *bar formula*) for the degrees of the spin characters analogous to the celebrated Hook Formula [**2**, Theorem 2.3.21] for the irreducible characters of $\mathfrak{S}_n$. The bar formula is a reformulation of the original degree formula proved by Schur in [**7**, § IX, p. 235]. We state the bar formula below. In the bar formula the role played by hooks and hook lengths of partitions is replaced by *bars* and *bar lengths* of bar partitions.

If $\lambda = (a_1 > \cdots > a_m > 0)$ is a bar partition of n, then the multiset of bar lengths in λ is

$$\mathcal{B}(\lambda) = \bigcup_{1 \leqslant i \leqslant m} \{1, \ldots, a_i\} \cup \{a_i + a_j \mid j > i\} \setminus \{a_i - a_j \mid j > i\}.$$

Writing $\pi\mathcal{B}(\lambda)$ for the product of all the bar lengths in λ, we then have the bar formula for the degree of a spin character ρ_λ of $\hat{\mathfrak{S}}_n$ labelled by λ:

$$\rho_\lambda(1) = 2^{\lfloor (n-m)/2 \rfloor} \frac{n!}{\pi\mathcal{B}(\lambda)},$$

where, for any rational number x, $\lfloor x \rfloor$ denotes the integral part of x.

For any odd integer $d \geqslant 3$, it is well known that the bar partition λ is uniquely determined by its *$\bar{d}$-core* $\bar{c}_d(\lambda)$ and its *$\bar{d}$-quotient* $\lambda^{(\bar{d})}$ (see, for example, [**6**, Proposition 4.2]). The $\bar{d}$-core partition $\bar{c}_d(\lambda)$ of λ is obtained by removing from λ all the bars of length divisible by d, while $\lambda^{(\bar{d})}$ encodes the information about these bars.

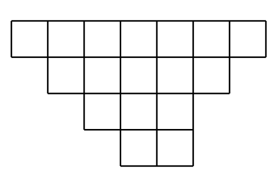
For any bar partition λ of n and odd integer $d \geqslant 3$, we denote by $\bar{q}_d(\lambda)$ the (unique) bar partition which has an *empty* $\bar{d}$-core and the same $\bar{d}$-quotient as λ. We refer to $\bar{q}_d(\lambda)$ as the *$\bar{d}$-quotient partition* of λ and have that $|\lambda| = |\bar{c}_d(\lambda)| + |\bar{q}_d(\lambda)|$ (see [**6**, Corollary 4.4]). This identity is reflected in our main result on the decomposition of the multiset of bar lengths (Theorem 4.1). It states that the multiset $\mathcal{B}(\lambda)$ of bar lengths in λ is the union of $\mathcal{B}(\bar{c}_d(\lambda))$ and $\tilde{\mathcal{B}}(\bar{q}_d(\lambda))$, where the multiset $\tilde{\mathcal{B}}(\bar{q}_d(\lambda))$ is obtained from $\mathcal{B}(\bar{q}_d(\lambda))$ by modifying its elements in an explicitly controlled way, depending on the $\bar{d}$-core of λ. As an immediate corollary we obtain that $\mathcal{B}(\bar{c}_d(\lambda))$ is contained in $\mathcal{B}(\lambda)$.

In § 2, we describe the *doubling* of bar partitions; this construction was first suggested by Macdonald [**3**], and then studied by Morris and Yaseen [**5**]. It allows us to see all the bar lengths in a bar partition as hook lengths in a larger partition. We present the construction, as well as interpretations of the bar core and bar quotient in this setting. In § 3, we introduce a number of subsets of the set of hooks in the doubled partition, and derive from Macdonald's construction a number of properties of hook lengths and bar lengths. In § 4, we then apply the results of [**1**] to deduce our main result, Theorem 4.1. We then finally apply the theorem to give a d-version of the bar formula (a relative bar formula).

2. The Macdonald construction

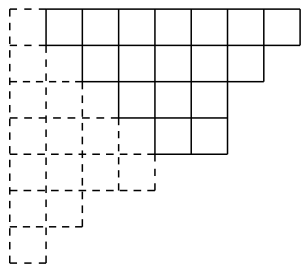
Let $n \geqslant 1$ be any integer, and let $\lambda = (a_1 > \cdots > a_m > 0)$ be a bar partition of n. In [**3**, Chapter III, p. 135], Macdonald presented a construction for the *doubling* of λ, which we present here using the example given by the bar partition $\lambda = (7, 5, 3, 2)$ of $n = 17$.

The *shifted Young diagram* $S(\lambda)$ of λ is obtained from the usual Young diagram of λ by moving, for each $i \geqslant 1$, the ith row $(i-1)$ squares to the right. In our example, this gives



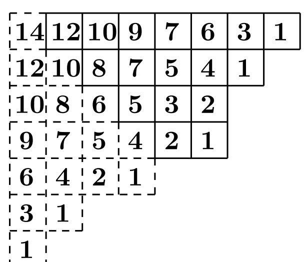
Equivalently, $S(\lambda)$ can be seen as the part above the diagonal in the Young diagram of the *doubled partition* $D(\lambda) = (a_1, \ldots a_m \mid a_1 - 1, \ldots, a_m - 1)$ of $2n$ (given in the

Frobenius notation; see, for example, [**3**, Chapter I]). In our example, we obtain the partition $D(\lambda) = (8, 7, 6, 6, 4, 2, 1)$ of $2n = 34$, which has the Young diagram



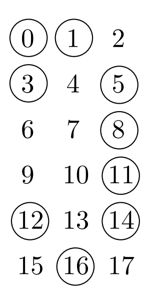
By filling the boxes of the Young diagram of $D(\lambda)$ with the corresponding hook lengths, we obtain that the bar lengths in λ are those hook lengths that appear in the subdiagram $S(\lambda)$. In our example, we get

$$\mathcal{B}(\lambda) = \{12, 10, 9, 8, 7, 7, 6, 5, 5, 4, 3, 3, 2, 2, 1, 1, 1\}:$$



We refer the reader to [**2**, § 2.7] or [**6**, § 1] for the basic facts about β sets for partitions and their relation to hooks. In particular, if X is a β set for the partition λ, then there is a canonical correspondence between the hooks z in λ and pairs (a, b) of non-negative integers, where $a \in X$, $b \notin X$ and $a > b$. The length $h(z)$ of the hook z is then $a - b$.

Now take any odd integer $d \geqslant 3$. We represent a *d-normalized* β set X for $D(\lambda)$ (i.e. $|X|$ is a multiple of d) by placing beads on an abacus with d runners. If $d = 3$, then in our example we can take $X = \{0, 1, 3, 5, 8, 11, 12, 14, 16\}$, and we obtain



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For any integer ℓ, we denote by $[\ell]_d$ the *d-residue* of ℓ, i.e. the least non-negative integer congruent to $\ell \pmod d$. We label each node in the Young diagram of $D(\lambda)$ by a d-residue as follows: the d-residue labelling the (i,j)-node is $[j-i]_d$. In particular, note that the *diagonal nodes* (which correspond to hooks whose lengths are twice the size of the parts of λ) all have residue 0. Writing $H(D(\lambda))$ for the set of hooks in $D(\lambda)$, we define, for each $0 \leqslant i,j \leqslant d-1$, the subset $H_{i\to j}(D(\lambda))$ of hooks of $D(\lambda)$ whose hand node and foot node have d-residues i and $[j+1]_d$, respectively. For any hook $z \in H(D(\lambda))$, we have that $z \in H_{i\to j}(D(\lambda))$ if and only if, in the abacus, z corresponds to a bead on the ith runner and an empty spot on the jth runner. In this case, the length $h(z)$ of z satisfies $h(z) \equiv i - j \pmod d$ (see [**2**, § 2.7] for details). For each $0 \leqslant i \neq j \leqslant d-1$, we write $H_{\{i\}}(D(\lambda))$ for $H_{i\to i}(D(\lambda))$ and $H_{\{ij\}}(D(\lambda))$ for $H_{i\to j}(D(\lambda)) \cup H_{j\to i}(D(\lambda))$. In particular,

$$\bigcup_{0\leqslant i\leqslant d-1} H_{\{i\}}(D(\lambda))$$

is the set of hooks of length divisible by d in $D(\lambda)$.

Each runner in the abacus for $D(\lambda)$ can be seen as representing a β set: the ith runner contains beads representing the elements of $\{x \in X \mid [x]_d = i\}$; the corresponding β set is $X_i = \{k \in \mathbb{Z} \mid kd+i \in X\}$. The d-quotient $D(\lambda)^{(d)}$ of $D(\lambda)$ is the d-tuple $(X_0,\dots,X_{d-1})$ of these β sets. The fact that we took a normalized β set ensures that the d-quotient we obtain is the same as the one we would obtain by considering the *star d-diagram* of $D(\lambda)$ [**2**, Theorem 2.7.37]. We may then reformulate [**5**, Theorem 4] as follows.

Theorem 2.1. *With the above notation, the d-quotient $D(\lambda)^{(d)}$ of $D(\lambda)$ has the form*

$$D(\lambda)^{(d)} = (D(\mu_0), \mu_1, \dots, \mu_{(d-1)/2}, \mu^*_{(d-1)/2}, \dots, \mu^*_1),$$

where μ_0 is a bar partition, $\mu_1,\dots,\mu_{(d-1)/2}$ are partitions and * denotes conjugation of partitions.

Furthermore, the $\bar{d}$-quotient of λ is $\lambda^{(\bar{d})} = (\mu_0, \mu_1, \dots, \mu_{(d-1)/2})$.

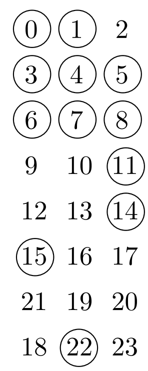
In our example, we find

$$D(\lambda)^{(3)} = ((2),(4),(1,1,1,1)) = (D((1)),(4),(4)^*) \quad \text{and} \quad \lambda^{(\bar{d})} = \lambda^{(\bar{3})} = ((1),(4)).$$

Removing all the hooks of length divisible by d in $D(\lambda)$ (or, equivalently, moving all the beads in the abacus of $D(\lambda)$ as far up as possible on their respective runners), we obtain the d-core $D(\lambda)_{(d)}$ of $D(\lambda)$. Then we see (cf. [**5**, p. 26]) that $D(\lambda)_{(d)} = D(\bar{c}_d(\lambda))$, where $\bar{c}_d(\lambda)$ is the $\bar{d}$-core of λ (which may also be obtained from λ by removing all the bars of length divisible by d; the removal of such a bar corresponds to removing a pair of d-hooks from $D(\lambda)$, one whose node is in $S(\lambda)$, and its counterpart in the lower half of the diagram). In our example, we find $D(\lambda)_{(3)} = (3,1) = D((2)) = D(\bar{c}_d(\lambda))$.

We define the *$\bar{d}$-quotient partition* of λ to be the (uniquely defined) bar partition $\bar{q}_d(\lambda)$ which has empty $\bar{d}$-core, and $\bar{d}$-quotient $\bar{q}_d(\lambda)^{(\bar{d})} = \lambda^{(\bar{d})}$. The doubled partition $D(\bar{q}_d(\lambda))$ therefore has empty d-core, and d-quotient $D(\bar{q}_d(\lambda))^{(d)} = D(\lambda)^{(d)}$. This proves that $D(\bar{q}_d(\lambda))$ is the *d-quotient partition* of $D(\lambda)$, which we write as $D(\bar{q}_d(\lambda)) = q_d(D(\lambda))$.

In the d-abacus of $D(\bar{q}_d(\lambda))$, the partition associated to each runner is the same as for $D(\lambda)$, but the corresponding β sets all have the same number of elements (which is the number of beads on the runners, and this is the same for each runner, since $D(\bar{q}_d(\lambda))$ has empty d-core). In our example, we can take this number to be 4, and we obtain

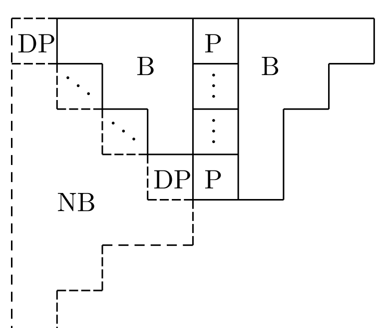


We therefore have $D(\bar{q}_3(\lambda)) = q_3(D(\lambda)) = (11, 5^2, 3, 1^6)$, and $\bar{q}_3(\lambda) = (10, 3, 2)$.

It is easy to see, using [**6**, Theorem 4.3], that since λ and $\bar{q}_d(\lambda)$ have the same $\bar{d}$-quotient, there is a length-preserving bijection between the sets of bars of length divisible by d in λ and $\bar{q}_d(\lambda)$, respectively. In our example, both multisets of lengths are $\{12, 9, 6, 3, 3\}$.

3. Multisets of bar lengths

We keep the notation as in § 2, and we take any bar partition μ (which we want to specialize to $\mu \in \{\lambda, \bar{c}_d(\lambda), \bar{q}_d(\lambda)\}$). We define several subsets of the set $H(D(\mu))$ of hooks and the multiset $\mathcal{H}(D(\mu))$ of hook lengths in $D(\mu)$. This is illustrated in the following diagram:



The subsets we will consider are defined as follows:

(i) $P(D(\mu))$ is the set of hooks corresponding to the parts of μ (denoted by 'P' above), and $\mathcal{P}(D(\mu))$ is the set of their lengths.

(ii) $DP(D(\mu))$ is the set of hooks corresponding to the doubled parts of μ (denoted by 'DP' above), and $\mathcal{DP}(D(\mu))$ is the set of their lengths.

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(iii) $B(D(\mu))$ is the set of hooks corresponding to bars in μ which are not parts (denoted by 'B' above), and $\mathcal{B}(D(\mu))$ is the multiset of their lengths.

(iv) $NB(D(\mu))$ is the set of hooks corresponding to 'non-bars' (denoted by 'NB' above), i.e. the counterparts in the lower half of the Young diagram of $B(D(\mu))$, and $\mathcal{NB}(D(\mu))$ is the multiset of their lengths. In particular, by construction, we have $\mathcal{NB}(D(\mu)) = \mathcal{B}(D(\mu))$.

We thus have the set equality

$$H(D(\mu)) = P(D(\mu)) \cup DP(D(\mu)) \cup B(D(\mu)) \cup NB(D(\mu))$$

and the multiset equality

$$\mathcal{H}(D(\mu)) = \mathcal{P}(D(\mu)) \cup \mathcal{DP}(D(\mu)) \cup \mathcal{B}(D(\mu)) \cup \mathcal{NB}(D(\mu)).$$

Note that $B(D(\mu))$ is the set of unmixed bars of type 1 and mixed bars (of type 3) in μ, while $P(D(\mu))$ is the set of unmixed bars of type 2 in μ (see, for example, [**6**, § 4]). In particular, we have, for the multiset of bar lengths in μ, that $\mathcal{B}(\mu) = \mathcal{P}(D(\mu)) \cup \mathcal{B}(D(\mu))$.

For any $0 \leqslant i, j \leqslant d-1$, we have defined in § 2 subsets $H_{i\to j}(D(\mu))$, $H_{\{i\}}(D(\mu))$ and $H_{\{ij\}}(D(\mu))$ (if $i \neq j$) of $H(D(\mu))$. Similarly, we define subsets

$$\begin{array}{llll} P_{i\to j}(D(\mu)), & P_{\{ij\}}(D(\mu)), & DP_{i\to j}(D(\mu)), & DP_{\{ij\}}(D(\mu)), \\ B_{i\to j}(D(\mu)), & B_{\{ij\}}(D(\mu)), & NB_{i\to j}(D(\mu)), & NB_{\{ij\}}(D(\mu)). \end{array}$$

As before, for any hook z in a partition, we let $h(z)$ denote its length. For any $z \in B(D(\mu))$, we denote by z^* the counterpart of z in $NB(D(\mu))$ (so that $h(z) = h(z^*)$). For any subset K of $B(D(\mu))$, we write K^* for the subset $\{z^*, z \in K\}$ of $NB(D(\mu))$. In particular, we have $NB(D(\mu)) = B(D(\mu))^*$. For any $1 \leqslant i \leqslant d-1$, we let $i^* = d - i$ (so that, in the d-quotient $(D(\mu_0), \mu_1, \dots, \mu_{d-1})$ of $D(\mu)$, we have, for each $1 \leqslant i \leqslant d-1$, $\mu_{i^*} = \mu_i^*$). We also let $0^* = 0$.

Lemma 3.1. *For any $z \in B(D(\mu))$, if $z \in B_{i\to j}(D(\mu))$ (for some $0 \leqslant i, j \leqslant d-1$), then $z^* \in NB_{j^*\to i^*}(D(\mu))$.*

Proof. Suppose $z^* \in NB_{k\to \ell}(D(\mu))$. By definition, z has hand residue i and foot residue $[j+1]_d$, while z^* has hand residue k and foot residue $[\ell+1]_d$. In particular, considering the lengths $h(z)$ and $h(z^*)$ of z and z^*, we have $h(z) \equiv i - j \pmod d$ and $h(z^*) \equiv k - \ell \pmod d$. But, since $h(z) = h(z^*)$, we have $i - j \equiv k - \ell \pmod d$.

Now, since $z \in B(D(\mu))$, the arm of z is in a row which corresponds to a part of μ, say the rth row of the Young diagram of $D(\mu)$. Then, by construction of $D(\mu)$, the counterpart z^* of z has its leg in the rth column of the Young diagram of $D(\mu)$. Also, by construction, this column is one node shorter than the rth row. It is then easy to see that if the rth row has end residue m, then the rth column has end residue $[d-(m-1)]_d = [(d-m)+1]_d = [m^*+1]_d$ (indeed, we know that the residues increase from left to right in a row, while they decrease from top to bottom in a column, and that the rth row and rth column intersect on a diagonal node of residue 0). Therefore, the foot residue of z^* is $[i^*+1]_d$, so that we have $\ell = i^*$. But then, from $h(z) = h(z^*)$, we obtain $i - j \equiv k - i^* \pmod d$, whence, since $i^* = d - i$, $k \equiv -j \pmod d$, and thus $k = j^*$. □

Corollary 3.2. *For any $0 \leqslant i, j \leqslant d-1$, we have $NB_{j^* \to i^*}(D(\mu)) = B_{i \to j}(D(\mu))^*$ and $\mathcal{NB}_{j^* \to i^*}(D(\mu)) = \mathcal{B}_{i \to j}(D(\mu))$.*

We now prove a symmetry property on the number of beads in the abacus of $D(\mu)$. We suppose that the d-abacus of $D(\mu)$ is *minimally normalized*, i.e. that the β set for $D(\mu)$ used to build the abacus has a multiple of d elements, and is minimal with respect to this property. For each $0 \leqslant i \leqslant d-1$, we write x_i for the number of beads on the ith runner of the (minimally normalized) d-abacus of $D(\mu)$.

Lemma 3.3. *For any $0 \leqslant i, j \leqslant d-1$, we have $x_i + x_{i^*} = x_j + x_{j^*}$.*

In the example of § 2, we have $x_0 = 3$, $x_1 = 2$ and $x_2 = 4$, whence $x_0 + x_{0^*} = x_1 + x_{1^*} = x_2 + x_{2^*} = 6$.

Proof. Let a be the largest part of μ and $[a]_d$ be its d-residue. Consider the *rim* R of the Young diagram of $D(\mu)$. Then R is composed of $a+1$ horizontal segments (of length 1) and a vertical segments.

We extend the rim horizontally to the top right and vertically to the bottom left as follows. Suppose $(k-1)d < a \leqslant kd$ for some $k \in \mathbb{N}$. If $a+1 \leqslant kd$ (i.e. $[a]_d \neq 0$), then extend R to an $\tilde{R}$ that has kd horizontal segments and kd vertical ones. If $a+1 > kd$ (i.e. $a = kd$), then extend R to an $\tilde{R}$ that has $(k+1)d$ horizontal segments and $(k+1)d$ vertical ones. In particular, $\tilde{R}$ always has ℓd vertical segments and ℓd horizontal ones, with $\ell d > a$. This implies that, while the horizontal extension of R may be empty (if $a+1 = kd$), the vertical one never is. In fact, the horizontal extension is always exactly one segment shorter than the vertical one, which is $\ell d - a$ segments long ($\ell d - a \neq 0$ and $\ell d - a - 1 = 0 \iff a + 1 = kd$).

Label the vertical segment at the end of the first row by its d-residue $[a]_d$. Complete the labelling of each segment of $\tilde{R}$ by residues modulo d by increasing by 1 for each step to the top or right, and decreasing by 1 for each step to the bottom or left. In particular, a row has end residue j if and only if the corresponding vertical segment of $\tilde{R}$ is labelled by j, while a column has end residue j if and only if the corresponding horizontal segment of $\tilde{R}$ is labelled by $[j-1]_d$.

For each $0 \leqslant j \leqslant d-1$, let V_j (respectively, H_j) be the set of vertical (respectively, horizontal) segments of $\tilde{R}$ labelled by j. We thus have

$$|V_j| = \begin{cases} x_j & \text{if } (k-1)d < a < kd, \\ x_j + 1 & \text{if } a = kd, \end{cases} \qquad 0 \leqslant j \leqslant d-1. \tag{3.1}$$

By construction of $D(\mu)$ (see also the proof of Lemma 3.1), we see that the horizontal segment at the bottom of the first column is labelled by $[a]_d^*$. Since the vertical extension of R has $\ell d - a$ segments, the bottom one is labelled by $[[a]_d^* - (\ell d - a)]_d = 0$. And,

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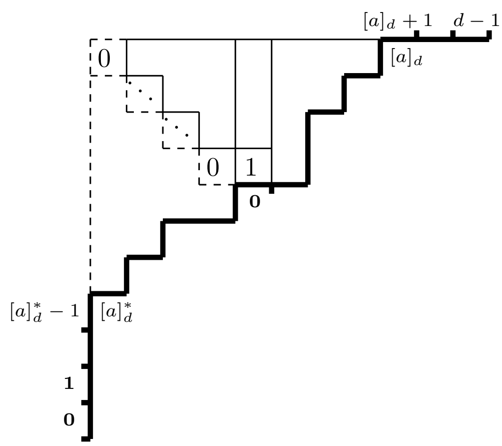
since the horizontal extension has $\ell d - a - 1$ segments, the last one (if it exists, i.e. if $\ell d - a - 1 \neq 0$) is labelled by $[a + \ell d - a - 1]_d = d - 1$.

Following $\tilde{R}$ from bottom left to top right, the labelling residues increase by one at each step, going (by the above) from 0 to $d-1$, and do so 2ℓ times (since $\tilde{R}$ has $2\ell d$ segments). This shows that, for each $0 \leqslant j \leqslant d-1$, the total number of segments of $\tilde{R}$ labelled by j is 2ℓ, i.e.

$$|V_j| + |H_j| = 2\ell, \quad 0 \leqslant j \leqslant d-1. \tag{3.2}$$

The nodes on the diagonal all have residue 0. Thus, the column of nodes which corresponds to the parts of μ has end residue 1 (since it is immediately to the right of the diagonal). Hence, the corresponding horizontal segment is labelled by 0. We have the following picture:



Now the portions of $\tilde{R}$ to the right and to the left of this 0 (except the bottom left segment) are symmetric (by construction of $D(\mu)$, and by the above considerations on the horizontal and vertical extensions). The vertical segments of one are in bijection with the horizontal segments of the other, and any label j is sent to a label j^*. Adding the (horizontal) 0 in the middle and the (vertical) 0 at the bottom, this proves that $|V_j| = |H_{j^*}|$ for each $0 \leqslant j \leqslant d-1$. Together with (3.2), this yields

$$|V_j| + |V_{j^*}| = |V_j| + |H_j| = 2\ell, \quad 0 \leqslant j \leqslant d-1.$$

Using (3.1), this implies the result. □

Remark 3.4. In the course of the above proof, we have established the following fact which will be useful later: if a node z in the Young diagram of $D(\mu)$ corresponds to a part of μ of length $h(z) \equiv i \pmod d$ for some $0 \leqslant i \leqslant d-1$, then z has hand residue i and foot residue 1 (i.e. $z \in P_{i \to 0}(D(\mu))$), and the diagonal node $z^{\times 2}$ corresponding to the associated doubled part has hand residue i and foot residue $[i^* + 1]_d$ (i.e. $z^{\times 2} \in DP_{i \to i^*}(D(\mu))$).

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Lemma 3.3 has several important consequences in our context. Let λ be any bar partition and let $\bar{q}_d(\lambda)$ be its $\bar{d}$-quotient partition. Following [**1**, Theorem 4.7], we have the following definition.

Definition 3.5. For each hook $z \in H(D(\bar{q}_d(\lambda)))$, the *modified hook length* $\bar{h}(z)$ of z is given by $\bar{h}(z) = h(z) + (x_i - x_j)d$ if z has hand residue i and foot residue $[j+1]_d$.

We then write $\bar{\mathcal{H}}(D(\bar{q}_d(\lambda))) = \{|\bar{h}(z)| \mid z \in H(D(\bar{q}_d(\lambda)))\}$ (note that the same multiset is denoted by $\mathrm{abs}(\bar{\mathcal{H}}(D(\bar{q}_d(\lambda))))$ in [**1**]). Subsets of modified hook lengths are defined similarly for the subsets of hooks we introduced earlier.

Corollary 3.6. *For any bar partition λ and any $z \in B(D(\bar{q}_d(\lambda)))$, we have $\bar{h}(z) = \bar{h}(z^*)$. In particular, for any $0 \leqslant i, j \leqslant d-1$, we have*

$$\bar{\mathcal{B}}_{i\to j}(D(\bar{q}_d(\lambda))) = \overline{\mathcal{NB}}_{j^*\to i^*}(D(\bar{q}_d(\lambda))).$$

Proof. Suppose $z \in B_{i\to j}(D(\bar{q}_d(\lambda)))$ (for some $0 \leqslant i, j \leqslant d-1$). Then, by Lemma 3.1, $z^* \in NB_{j^*\to i^*}(D(\bar{q}_d(\lambda)))$. We thus have

$$\bar{h}(z) = h(z) + (x_i - x_j)d \quad \text{and} \quad \bar{h}(z^*) = h(z^*) + (x_{j^*} - x_{i^*})d.$$

But $h(z) = h(z^*)$ and, by Lemma 3.3, $x_i - x_j = x_{j^*} - x_{i^*}$. Hence, $\bar{h}(z) = \bar{h}(z^*)$. Corollary 3.2 concludes the proof. □

Corollary 3.7. *For any bar partition λ and any $z \in P(D(\bar{q}_d(\lambda)))$, we let $z^{\times 2}$ be the corresponding element in $DP(D(\bar{q}_d(\lambda)))$, satisfying $h(z^{\times 2}) = 2h(z)$. We then have $\bar{h}(z^{\times 2}) = 2\bar{h}(z)$.*

Proof. Take any $z \in P(D(\bar{q}_d(\lambda)))$ and suppose $h(z) \equiv i \pmod d$ for some $0 \leqslant i \leqslant d-1$. Then, by Remark 3.4, we have $z \in P_{i\to 0}(D(\bar{q}_d(\lambda)))$ and $z^{\times 2} \in DP_{i\to i^*}(D(\bar{q}_d(\lambda)))$. By Definition 3.5, we therefore get that $\bar{h}(z) = h(z) + d(x_i - x_0)$ and $\bar{h}(z^{\times 2}) = 2h(z) + d(x_i - x_{i^*})$. Now, by Lemma 3.3, we see that $x_i - x_{i^*} = 2(x_i - x_0)$, so that $\bar{h}(z^{\times 2}) = 2h(z) + d(x_i - x_{i^*}) = 2h(z) + 2d(x_i - x_0) = 2\bar{h}(z)$. □

Corollary 3.8. *For any bar partition λ, the elements of $\bar{\mathcal{P}}(D(\bar{q}_d(\lambda)))$ are distinct, and the elements of $\overline{\mathcal{DP}}(D(\bar{q}_d(\lambda)))$ are distinct.*

Proof. Suppose $z \in P_{i\to 0}(D(\bar{q}_d(\lambda)))$, $z' \in P_{j\to 0}(D(\bar{q}_d(\lambda)))$ and $|\bar{h}(z)| = |\bar{h}(z')|$. We want to show that $h(z) = h(z')$, which then implies $z = z'$. We see from Definition 3.5 that $|\bar{h}(z)| \equiv \pm i \pmod d$ and $|\bar{h}(z')| \equiv \pm j \pmod d$. Thus, we must have $j = i$ or $j = i^*$. If $i = 0$, then $j = 0$, so that $\bar{h}(z) = h(z)$ and $\bar{h}(z') = h(z')$ and we are done. If we have $j = i \neq 0$, then if $\bar{h}(z) = \bar{h}(z')$ we obviously have that $h(z) = h(z')$. If $\bar{h}(z) = -\bar{h}(z')$, we get $i \equiv -i \pmod d$, which is impossible. Consider the case in which $j = i^*$, $h(z) \equiv i \pmod d$ and $h(z') \equiv i^* \pmod d$. Then $\bar{h}(z) = -\bar{h}(z')$, i.e. $h(z) + d(x_i - x_0) = -h(z') - d(x_{i^*} - x_0)$. Thus, $h(z) + h(z') = d(x_0 - x_i) + d(x_0 - x_{i^*})$. But the right-hand side of this is 0, by Lemma 3.3, which is impossible.

We have now shown that the elements of $\bar{\mathcal{P}}(D(\bar{q}_d(\lambda)))$ are distinct. Corollary 3.7 shows that the elements of $\overline{\mathcal{DP}}(D(\bar{q}_d(\lambda)))$ are distinct. □

4. Main result

We are now in a position to prove our main result. Recall that, if λ is a bar partition, then the multiset of bar lengths in λ is $\mathcal{B}(\lambda) = \mathcal{B}(D(\lambda)) \cup \mathcal{P}(D(\lambda))$. We will also write $\mathcal{P}(\lambda)$ for the set $\mathcal{P}(D(\lambda))$ of parts of λ. Also $\bar{\mathcal{B}}(\bar{q}_d(\lambda))$ is the multiset of absolute values of the modified hook lengths $\bar{h}(z)$, $z \in B(\bar{q}_d(\lambda))$. Finally, for any multiset A, we denote by $A^{\times 2}$ the multiset given by $A^{\times 2} = \{2a \mid a \in A\}$.

Theorem 4.1. *Take any bar partition λ and any odd integer $d \geqslant 3$, and let $T(\lambda) = \mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))$. Then*

$$\mathcal{B}(\lambda) = \mathcal{B}(\bar{c}_d(\lambda)) \cup \tilde{\mathcal{B}}(\bar{q}_d(\lambda)),$$

where $\tilde{\mathcal{B}}(\bar{q}_d(\lambda)) = [\bar{\mathcal{B}}(\bar{q}_d(\lambda)) \setminus T(\lambda)] \cup T(\lambda)^{\times 2}$.

Proof. We write

$$B(\lambda) = \bigcup_{0 \leqslant i \leqslant d-1} B_{\{i\}}(\lambda) \cup \bigcup_{0 \leqslant i < j \leqslant d-1} B_{\{ij\}}(\lambda),$$

where we have

$$B_{\{i\}}(\lambda) = B_{i \to i}(\lambda) = B_{i \to i}(D(\lambda)) \cup P_{i \to i}(D(\lambda))$$

and

$$B_{\{ij\}}(\lambda) = B_{i \to j}(\lambda) \cup B_{j \to i}(\lambda) = B_{\{ij\}}(D(\lambda)) \cup P_{\{ij\}}(D(\lambda)).$$

Now, $\bigcup_{0 \leqslant i \leqslant d-1} B_{\{i\}}(\lambda)$ is exactly the set of bars of length divisible by d in λ. By construction (see § 2), we thus have

$$\bigcup_{0 \leqslant i \leqslant d-1} \mathcal{B}_{\{i\}}(\lambda) = \bigcup_{0 \leqslant i \leqslant d-1} \mathcal{B}_{\{i\}}(\bar{q}_d(\lambda)) = \bigcup_{0 \leqslant i \leqslant d-1} \bar{\mathcal{B}}_{\{i\}}(\bar{q}_d(\lambda))$$

and

$$\bigcup_{0 \leqslant i \leqslant d-1} \mathcal{B}_{\{i\}}(\bar{c}_d(\lambda)) = \emptyset.$$

We also want to examine separately the case of parts of length divisible by d in λ. These are the bars of type 2 (see [**6**, § 4]) of length divisible by d in λ, and, if we write

$$\lambda^{(\bar{d})} = \bar{q}_d(\lambda)^{(\bar{d})} = (\mu_0, \mu_1, \dots, \mu_{(d-1)/2}),$$

then they correspond bijectively to the parts of μ_0 (and are d times as long). Since $\bar{q}_d(\lambda)$ has the same $\bar{d}$-quotient as λ, we see that it has the same parts of length divisible by d.

To prove our result, it is now sufficient to consider the bars whose length is not divisible by d, i.e. which correspond to a bead and an empty spot on distinct runners i and j in the abacus. We distinguish between three cases, corresponding to the three possible cardinalities of the set $\{i, j, i^*, j^*\}$.

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In each case, we will apply [**1**, Theorem 4.7], which, in our context, may be formulated as

$$\mathcal{H}(D(\lambda)) = \mathcal{H}(D(\bar{c}_d(\lambda))) \cup \bar{\mathcal{H}}(D(\bar{q}_d(\lambda))).$$

Note that [**1**, Theorem 4.7] can be refined to pairs of runners using the proof of [**1**, Theorem 3.2].

Case 1. Take any $0 \leqslant i < j \leqslant d-1$ such that $|\{i, j, i^*, j^*\}| = 4$. In particular, $j \neq i^*$, so that $DP_{\{ij\}}(D(\lambda)) = \emptyset = DP_{\{i^*j^*\}}(D(\lambda))$, and $0 \notin \{i, j, i^*, j^*\}$, so that $P_{\{ij\}}(D(\lambda)) = \emptyset = P_{\{i^*j^*\}}(D(\lambda))$ (and similarly for $D(\bar{c}_d(\lambda))$ and $D(\bar{q}_d(\lambda))$).

By [**1**, Theorem 4.7], we have

$$\mathcal{H}_{\{ij\}}(D(\lambda)) = \mathcal{H}_{\{ij\}}(D(\bar{c}_d(\lambda))) \cup \bar{\mathcal{H}}_{\{ij\}}(D(\bar{q}_d(\lambda))).$$

Now $\mathcal{H}_{\{ij\}}(D(\lambda)) = \mathcal{H}_{i\to j}(D(\lambda)) \cup \mathcal{H}_{j\to i}(D(\lambda))$, and we have

$$\mathcal{H}_{i\to j}(D(\lambda)) = \mathcal{B}_{i\to j}(D(\lambda)) \cup \mathcal{NB}_{i\to j}(D(\lambda))$$

and

$$\mathcal{H}_{j\to i}(D(\lambda)) = \mathcal{B}_{j\to i}(D(\lambda)) \cup \mathcal{NB}_{j\to i}(D(\lambda)).$$

Also, by Corollary 3.2,

$$\mathcal{NB}_{i\to j}(D(\lambda)) = \mathcal{B}_{j^*\to i^*}(D(\lambda)) \quad \text{and} \quad \mathcal{NB}_{j\to i}(D(\lambda)) = \mathcal{B}_{i^*\to j^*}(D(\lambda)),$$

whence we obtain

$$\mathcal{H}_{\{ij\}}(D(\lambda)) = \mathcal{B}_{\{ij\}}(D(\lambda)) \cup \mathcal{B}_{\{i^*j^*\}}(D(\lambda)) = \mathcal{B}_{\{ij\}}(\lambda) \cup \mathcal{B}_{\{i^*j^*\}}(\lambda).$$

Similarly,

$$\mathcal{H}_{\{ij\}}(D(\bar{c}_d(\lambda))) = \mathcal{B}_{\{ij\}}(\bar{c}_d(\lambda)) \cup \mathcal{B}_{\{i^*j^*\}}(\bar{c}_d(\lambda)),$$

and, using Corollary 3.6,

$$\bar{\mathcal{H}}_{\{ij\}}(D(\bar{q}_d(\lambda))) = \bar{\mathcal{B}}_{\{ij\}}(\bar{q}_d(\lambda)) \cup \bar{\mathcal{B}}_{\{i^*j^*\}}(\bar{q}_d(\lambda)).$$

Hence, in this case,

$$\mathcal{B}_{\{ij\}}(\lambda) \cup \mathcal{B}_{\{i^*j^*\}}(\lambda) = \mathcal{B}_{\{ij\}}(\bar{c}_d(\lambda)) \cup \mathcal{B}_{\{i^*j^*\}}(\bar{c}_d(\lambda)) \cup \bar{\mathcal{B}}_{\{ij\}}(\bar{q}_d(\lambda)) \cup \bar{\mathcal{B}}_{\{i^*j^*\}}(\bar{q}_d(\lambda)).$$

Case 2. Take any $0 \leqslant i < j \leqslant d-1$ such that $|\{i, j, i^*, j^*\}| = 3$. This means that $i = i^* = 0$ and $j \neq j^*$ (in particular, $j \neq 0$ and $j \neq i^*$), so that $P_{i\to j}(D(\lambda)) = \emptyset$ and $DP_{\{ij\}}(D(\lambda)) = \emptyset = DP_{\{i^*j^*\}}(D(\lambda))$ (and similarly for $D(\bar{c}_d(\lambda))$ and $D(\bar{q}_d(\lambda))$).

This time, we have

$$\mathcal{H}_{i\to j}(D(\lambda)) = \mathcal{H}_{0\to j}(D(\lambda)) = \mathcal{B}_{0\to j}(D(\lambda)) \cup \mathcal{NB}_{0\to j}(D(\lambda))$$

and

$$\mathcal{H}_{j\to 0}(D(\lambda)) = \mathcal{P}_{j\to 0}(D(\lambda)) \cup \mathcal{B}_{j\to 0}(D(\lambda)) \cup \mathcal{NB}_{j\to 0}(D(\lambda)).$$

Also, by Corollary 3.2, we have

$$\mathcal{NB}_{0\to j}(D(\lambda)) = \mathcal{B}_{j^*\to 0}(D(\lambda)) \quad \text{and} \quad \mathcal{NB}_{j\to 0}(D(\lambda)) = \mathcal{B}_{0\to j^*}(D(\lambda)),$$

whence we obtain $\mathcal{H}_{\{0j\}}(D(\lambda)) = \mathcal{P}_{\{0j\}}(D(\lambda)) \cup \mathcal{B}_{\{0j\}}(D(\lambda)) \cup \mathcal{B}_{\{0j^*\}}(D(\lambda))$.

Now, $0 < j^* \leqslant d-1$, and $(0, j^*)$ satisfies the same condition as $(0, j)$ (and, in fact, $\{0, j, 0^*, j^*\} = \{0, j^*, 0^*, (j^*)^*\}$). Thus, we also have

$$\mathcal{H}_{\{0j^*\}}(D(\lambda)) = \mathcal{P}_{\{0j^*\}}(D(\lambda)) \cup \mathcal{B}_{\{0j^*\}}(D(\lambda)) \cup \mathcal{B}_{\{0j\}}(D(\lambda)).$$

Writing $\mathcal{H}^{\lambda}_{\{0jj^*\}}$ for $\mathcal{H}_{\{0j\}}(D(\lambda)) \cup \mathcal{H}_{\{0j^*\}}(D(\lambda))$, and using similar notation for parts and bars, we hence obtain

$$\mathcal{H}^{\lambda}_{\{0jj^*\}} = \mathcal{P}^{\lambda}_{\{0jj^*\}} \cup 2\mathcal{B}^{\lambda}_{\{0jj^*\}},$$

where, for any multiset A, we write $2A$ for $A \cup A$. Similarly, and with analogous notation, we have

$$\mathcal{H}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} = \mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cup 2\mathcal{B}^{\bar{c}_d(\lambda)}_{\{0jj^*\}}.$$

And, using Corollary 3.6, we also obtain

$$\bar{\mathcal{H}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}} = \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}} \cup 2\bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}.$$

Finally, by [**1**, Theorem 4.7], we have

$$\mathcal{H}^{\lambda}_{\{0jj^*\}} = \mathcal{H}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cup \bar{\mathcal{H}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}.$$

Rewriting this equality using the expressions we found above, we obtain

$$\begin{aligned}\mathcal{P}^{\lambda}_{\{0jj^*\}} \cup 2\mathcal{B}^{\lambda}_{\{0jj^*\}} &= \mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cup 2\mathcal{B}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cup \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}} \cup 2\bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}} \\ &= \mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \circ \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}} \cup 2\Big[\Big(\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cap \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}\Big) \cup \mathcal{B}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cup \bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}\Big],\end{aligned}$$

where $\circ$ denotes symmetric difference.

Now any multiset Q has a unique decomposition of the form $Q = R + 2S$, where R and S are sub-multisets, and the elements of R are distinct. The elements of $\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}}$ are distinct, since $\bar{c}_d(\lambda)$ is a bar partition, and those of $\bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}$ are distinct by Corollary 3.8, whence we find that the elements of $\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \circ \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}$ are distinct. This implies that

$$\mathcal{P}^{\lambda}_{\{0jj^*\}} = \mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \circ \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}$$

and

$$\mathcal{B}^{\lambda}_{\{0jj^*\}} = (\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cap \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}) \cup \mathcal{B}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cup \bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}.$$

In particular, we obtain, for each $1 \leqslant j \leqslant \frac{1}{2}(d-1)$,

$$\mathcal{B}_{\{0jj^*\}}(\lambda) = \mathcal{B}_{\{0jj^*\}}(\bar{c}_d(\lambda)) \cup \bar{\mathcal{B}}_{\{0jj^*\}}\Big(\bar{q}_d(\lambda)) \setminus (\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0jj^*\}} \cap \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0jj^*\}}\Big).$$

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Case 3. Finally, take any $0 \leqslant i < j \leqslant d-1$ such that $|\{i,j,i^*,j^*\}| = 2$. This means that $i \neq 0$, and $j = i^*$. In particular,

$$P_{\{ij\}}(D(\lambda)) = P_{\{i^*j^*\}}(D(\lambda)) = P_{\{ii^*\}}(D(\lambda)) = \emptyset,$$

while all the doubled parts non-divisible by d fit in this case. We have

$$\mathcal{H}_{i\to i^*}(D(\lambda)) = \mathcal{DP}_{i\to i^*}(D(\lambda)) \cup \mathcal{B}_{i\to i^*}(D(\lambda)) \cup \mathcal{NB}_{i\to i^*}(D(\lambda))$$

and

$$\mathcal{H}_{i^*\to i}(D(\lambda)) = \mathcal{DP}_{i^*\to i}(D(\lambda)) \cup \mathcal{B}_{i^*\to i}(D(\lambda)) \cup \mathcal{NB}_{i^*\to i}(D(\lambda)),$$

whence, by Corollary 3.2, we obtain $\mathcal{H}_{\{ii^*\}}(D(\lambda)) = \mathcal{DP}_{\{ii^*\}}(D(\lambda)) \cup 2\mathcal{B}_{\{ii^*\}}(D(\lambda))$. Similarly,

$$\mathcal{H}_{\{ii^*\}}(D(\bar{c}_d(\lambda))) = \mathcal{DP}_{\{ii^*\}}(D(\bar{c}_d(\lambda))) \cup 2\mathcal{B}_{\{ii^*\}}(D(\bar{c}_d(\lambda)))$$

and, by Corollary 3.6, $\bar{\mathcal{H}}_{\{ii^*\}}(D(\bar{q}_d(\lambda))) = \overline{\mathcal{DP}}_{\{ii^*\}}(D(\bar{q}_d(\lambda))) \cup 2\bar{\mathcal{B}}_{\{ii^*\}}(D(\bar{q}_d(\lambda)))$.

Applying [**1**, Theorem 4.7] and using similar notation to that in case 2, we obtain

$$\begin{aligned}\mathcal{DP}^{\lambda}_{\{ii^*\}} \cup 2\mathcal{B}^{\lambda}_{\{ii^*\}} &= \mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cup 2\mathcal{B}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cup \overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii^*\}} \cup 2\bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{ii^*\}} \\ &= \mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \circ \overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii^*\}} \cup 2\Big[\Big(\mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cap \overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii^*\}}\Big) \cup \mathcal{B}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cup \bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{ii^*\}}\Big].\end{aligned}$$

Now the elements of $\mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}}$ are distinct, and those of $\overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii*\}}$ are distinct by Corollary 3.8, whence the elements of $\mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \circ \overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii^*\}}$ are distinct. This implies that

$$\mathcal{B}^{\lambda}_{\{ii^*\}} = \Big(\mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cap \overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii^*\}}\Big) \cup \mathcal{B}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cup \bar{\mathcal{B}}^{\bar{q}_d(\lambda)}_{\{ii^*\}}.$$

Note that (using Corollary 3.7) we have

$$\mathcal{DP}^{\bar{c}_d(\lambda)}_{\{ii^*\}} \cap \overline{\mathcal{DP}}^{\bar{q}_d(\lambda)}_{\{ii^*\}} = \Big[\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0ii^*\}} \cap \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0ii^*\}}\Big]^{\times 2} := \Big\{2h \,\Big|\, h \in \mathcal{P}^{\bar{c}_d(\lambda)}_{\{0ii^*\}} \cap \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0ii^*\}}\Big\}.$$

Thus, for each $1 \leqslant i \leqslant \frac{1}{2}(d-1)$, we have

$$\mathcal{B}_{\{ii^*\}}(\lambda) = \mathcal{B}_{\{ii^*\}}(\bar{c}_d(\lambda)) \cup \bar{\mathcal{B}}_{\{ii^*\}}(\bar{q}_d(\lambda)) \cup \Big[\mathcal{P}^{\bar{c}_d(\lambda)}_{\{0ii^*\}} \cap \bar{\mathcal{P}}^{\bar{q}_d(\lambda)}_{\{0ii^*\}}\Big]^{\times 2}.$$

Finally, we see that our three cases cover all the bars between any pair of (distinct) runners, because

$$\begin{aligned}\bigcup_{\substack{0\leqslant i<j\leqslant d-1,\\ |\{i,j,i^*,j^*\}|=4}} \{i,j\} \cup \{i^*,j^*\} \cup \bigcup_{1\leqslant j\leqslant (d-1)/2} \{0,j\} \cup \{0,j^*\} \cup \bigcup_{1\leqslant i\leqslant (d-1)/2} \{i,i^*\} \\ = \bigcup_{1\leqslant i<j\leqslant d-1} \{i,j\}.\end{aligned}$$

When we take the union of all the subsets of bars we computed, we see that all the parts of $\bar{c}_d(\lambda)$ and $\bar{q}_d(\lambda)$ which are not divisible by d appear exactly once in case 2 and

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their doubles appear once in case 3. Since $\bar{c}_d(\lambda)$ has no part divisible by d, we therefore obtain, together with the case of bars on a single runner,

$$\mathcal{B}(\lambda) = [\mathcal{B}(\bar{c}_d(\lambda)) \cup \bar{\mathcal{B}}(\bar{q}_d(\lambda))] \setminus [\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))] \cup [\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))]^{\times 2},$$

i.e. $\mathcal{B}(\lambda) = \mathcal{B}(\bar{c}_d(\lambda)) \cup \tilde{\mathcal{B}}(\bar{q}_d(\lambda))$, as claimed. □

For any bar partition μ, we denote by $m(\mu)$ the number of parts of μ.

Corollary 4.2. *For any bar partition λ and any odd integer $d \geqslant 3$, the following hold:*

(1) $\mathcal{B}(\bar{c}_d(\lambda)) \subset \mathcal{B}(\lambda)$;

(2) $\bar{\mathcal{B}}(\bar{q}_d(\lambda)) \subset \mathcal{B}(\lambda)$;

(3) $\mathcal{P}(\lambda) = \mathcal{P}(\bar{c}_d(\lambda)) \circ \bar{\mathcal{P}}(\bar{q}_d(\lambda))$;

(4) $m(\bar{c}_d(\lambda)) + m(\bar{q}_d(\lambda)) = m(\lambda) + 2|\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))|$;

(5) $\mathcal{B}(\lambda) = \mathcal{B}(\bar{c}_d(\lambda)) \cup \bar{\mathcal{B}}(\bar{q}_d(\lambda))$ *if and only if* $m(\lambda) = m(\bar{c}_d(\lambda)) + m(\bar{q}_d(\lambda))$.

Proof. Part (1) is immediate from Theorem 4.1. To obtain (2), one just has to rewrite the result as

$$\mathcal{B}(\lambda) = \bar{\mathcal{B}}(\bar{q}_d(\lambda)) \cup (\mathcal{B}(\bar{c}_d(\lambda)) \setminus [\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))]) \cup [\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))]^{\times 2}.$$

Part (3) is visible in the proof of Theorem 4.1: the parts of length divisible by d are the same in λ and $\bar{q}_d(\lambda)$ (while $\bar{c}_d(\lambda)$ has none), and those of length not divisible by d in λ are examined in case 2 of the proof. Looking at the cardinalities of the sets involved, (4) is a direct consequence of (3). By (4), $m(\lambda) = m(\bar{c}_d(\lambda)) + m(\bar{q}_d(\lambda))$ if and only if $\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda)) = \emptyset$. This, in turn, is, by Theorem 4.1, equivalent to $\mathcal{B}(\lambda) = \mathcal{B}(\bar{c}_d(\lambda)) \cup \bar{\mathcal{B}}(\bar{q}_d(\lambda))$ (as $A^{\times 2} \neq A$ for any non-empty multiset A). □

Remark 4.3. Note that the situation given in (5) above does occur, for instance, in the example we introduced in § 2.

We now illustrate Theorem 4.1 by an explicit example. As the above remark shows, the example we introduced in § 2 does not fully illustrate the extent of Theorem 4.1. We therefore consider instead the bar partition $\lambda = (13, 10, 4)$ of $n = 27$, and $d = 3$. Below is the shifted diagram of λ, filled in with the corresponding bar lengths:

23	17	**13**	12	11	10	8	7	6	5	4	2	1
	14	**10**	9	8	**7**	5	4	3	2	1		
		4	3	**2**	**1**							

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We then have $\bar{c}_3(\lambda) = (7, 4, 1)$, with corresponding shifted diagram:

11	8	**7**	5	4	2	1
	5	**4**	2	1		
		1				

The $\bar{d}$-quotient partition of λ is $\bar{q}_3(\lambda) = (8, 4, 2, 1)$. It has the following shifted diagram, where we indicate alongside the rim the runners to consider (i.e. the hand residue at the end of rows, and the foot residue decreased by 1 at the end of columns):

12	10	9	8	5	3	2	1	2
2	6	5	4	1	1 2	0	1	
	1	3	2	2 0				
		2	1	1				
			0					

We can now compute the modified bar lengths given by Definition 3.5. The normalized β set $\{0, 1, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 19, 25, 28\}$ for $D(\lambda)$ gives us $x_0 = 5$, $x_1 = 8$ and $x_2 = 2$. For $0 \leqslant i, j \leqslant 2$ and $z \in \mathcal{B}_{i \to j}(\bar{q}_3(\lambda))$, we have $\bar{h}(z) = h(z) + 3(x_i - x_j)$, and $\bar{\mathcal{B}}_{i \to j}(\bar{q}_3(\lambda)) = \{|\bar{h}(z)| \mid z \in \mathcal{B}_{i \to j}(\bar{q}_3(\lambda))\}$. This gives the following:

12	8	9	**1**	4	3	7	17
	6	23	**13**	10			
		3	**7**				
			10				

It is then easy to check that the result announced by Theorem 4.1 does hold. We just explicitly describe the case of parts (indicated in bold in the above diagrams). We see that, in accordance with Corollary 4.2, $\mathcal{P}(\lambda) = \{13, 10, 4\} = \mathcal{P}(\bar{c}_3(\lambda)) \circ \bar{\mathcal{P}}(\bar{q}_3(\lambda))$. And, for the last four bars in bold in the diagram of λ, we have

$$\{1, 2, 7, 14\} = \{1, 7\} \cup \{2, 14\} = [\mathcal{P}(\bar{c}_3(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_3(\lambda))] \cup [\mathcal{P}(\bar{c}_3(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_3(\lambda))]^{\times 2}.$$

In [**1**, Corollary 4.12], a generalization of a relative hook formula discovered by Malle and Navarro was presented. We finish this paper by stating the bar analogue of [**1**, Corollary 4.12].

If λ is a bar partition of n, we again let ρ_λ be an irreducible spin character of $\hat{\mathfrak{S}}_n$ labelled by λ. We define $\sigma(\lambda) = |\lambda| - m(\lambda)$, and $\delta(\lambda) = \lfloor \sigma(\lambda)/2 \rfloor$ so that the bar formula reads

$$\rho_\lambda(1) = 2^{\delta(\lambda)} \frac{n!}{\pi \mathcal{B}(\lambda)}.$$

By Corollary 4.2 (3), we have that

$$m(\lambda) = m(\bar{c}_d(\lambda)) + m(\bar{q}_d(\lambda)) - 2\delta,$$

where $\delta = |\mathcal{P}(\bar{c}_d(\lambda)) \cap \bar{\mathcal{P}}(\bar{q}_d(\lambda))|$. It follows from this and from $|\lambda| = |\bar{q}_d(\lambda)| + |\bar{c}_d(\lambda)|$ that

$$\sigma(\lambda) = \sigma(\bar{c}_d(\lambda)) + \sigma(\bar{q}_d(\lambda)) + 2\delta.$$

We thus have

$$\delta(\lambda) = \delta(\bar{q}_d(\lambda)) + \delta(\bar{c}_d(\lambda)) + \delta + \varepsilon, \tag{4.1}$$

where $\varepsilon = 0$ if $\sigma(\lambda)$ is odd or if $\sigma(\lambda)$ and $\sigma(\bar{c}_d(\lambda))$ are both even, and $\varepsilon = 1$ otherwise. Theorem 4.1 now implies that

$$\pi\mathcal{B}(\lambda) = \pi\tilde{\mathcal{B}}(\bar{q}_d(\lambda))\pi\mathcal{B}(\bar{c}_d(\lambda)) = 2^{\delta}\pi\bar{\mathcal{B}}(\bar{q}_d(\lambda))\pi\mathcal{B}(\bar{c}_d(\lambda)).$$

Combining this with formula (4.1), we get a relative bar formula.

Corollary 4.4. *With the above notation,*

$$\rho_\lambda(1) = \frac{|\lambda|!}{|\bar{c}_d(\lambda)|!}\frac{2^{\delta(\bar{q}_d(\lambda))+\varepsilon}}{\pi\bar{\mathcal{B}}(\bar{q}_d(\lambda))}\rho_{\bar{c}_d(\lambda)}(1).$$

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