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TESTING ON NULL SEQUENCES IS ENOUGH FOR BOCHNER INTEGRABILITY

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Dedicated to Paul R. Halmos on the occasion of his eightieth birthday.

Let E be a normed space, a Fréchet space or a complete (DF)-space satisfying the dual density condition. Let Ω be a Radon measure space. We prove that a function $f : \Omega \to E$ is Bochner p-integrable if (and only if) f is p-integrable with respect to the topology of uniform convergence on the norm-null sequences from E'.

1. INTRODUCTION

Our first question was: Can one deduce that a function with values in a Banach space is Bochner integrable from the fact that it is integrable for a coarser topology? Of course the answer is negative for the weak topology (see [**3**, II.3.3 on p.53] for a concrete example or use the Dvoretzky-Rogers theorem in general). In this paper, we want to show that the answer is "Yes" for the topology of uniform convergence on the null sequences of the dual.

Let Ω be the measure space, E be the Banach space and τ be the coarser topology. If $f : \Omega \to E$ is the τ-integrable function candidate to be Bochner integrable, two problems are involved here: to prove that absolute integrability with respect to the τ-seminorms implies that $t \to \|f(t)\|$ is in L^1 and to show that f is norm-measurable if it is τ-measurable. The aim of this paper is to show that there is a (somehow natural) class of spaces for which these two problems have a solution and that this class includes normed spaces, Fréchet spaces, strict (LF)-spaces and complete (DF)-spaces satisfying the dual density condition of Bierstedt and Bonet.

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2. Terminology and Notation

In what follows (Ω, Σ, μ) stands for a σ-finite Radon measure space, where Ω is a locally compact and σ-compact topological space. Let (E, τ) be a locally convex space with a topology defined by a family of continuous seminorms $\mathcal{Q}(E, \tau)$. We consider measurability of functions in the sense of Lusin: we say that a function $f : \Omega \to E$ is τ-measurable if there is a sequence (K_n) (that we may take either disjoint or increasing) of compact sets such that the restriction $f|_{K_n}$ is continuous for every $n \in \mathbb{N}$, and $\mu\Bigl(\Omega \setminus \bigcup_n K_n\Bigr) = 0$. When (E, τ) is metrisable, the notion of a τ-measurable function in the sense of Lusin coincides with the usual definition of a strongly measurable function as the μ-almost everywhere limit of a sequence of simple functions. If τ_1 and τ_2 are two topologies defined on E, the identity $(E, \tau_1) \to (E, \tau_2)$ is said to be universally measurable if (among several equivalent conditions) every τ_1-measurable function is also τ_2-measurable (for arbitrary Ω and μ).

A function $f : \Omega \to E$ is said to be integrable with respect to τ, or simply τ-integrable, if it is τ-measurable and the scalar functions $q(f) : t \in \Omega \to q\bigl(f(t)\bigr) \in \mathbb{R}$ are in $L^1(\mu)$ for every $q \in \mathcal{Q}(E, \tau)$. When (E, τ) is a Banach space, τ-integrability equals Bochner integrability. $L^1(E, \tau)$ will denote the space of all (classes of μ-almost everywhere equal) τ-integrable functions endowed with the locally convex topology defined by the family of seminorms $f \to \|q(f)\|_1$ as $q \in \mathcal{Q}(E, \tau)$. For $1 < p \leqslant \infty$, the space $L^p(E, \tau)$ is defined in the analogous way.

We say that a locally convex space (E, τ) has property (B) of Pietsch if for each bounded subset M of the space $\ell^1\{E, \tau\}$ of all absolutely summable sequences in (E, τ), there exists a disc $B \subset E$ such that for all $(x_n) \in M$ the following hold: $x_n \in E_B$ for each n and $\sum_n p_B(x_n) \leqslant 1$, where E_B is the linear span of B and p_B is its natural norm, the gauge of B. In other words, each bounded subset of $\ell^1\{E, \tau\}$ is a bounded subset of some $\ell^1\{E_B, p_B\}$. Metrisable and (df)-spaces have property (B), for instance. A locally convex space (E, τ) is said to have property (BM) if it has property (B) and the topology τ is metrisable when restricted to bounded sets. Metrisable or, more generally, strict (LF)-spaces have property (BM). For a quasi-complete locally convex space (E, τ) with property (BM) the identity $(E, \sigma(E, E')) \to (E, \tau)$ is universally measurable [4, 4.13].

We introduced in [4, 3.5] the notion of fundamental L^p-boundedness as an extension of property (B). Let $1 \leqslant p \leqslant \infty$. A locally convex space (E, τ) is said to be fundamentally L^p-bounded, with respect to (Ω, Σ, μ), if each bounded subset M of $L^p\{E, \tau\}$ is contained in a bounded set of the form

$$[U_p, B] := \{f \in L^p\{E, \tau\} : f(t) \in E_B \text{ almost everywhere and } p_B(f) \in U_p\},$$

where B is a disc in E and U_p stands for the unit ball of $L^p(\mu)$. This definition applied to the particular case of the counting measure on the power set of $\mathbb{N}$, tells us that fundamental ℓ^1-boundedness is just property (B) (this is the terminology of [7], by the way).

The dual density condition was introduced by Bierstedt and Bonet in connection with their solution to the problem of when a Köthe echelon space is distinguished. They proved [1, Theorem 5] that a (DF)-space E satisfies the dual density condition if and only if every bounded subset of E is metrisable, or if and only if $\ell^\infty(E,\tau)$ is quasi-barrelled. (DF)-spaces satisfying the dual density condition are quasi-barrelled (but not the opposite!). In particular, for (DF)-spaces property (BM) equals the dual density condition.

We refer the reader to the books by Jarchow [5], Köthe [6], Pérez Carreras and Bonet [7] or Robertson and Robertson [8] for the terminology about locally convex spaces and to the monographs by Bourbaki [2], Diestel and Uhl [3], Schwartz [9] or Thomas [10] for the properties of measurable functions. Our paper [4] contains several results about localisation of bounded sets in $L^p(E,\tau)$ and Radon-Nikodym theorems.

3. Results

Main Theorem. *Let (E,τ) be a locally convex space. Let τ_0 be another locally convex topology on E coarser that τ and such that*

(1) *the identity $(E,\tau_0) \to (E,\tau)$ is universally measurable,*
(2) *every τ_0-bounded subset of E is also τ-bounded,*
(3) *the space (E,τ_0) is fundamentally L^p-bounded for some $p \in [1,\infty)$.*

Then a function $f : \Omega \to E$ is p-integrable with respect to τ if (and only if) f is p-integrable with respect to τ_0, that is

$$L^p(E,\tau) = L^p(E,\tau_0)$$

holds as an equality of vector spaces.

Proof: Let $f : \Omega \to E$ be τ_0-integrable. Since the identity $(E,\tau_0) \to (E,\tau)$ is universally measurable, it follows that the function f is τ-measurable. It remains to prove that for every τ-continuous seminorm q the scalar function $t \to q\big(f(t)\big)$ is in $L^p(\mu)$. The space (E,τ_0) is fundamentally L^p-bounded, therefore there exists a disc B in (E,τ_0) such that the scalar function $t \to p_B(f(t))$ is in $L^p(\mu)$. Since B is also τ-bounded, it is contained in some multiple of the unit ball of q, hence $t \to q\big(f(t)\big)$ is in $L^p(\mu)$, as desired. ∎

We shall give several applications of this theorem.

Corollary 1. *Let E be a normed space and let τ_0 be the topology of uniform convergence on the sequences that converge to zero in E'. Let $p \in [1,\infty)$. Then a function $f : \Omega \to E$ is Bochner p-integrable if (and only if) $f \in L^p(E,\tau_0)$.*

PROOF: We only have to check that conditions (1) and (3) above hold in this case. A consequence of the Grothendieck-Phillips theorem [**9**, Part II, I.1. Theorem 3 on p.162] or [**10**, p.50], is that for a Banach space E the identity $(E,\sigma(E,E')) \to (E,\|\cdot\|)$ is universally measurable. It is easy to see, by passing to its completion, that the same holds when E is a non-complete normed space. This proves (1). To see (3) note that (E,τ_0) is a (df)-space [**5**, 12.4–5], that is, it has a fundamental sequence of bounded sets (the integer multiples of the unit ball) and every norm-null sequence in E' is equicontinuous. Now [**4**, 3.10] states that (df)-spaces are fundamentally L^p-bounded for every $p \in [1,\infty)$. ∎

REMARKS. Given $p \in [1,\infty)$, Corollary 1 tells us, in other words, that if E is a normed space and $f : \Omega \to E$ is strongly measurable then f is Bochner p-integrable provided that for every null sequence (x'_n) from E', the scalar function

$$t \in \Omega \to \sup\{|\langle f(t), x'_n\rangle| : n \in \mathbb{N}\}$$

is p-integrable.

The (df)-space (E,τ_0) is not complete if E is not reflexive [**5**, 12.5.1 and 2]. If A is a measurable set, the integral $\int_A f\,d\mu$ of a function $f \in L^1(E,\tau_0)$ is obtained as the limit of a net of Riemann sums so that they belong, a priori, to the completion of (E,τ_0) and this completion coincides (as a vector space) with the bidual E'' [**5**, 12.5.1]. However, it follows from Corollary 1 that these integrals are, indeed, elements of E.

Corollary 1 also holds for every locally convex topology between τ_0 and the norm topology. For all of these topologies E is again a (df)-space.

Let us consider now the situation on the dual E' of a Banach space E. The difficulty is to lift integrability from the topology τ'_0 of uniform convergence on the null sequences on E to the norm topology in E'. The main problem will be that the behaviour of the measurability is not so good; a measurable function with respect the weak*-topology $\sigma(E',E)$ may not be measurable with respect the norm topology on E' as the cases [**9**, Exercise 1 and 2 on p.168] $E = \ell^1$ or $E = C[0,1]$ show.

COROLLARY 2. *Let E be a Banach space with dual E' and $p \in [1,\infty)$. Then the following hold.*

(a) *A strongly measurable function $f : \Omega \to E'$ is Bochner p-integrable if and only if for every null sequence (x_n) in E the scalar function*

$$t \in \Omega \to \sup\{|\langle x_n, f(t)\rangle| : n \in \mathbb{N}\}$$

is in $L^p(\mu)$.

(b) *If E' is separable, then a function $f : \Omega \to E'$ is Bochner p-integrable if and only if it is p-integrable with respect to the topology τ'_0 of uniform convergence on the norm-null sequences in E.*

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PROOF: Part (a) can be proved as in the Main Theorem using the fact that (E',τ_0') is also a (df)-space and so is fundamentally L^p-bounded. Part (b) follows from a theorem due to Meyer and Schwartz [**9**, Part I, II.3 Corollary 2 of Theorem 10 on pp.122–124] [**10**, p.51] stating that if E' is separable then the identity $(E',\sigma(E',E)) \to (E',\|\cdot\|)$ is universally measurable. ∎

By the Banach-Dieudonné theorem, τ_0' equals the topology of uniform convergence on the compact subsets of E [**5**, 9.4.3]. Moreover, (E',τ_0') is not only a (df)-space; it is also a complete, Schwartz (gDF)-space [**5**, 9.4.1–3, 11.1.4 and 12.5.2 and 6].

COROLLARY 3. *Let $p \in [1,\infty)$ and (E,τ) be a complete (DF)-space with the dual density condition. Let τ_0 be the topology of uniform convergence on the sequences from E' that converge to zero in the strong topology $\beta(E',E)$. Then a function $f : \Omega \to E$ is p-integrable with respect to τ if (and only if) f is p-integrable with respect to τ_0.*

PROOF: Since every (DF)-space with the dual density condition has property (BM), condition (1) is a particular case of [**4**, 4.13]. On the other hand, it is clear that if (E,τ) is a (DF)-space then (E,τ_0) is a (df)-space and the proof finishes as in the proof of Corollary 1. ∎

Corollary 3 can be also obtained as a consequence of Corollary 4 below — the corresponding result for quasi-complete spaces having property (BM) — but the proof of the latter requires more work.

LEMMA. *Let (E,τ) be a quasi-complete locally convex space with property (BM) and let τ_0 be the topology of uniform convergence on the sequences that converge to zero in $(E',\beta(E',E))$. Then (E,τ_0) is fundamentally L^p-bounded for each $p \in [1,\infty)$.*

PROOF: We start by proving that (E,τ_0) has property (B). Since (E,τ) has property (B), it will be enough to prove that if M is a bounded subset of $\ell^1\{E,\tau_0\}$ then M is also bounded in $\ell^1\{E,\tau\}$. Let q be any τ-continuous seminorm and assume that

$$\sup\left\{\sum_{n=1}^{\infty} q(x_n) : (x_n) \in M\right\} = \infty.$$

Then there exists a sequence $\left\{\left(x_n^{(k)}\right)_n : k = 1,2,\dots\right\} \subset M$ and an increasing sequence of indices (n_k) such that

$$\sum_{n=1+n_k}^{n_{k+1}} q\left(x_n^{(k)}\right) > 2^{2k}.$$

Let $V \subset E'$ be the polar of the unit ball associated to q. Then we can find a sequence

$(v_n) \subset V$ such that

$$\sum_{n=1+n_k}^{n_{k+1}} \langle x_n^{(k)}, v_n \rangle > 2^{2k} \qquad \text{for every } k = 1, 2, \ldots .$$

Since V is $\beta(E', E)$-bounded it follows that the sequence

$$\frac{v_1}{2^0}, \ldots, \frac{v_{n_1}}{2^0}, \frac{v_{1+n_1}}{2^1}, \ldots, \frac{v_{n_2}}{2^1}, \frac{v_{1+n_2}}{2^2}, \ldots, \frac{v_{n_3}}{2^2}, \ldots$$

converges to zero in the strong topology $\beta(E', E)$ and satisfies

$$\sum_{n=1+n_k}^{n_{k+1}} \left\langle x_n^{(k)}, \frac{v_n}{2^k} \right\rangle > 2^k \qquad \text{for every } k = 1, 2, \ldots,$$

contradicting the fact that M is bounded in $\ell^1\{E, \tau_0\}$.

We now prove that (E, τ_0) is fundamentally $L^1(\mu)$-bounded. By [**4**, 4.13] the identity $(E, \sigma(E, E')) \to (E, \tau)$ is universally measurable so that the identity $(E, \tau_0) \to (E, \tau)$ will also be universally measurable. (This proves, by the way, that condition (1) of the Main Theorem is satisfied.) Therefore, given $f \in L^1(E, \tau_0)$ there is a disjoint sequence (K_n) of compact sets such that the restriction $f|_{K_n}$ is τ-continuous for every $n \in \mathbb{N}$, and $\mu(\Omega \setminus \cup_n K_n) = 0$. In particular (see [**2**]), f will be integrable on every measurable set A contained in some K_n; that is, there is an element $\int_A f\, d\mu \in E$ such that

$$\langle \textstyle\int_A f\, d\mu,\, v \rangle = \int_A \langle f(t), v \rangle \, d\mu \qquad \text{for every } v \in E'.$$

Let $F \subset L^1(E, \tau_0)$ be a bounded set. For each $q \in \mathcal{Q}(E, \tau_0)$ take

$$\rho_q := \sup \left\{ \int_\Omega q(f)\, d\mu : f \in F \right\} < \infty.$$

Let F_0 be the set of all E-valued sequences of the form

$$\left(\textstyle\int_{A_1} f\, d\mu,\ \int_{A_2} f\, d\mu,\ \ldots \right)$$

where $f \in F$ and $A_1, A_2, \ldots$ is a sequence of pairwise disjoint, measurable sets with positive and finite measure such that each one of them is contained in some compact set where f is τ-continuous. As pointed out above, for each $f \in F$ there is at least one such sequence (A_n). For each seminorm $q \in \mathcal{Q}(E, \tau)$ and each of these sequences, we have

$$\sum_{n=1}^{\infty} q\left(\textstyle\int_{A_n} f\, d\mu\right) \leqslant \sum_{n=1}^{\infty} \int_{A_n} q(f)\, d\mu \leqslant \int_\Omega q(f)\, d\mu \leqslant \rho_q.$$

This tells us that F_0 is a bounded subset of $\ell^1\{E,\tau_0\}$. We have proved above that (E,τ_0) has property (B), hence there is a closed disc $B \subset E$ such that for every sequence $(x_n) \in F_0$ we have $x_n \in E_B$ for each $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} p_B(x_n) \leqslant 1$. We shall prove that for every $f \in F$ we have (i) $f(t) \in E_B$ almost everywhere and (ii) the function $t \to p_B\big(f(t)\big)$ is in the unit ball of $L^1(\mu)$.

(i) If there is $f \in F$ such that $\mu\{t \in \Omega : f(t) \notin E_B\} > 0$, then there is a compact set $K \subset \Omega$ with positive measure such that $f : K \to E$ is τ-continuous and $f(t) \notin E_B$ for all $t \in K$. By [4, 3.7] for every $n \in \mathbb{N}$ there exists a simple function $z_n : K \to B^\circ \subset E'$ such that Re $\langle f(t), z_n(t)\rangle > n$ for all $t \in K$. If we write $z_n = \sum_{i=1}^{k} v_i \chi_{A_i}$, where $\{A_1, A_2, \dots, A_k\}$ is a measurable partition of K and $\{v_1, v_2, \dots, v_k\} \subset B^\circ$, then the sequence

$$\Big(\int_{A_1} f\,d\mu,\ \int_{A_2} f\,d\mu,\ \dots, \int_{A_k} f\,d\mu,\ 0,\ 0,\ \dots\Big)$$

is in F_0. However, we also have

$$\sum_{i=1}^{k} p_B\Big(\int_{A_i} f\,d\mu\Big) \geqslant \sum_{i=1}^{k} \Big| \text{ Re } \langle \int_{A_i} f\,d\mu,\ v_i\rangle \Big| = \sum_{i=1}^{k} \Big| \int_{A_i} \text{ Re } \langle f, v_i\rangle\ d\mu\Big|$$
$$\geqslant n \sum_{i=1}^{k} \mu(A_i) = n\mu(K),$$

contradicting the boundedness of F_0 in $\ell^1\{E_B, p_B\}$.

(ii) Assume that the set of functions $\{p_B(f) : f \in F\}$ is not contained in the unit ball of $L^1(\mu)$. This can happen because this set is not contained in $L^1(\mu)$ at all, or simply because $\|p_B(f)\|_1 > 1$ for some $f \in F$. In either case, we can find a function $f \in F$ and a compact set $K \subset \Omega$, such that the functions $f : K \to E$ and $p_B(f) : K \to \mathbb{R}$ are τ-continuous, and $\|p_B(f) \cdot \chi_K\|_1 > 1 + \delta$, for some positive δ. It is well-known that for $\varphi \in L^1(\mu)$ one has

$$\|\varphi\|_1 = \sup\left\{ \left| \int_\Omega \varphi \cdot \theta\, d\mu \right| : \theta \text{ a simple function with } \|\theta\|_\infty \leqslant 1 \right\},$$

so we can find a simple function θ in the unit ball of $L^\infty(\mu)$ such that $\int_K p_B(f)\cdot\theta\,d\mu > 1 + \delta$; note that we may assume that θ is non-negative. Again by [4, 3.7], given $\varepsilon > 0$ there is a simple function $z : K \to B^\circ \subset E'$ such that

$$p_B\big(f(t)\big) < \text{ Re } \langle f(t), z(t)\rangle + \varepsilon \qquad \text{for all } t \in K.$$

Write θ and z as

$$\theta = \sum_{i=1}^{k} \alpha_i \chi_{A_i} \qquad z = \sum_{i=1}^{k} v_i \chi_{A_i}$$

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where the sets (A_i) are pairwise disjoint and have positive finite measure, and each α_i is in $[0,1]$. Take the sequence $(x_i) \subset E$ defined by $x_i := \int_{A_i} f\,d\mu$, for $i = 1,2,\ldots,k$ and $x_i = 0$ afterwards. Then (x_i) is in F_0 because each A_i is contained in K, where f is τ-continuous. Now, since each α_i is in $[0,1]$ and F_0 is contained in the unit ball of $\ell^1\{E_B, p_B\}$, we have that

$$\begin{aligned}
1 &\geqslant \sum_{i=1}^{k} \alpha_i p_B(x_i) = \sum_{i=1}^{k} \alpha_i p_B\big(\textstyle\int_{A_i} f\,d\mu\big) \geqslant \sum_{i=1}^{k} \alpha_i \operatorname{Re}\langle \textstyle\int_{A_i} f\,d\mu,\, v_i\rangle \\
&= \sum_{i=1}^{k} \alpha_i \int_{A_i} \operatorname{Re}\langle f, v_i\rangle\,d\mu \geqslant \sum_{i=1}^{k} \alpha_i \int_{A_i} \big(p_B(f) - \varepsilon\big)\,d\mu \\
&= \int_K \theta p_B(f)\,d\mu - \varepsilon \left\|\theta\chi_K\right\|_1 > 1 + \delta - \varepsilon\mu(K),
\end{aligned}$$

where the last inequality holds because $\int_K p_B(f)\cdot\theta\,d\mu > 1+\delta$, on the one hand, and $\|\theta\chi_K\|_1 \leqslant \|\theta\|_\infty \|\chi_K\|_1 \leqslant \mu(K)$, on the other. Since ε was arbitrary, we obtain a contradiction.

To finish the proof of the Lemma apply [**4**, 3.6]; this result tells us that fundamental $L^1(\mu)$-boundedness implies fundamental L^p-boundedness for every $p \in [1,\infty]$. ∎

Corollary 4. *Let $p \in [1,\infty)$ and let (E,τ) be a Fréchet space or, more generally, a strict (LF)-space. Let τ_0 be the topology of uniform convergence on the sequences from E' that converge to zero in the strong topology $\beta(E',E)$. Then a function $f : \Omega \to E$ is p-integrable with respect to τ if (and only if) f is p-integrable with respect to τ_0.*

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