Bull. Aust. Math. Soc. **99** (2019), 101–113
doi:10.1017/S0004972718001120

MULTIPLICITIES OF EIGENVALUES OF THE DIFFUSION OPERATOR WITH RANDOM JUMPS FROM THE BOUNDARY

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(Received 4 June 2018; accepted 12 September 2018; first published online 28 November 2018)

Abstract

This paper deals with a non-self-adjoint differential operator which is associated with a diffusion process with random jumps from the boundary. Our main result is that the algebraic multiplicity of an eigenvalue is equal to its order as a zero of the characteristic function $\Delta(\lambda)$. This is a new criterion for determining the multiplicities of eigenvalues for concrete operators.

2010 *Mathematics subject classification*: primary 34L15; secondary 47A10, 60J60.

Keywords and phrases: diffusion, eigenvalue, non-self-adjoint, multiplicity.

1. Introduction

We investigate the non-self-adjoint differential operator L in $L^2_w(J,\mathbb{C})$ generated by the differential expression

$$Ly = ly := b_0(x)y'' + b_1(x)y'$$

and

$$\operatorname{dom}(L) := \left\{ y \in L^2_w(J,\mathbb{C}) \;\middle|\; \begin{array}{l} y, y' \in AC[0,1], Ly \in L^2_w(J,\mathbb{C}) \\ y(0) = \int_0^1 y(x)\,d\nu_0(x), \quad y(1) = \int_0^1 y(x)\,d\nu_1(x) \end{array} \right\}.$$

Here ν_0, ν_1 are probability distributions on $J := (0,1)$ and

$$w := -\frac{1}{b_0}, \quad \frac{b_1}{b_0} \in L^1(J,\mathbb{R}), \quad b_0 < 0 \text{ a.e. on } (0,1).$$

It is well known that the operator L is associated with a diffusion process with jumping boundary. It has attracted great interest recently in connection with probability

The work was done at the University of Vienna while the first author was visiting the Fakultät für Mathematik, supported by the China Scholarship Council. This research was also supported by the National Natural Science Foundation of China (Grant No. 11601372) and the Science and Technology Research Project of Higher Education in Hebei Province (Grant No. QN2017044).

theory and practical problems in genetics (see, for example, [1, 3, 4, 6, 7, 10–13] and the references therein). In this process, whenever the boundary of the interval $[0, 1]$ is reached, the diffusion is redistributed in $(0, 1)$ according to the probability distributions ν_0, ν_1, runs again until it hits the boundary, is redistributed and repeats this behaviour forever. Such nonlocal boundary conditions can already be found in the fundamental work of Feller [5], which characterised completely the analytic structure of one-dimensional diffusion processes.

This family of non-self-adjoint differential operators has interesting spectral properties (see, for example, [3, 10–13]). In the case $b_0(x) \equiv -1$, $b_1(x) \equiv 0$, Leung *et al.* [13] discovered that the whole spectrum is real despite the fact that the operator L is non-self-adjoint. In addition, in this case the spectral gap is bounded between the lowest and the second Dirichlet eigenvalues. Recently, Kolb and Krejčiřík [10] analysed the geometric and algebraic multiplicities of the eigenvalues from a purely operator-theoretic perspective in the case $b_0(x) \equiv -1$, $b_1(x) \equiv 0$, $\nu_0 = \nu_1 = \delta_a$, $a \in (0, 1)$, and showed that all the eigenvalues of L are algebraically simple if and only if $a \notin \mathbb{Q}$. Based on this, they studied the basis properties of L. Many of these papers assume that the coefficients of L are constant (and sometimes in addition that $b_1 = 0$), and/or that the measures ν_0, ν_1 coincide and/or that these measures are degenerate measures. In this paper, there are no such assumptions and an interesting result on the multiplicities of the eigenvalues of L is developed.

Let $y_1(x, \lambda)$ and $y_2(x, \lambda)$ be the fundamental solutions of

$$b_0(x)y''(x) + b_1(x)y'(x) = \lambda y(x) \tag{1.1}$$

determined by the initial conditions

$$y_1(0, \lambda) = y_2'(0, \lambda) = 1, \quad y_1'(0, \lambda) = y_2(0, \lambda) = 0.$$

Denote

$$\Delta(\lambda) := \det\begin{pmatrix} \int_0^1 y_1(x, \lambda)\, d\nu_0(x) - 1 & \int_0^1 y_2(x, \lambda)\, d\nu_0(x) \\ \int_0^1 y_1(x, \lambda)\, d\nu_1(x) - y_1(1, \lambda) & \int_0^1 y_2(x, \lambda)\, d\nu_1(x) - y_2(1, \lambda) \end{pmatrix}. \tag{1.2}$$

By direct calculation, λ is an eigenvalue of L if and only if $\Delta(\lambda) = 0$.

We now present the main theorem of this paper.

THEOREM 1.1. *Assume that λ_0 is an eigenvalue of L with algebraic multiplicity $\chi(\lambda_0)$. Let n_0 denote the order of λ_0 as a zero of $\Delta(\lambda)$. Then $\chi(\lambda_0) = n_0$.*

This theorem is useful for identifying the multiplicities of eigenvalues of the operator L. For example, it provides a straightforward method to obtain one of the main results in [10] (see Remark 3.2). Moreover, if $b_0 < 0$ and $b_1 \neq 0$ are constants and $\nu_0 = \nu_1 = \delta_{1/2}$, then, as a consequence of Theorem 1.1, Remark 3.3 shows that all the eigenvalues of L are algebraically simple. This partially answers the last open problem in [10, Section 8].

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2. Basic properties and preliminaries

Let us first recall some notation and definitions.

NOTATION 2.1. Let T be a linear operator in a Hilbert space H. In what follows, $\operatorname{dom}(T)$ and $\ker(T)$ are the domain and kernel of T, respectively; $\sigma(T)$, $\sigma_p(T)$ and $\rho(T)$ denote the spectrum, point spectrum and the resolvent set of T, respectively; $R_\lambda(T) := (T - \lambda I)^{-1}$ for $\lambda \in \rho(T)$ is the resolvent of T.

DEFINITION 2.2. Let T be a linear operator in a Hilbert space H. The smallest integer $p > 0$ such that $\ker(T^p) = \ker(T^{p+1})$ is called the ascent of T and it is denoted by $\alpha(T)$.

DEFINITION 2.3. Let T be a closed linear operator in a Hilbert space H and let λ_0 be an eigenvalue of T. The space $\ker(T - \lambda_0 I)$ is called the eigenspace of T corresponding to λ_0 and its dimension is called the *geometric multiplicity* of λ_0. More generally, the space $\bigcup_{n=1}^{\infty} \ker((T - \lambda_0 I)^n)$ is called the generalised eigenspace of T corresponding to λ_0 and its dimension is the *algebraic multiplicity* of λ_0.

Most of this section is devoted to proving the following proposition, which will be used in the proof of our main theorem.

PROPOSITION 2.4. *The operator L is closed and has a purely discrete spectrum. Moreover, $\alpha(L - \lambda_0 I)$ is finite for any point $\lambda_0 \in \sigma(L)$.*

In order to prove Proposition 2.4, we first consider the differential operator L_0 in $L^2_w(J, \mathbb{C})$ defined by

$$L_0 y := b_0(x)y'' + b_1(x)y',$$
$$\operatorname{dom}(L_0) := \left\{ y \in L^2_w(J, \mathbb{C}) \;\middle|\; \begin{array}{l} y, y' \in AC[0,1], L_0 y \in L^2_w(J, \mathbb{C}), \\ y(0) = y(1) = 0 \end{array} \right\}.$$

It is well known that

$$(R_\lambda(L_0) f)(x) = \int_0^1 G^0_\lambda(x,t) f(t)\, dt \quad \text{for } x \in [0,1], f \in L^2_w(J, \mathbb{C}),$$

where

$$G^0_\lambda(x,t) = \begin{cases} \dfrac{y_2(t,\lambda)[y_2(x,\lambda)y_1(1,\lambda) - y_1(x,\lambda)y_2(1,\lambda)]}{b_0(t)W(t)y_2(1,\lambda)} & \text{for } 0 \le t \le x, \\[2ex] \dfrac{y_2(x,\lambda)[y_2(t,\lambda)y_1(1,\lambda) - y_1(t,\lambda)y_2(1,\lambda)]}{b_0(t)W(t)y_2(1,\lambda)} & \text{for } x \le t \le 1. \end{cases}$$

Here $W(x) = \exp(-\int_0^x b_1(t)/b_0(t)\, dt)$ is the Wronskian of y_1 and y_2. It is obvious that $R_\lambda(L_0) : L^2_w(J, \mathbb{C}) \longrightarrow \operatorname{dom}(L_0)$ is a compact operator for $\lambda \in \rho(L_0) = \mathbb{C} \backslash \sigma(L_0)$, where $\sigma(L_0) = \{\lambda_n\}$ and λ_n are the zeros of the entire function $y_2(1, \lambda)$ (see [9, Section III, Example 6.11]).

Next, we give the formula for the resolvent $R_\lambda(L)$, following which Proposition 2.4 can be proved directly.

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LEMMA 2.5. *For every* $\lambda \in \mathbb{C}\backslash[\sigma(L_0) \cup \sigma_p(L)]$, *the resolvent* $R_\lambda(L)$ *of* L *admits the decomposition*

$$(R_\lambda(L)f)(x) = (R_\lambda(L_0)f)(x) + g_0(x)\int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_0(x) + g_1(x)\int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_1(x) \tag{2.1}$$

for each $f \in L^2_w(J,\mathbb{C})$ *and* $x \in [0,1]$, *where*

$$g_0(x) = \frac{[y_2(1,\lambda) - \int_0^1 y_2(x,\lambda)\,d\nu_1(x)]y_1(x,\lambda) - [y_1(1,\lambda) - \int_0^1 y_1(x,\lambda)\,d\nu_1(x)]y_2(x,\lambda)}{\Delta(\lambda)}$$

and

$$g_1(x) = \frac{[1 - \int_0^1 y_1(x,\lambda)\,d\nu_0(x)]y_2(x,\lambda) + y_1(x,\lambda)\int_0^1 y_2(x,\lambda)\,d\nu_0(x)}{\Delta(\lambda)}.$$

PROOF. Firstly, it is easy to see that $R_\lambda(L)$ is a bounded operator on $L^2_w(J,\mathbb{C})$. In fact, the last two terms of the decomposition represent finite-rank perturbations of the compact operator $R_\lambda(L_0)$. More specifically, for $i = 0, 1$,

$$g_i(x)\int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_i(x) = g_i(x)\int_0^1\int_0^1 G^0_\lambda(x,t)f(t)\,dt\,d\nu_i(x)$$

are continuous on $[0,1]$ for $\lambda \in \mathbb{C}\backslash[\sigma(L_0) \cup \sigma_p(L)]$.

Next, we prove that $R_\lambda(L)f \in \mathrm{dom}(L)$. Indeed,

$$(R_\lambda(L_0)f)(0) = (R_\lambda(L_0)f)(1) = 0$$

yields

$$\begin{aligned}(R_\lambda(L)f)(0) &= \int_0^1 (R_\lambda(L)f)(x)\,d\nu_0(x)\\ &= \frac{y_2(1,\lambda) - \int_0^1 y_2(x,\lambda)\,d\nu_1(x)}{\Delta(\lambda)}\int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_0(x)\\ &\quad + \frac{\int_0^1 y_2(x,\lambda)\,d\nu_0(x)}{\Delta(\lambda)}\int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_1(x)\end{aligned}$$

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and

$$
\begin{aligned}
(R_\lambda(L)f)(1) &= \int_0^1 (R_\lambda(L)f)(x)\,d\nu_1(x)\\
&= \frac{y_2(1,\lambda)\int_0^1 y_1(x,\lambda)\,d\nu_1(x) - y_1(1,\lambda)\int_0^1 y_2(x,\lambda)\,d\nu_1(x)}{\Delta(\lambda)}\\
&\qquad\times \int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_0(x)\\
&\quad + \frac{[1-\int_0^1 y_1(x,\lambda)\,d\nu_0(x)]y_2(1,\lambda) + y_1(1,\lambda)\int_0^1 y_2(x,\lambda)\,d\nu_0(x)}{\Delta(\lambda)}\\
&\qquad\times \int_0^1 (R_\lambda(L_0)f)(x)\,d\nu_1(x).
\end{aligned}
$$

Moreover, it is easy to deduce that

$$b_0(x)(R_\lambda(L)f)'' + b_1(x)(R_\lambda(L)f)' - \lambda(R_\lambda(L)f) = f \in L^2_w(J,\mathbb{C}).$$

Therefore, $R_\lambda(L)$ is a bounded operator from $L^2_w(J,\mathbb{C})$ to $\mathrm{dom}(L)$ and $R_\lambda(L)$ is the right inverse of $L-\lambda$. It remains to show that $R_\lambda(L)$ is the left inverse of $L-\lambda$. In fact, for every $\Psi \in \mathrm{dom}(L)$, write

$$\Psi_0(x) = [R_\lambda(L_0)(L-\lambda)\Psi](x), \quad \widetilde{\Psi} := \Psi_0 - \Psi.$$

Then

$$
\begin{cases}
b_0(x)\widetilde{\Psi}'' + b_1(x)\widetilde{\Psi}' = \lambda\widetilde{\Psi},\\
\widetilde{\Psi}(0) = \int_0^1 \widetilde{\Psi}(x)\,d\nu_0(x) - \int_0^1 \Psi_0(x)\,d\nu_0(x),\\
\widetilde{\Psi}(1) = \int_0^1 \widetilde{\Psi}(x)\,d\nu_1(x) - \int_0^1 \Psi_0(x)\,d\nu_1(x).
\end{cases}
$$

This yields

$$\widetilde{\Psi}(x) = -g_0(x)\int_0^1 \Psi_0(x)\,d\nu_0(x) - g_1(x)\int_0^1 \Psi_0(x)\,d\nu_1(x).$$

Thus, for every $\Psi \in \mathrm{dom}(L)$ and $\lambda \in \mathbb{C}\backslash[\sigma(L_0)\cup\sigma_p(L)]$, it follows from (2.1) that

$$
\begin{aligned}
[R_\lambda(L)(L-\lambda)\Psi](x) &= \Psi_0(x) + g_0(x)\int_0^1 [R_\lambda(L_0)(L-\lambda)\Psi](x)\,d\nu_0(x)\\
&\qquad + g_1(x)\int_0^1 [R_\lambda(L_0)(L-\lambda)\Psi](x)\,d\nu_1(x)\\
&= \Psi_0(x) - \widetilde{\Psi}(x) = \Psi(x).
\end{aligned}
$$

This completes the proof. □

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Remark 2.6. We mention here that in [1, 4], the compactness of the semigroup generated by the operator L has already been studied even in higher dimensions. However, more stringent assumptions on the coefficients are needed and the authors prefer often to work in different spaces.

In addition, let us recall several basic facts.

Lemma 2.7 [15]. *The initial problem consisting of (1.1) and the initial conditions*

$$y(0,\lambda)=h, \quad y'(0,\lambda)=k,$$

where $h,\ k\in\mathbb{C}$, has a unique solution $y(x,\lambda)$ and each of the functions $y(x,\lambda)$ and $y'(x,\lambda)$ is continuous on $[0,1]\times\mathbb{C}$. In particular, the functions $y(x,\lambda)$ and $y'(x,\lambda)$ are entire functions of $\lambda\in\mathbb{C}$.

Remark 2.8. In fact, from [15], the derivative of $y(x,\lambda)$ with respect to λ is given by

$$y'_\lambda(x,\lambda)=\int_0^x \frac{y_2(x,\lambda)y_1(t,\lambda)-y_1(x,\lambda)y_2(t,\lambda)}{b_0(t)\exp(-\int_0^t b_1(s)/b_0(s)\,ds)}y(t,\lambda)\,dt.$$

Remark 2.9. Lemma 2.7 implies that $\Delta(\lambda)$ is an entire function of $\lambda\in\mathbb{C}$.

Remark 2.10. Consider the differential operator $\widetilde{L}_0$ in $L^2_w(J,\mathbb{C})$ defined by

$$\widetilde{L}_0 y := b_0(x)y''+b_1(x)y',$$

$$\operatorname{dom}(\widetilde{L}_0) := \left\{ y\in L^2_w(J,\mathbb{C}) \;\middle|\; \begin{array}{l} y,y'\in AC[0,1], \widetilde{L}_0 y\in L^2_w(J,\mathbb{C}),\\ y(0)=y'(0)=0 \end{array} \right\}.$$

It is obvious that the resolvent set $\rho(\widetilde{L}_0)=\mathbb{C}$. Moreover, by direct calculation, for each $f\in L^2_w(J,\mathbb{C})$ and $x\in[0,1]$,

$$(R_\lambda(\widetilde{L}_0)f)(x)=\int_0^x \frac{y_2(x,\lambda)y_1(t,\lambda)-y_1(x,\lambda)y_2(t,\lambda)}{b_0(t)\exp(-\int_0^t b_1(s)/b_0(s)\,ds)}f(t)\,dt.$$

Therefore, $(R_\lambda(\widetilde{L}_0)f)(x)$ is an entire function of $\lambda\in\mathbb{C}$.

Now we are in a position to prove Proposition 2.4.

Proof of Proposition 2.4. From Lemma 2.5, it follows that the resolvent $R_\lambda(L)$ of L is a compact operator; thus, the operator L is closed and its spectrum is purely discrete. Moreover, for any point $\lambda_0\in\sigma(L)$, it follows from Lemma 2.5 that λ_0 is a pole of $R_\lambda(L)$ of finite order. Thus, the finiteness of $\alpha(L-\lambda_0 I)$ follows from [14, Ch. V, Theorem 10.1]. This proves Proposition 2.4. □

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3. Proof of Theorem 1.1 and remarks

Based on the statements in the previous section, we present the proof of Theorem 1.1 in this section and use this result to solve several problems.

PROOF OF THEOREM 1.1. Let m_0 denote the ascent of the operator $L - \lambda_0 I$, so that $\chi(\lambda_0) = \dim\ker((L - \lambda_0 I)^{m_0})$. Denote the geometric multiplicity of the eigenvalue λ_0 by m. It is obvious that $m \leq 2$. We will focus on the proof of the theorem in the case $m = 1$, since the proof for $m = 2$ can be given with only a slight modification.

When $m = 1$, the proof can be divided into two steps.

Step 1. For λ sufficiently close to λ_0, we first construct two linearly independent solutions $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ of the equation $(l - \lambda I)y = 0$ via the generalised eigenfunctions of λ_0. Recall that $ly = b_0(x)y'' + b_1(x)y'$.

Define a linear operator F on the finite-dimensional space $\ker((L - \lambda_0 I)^{m_0})$ by

$$F = (L - \lambda_0 I) \mid \ker((L - \lambda_0 I)^{m_0}).$$

Then $F^{m_0} = 0$ and $F^{m_0 - 1} \neq 0$, that is, F is nilpotent with index m_0. It follows from [8, Ch. 57, Theorem 2] that there exist functions

$$\eta, F\eta, \ldots, F^{m_0-1}\eta$$

which form a basis of the generalised space $\ker((L - \lambda_0 I)^{m_0})$. Note that in this case $m_0 = \chi(\lambda_0)$. Write

$$\xi_{0,1} := F^{m_0-1}\eta, \quad \xi_{1,1} := F^{m_0-2}\eta, \ldots, \xi_{m_0-1,1} := \eta.$$

Select another solution $\xi_{0,2}$ of the equation $(l - \lambda_0 I)y = 0$ such that $\xi_{0,1}$ and $\xi_{0,2}$ are fundamental solutions of $(l - \lambda_0 I)y = 0$. For $\lambda \in \mathbb{C}$, define

$$\phi_1(x, \lambda) = \sum_{k=0}^{m_0-1} (\lambda - \lambda_0)^k \xi_{k,1}(x) + (\lambda - \lambda_0)^{m_0} (R_\lambda(\widetilde{L}_0)\eta)(x),$$

$$\phi_2(x, \lambda) = \xi_{0,2}(x) + (\lambda - \lambda_0)(R_\lambda(\widetilde{L}_0)\xi_{0,2})(x).$$

Note that $\phi_i(x, \lambda_0) = \xi_{0,i}(x)$ for $i = 1, 2$. We will show that $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ are linearly independent solutions of the equation $(l - \lambda I)y = 0$. In fact,

$$\begin{aligned}
&((l - \lambda I)\phi_1)(x, \lambda) \\
&\quad = \sum_{k=1}^{m_0-1} (\lambda - \lambda_0)^k ((l - \lambda_0 I)\xi_{k,1})(x) - \sum_{k=0}^{m_0-1} (\lambda - \lambda_0)^{k+1} \xi_{k,1}(x) + (\lambda - \lambda_0)^{m_0} \eta(x) \\
&\quad = \sum_{k=1}^{m_0-1} (\lambda - \lambda_0)^k \xi_{k-1,1}(x) - \sum_{k=0}^{m_0-1} (\lambda - \lambda_0)^{k+1} \xi_{k,1}(x) + (\lambda - \lambda_0)^{m_0} \xi_{m_0-1,1}(x) = 0
\end{aligned}$$

and

$$((l - \lambda I)\phi_2)(x, \lambda) = ((l - \lambda I)\xi_{0,2})(x) + (\lambda - \lambda_0)\xi_{0,2}(x) = 0.$$

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Moreover, since $\xi_{0,1}$ and $\xi_{0,2}$ are linearly independent solutions of $(l-\lambda_0 I)y=0$,

$$\det\begin{pmatrix}\phi_1(0,\lambda_0) & \phi_2(0,\lambda_0)\\ \phi_1'(0,\lambda_0) & \phi_2'(0,\lambda_0)\end{pmatrix}=\det\begin{pmatrix}\xi_{0,1}(0) & \xi_{0,2}(0)\\ \xi_{0,1}'(0) & \xi_{0,2}'(0)\end{pmatrix}\neq 0.$$

As a consequence of Remark 2.10, $\phi_i(0,\lambda)$ and $\phi_i'(0,\lambda)$ for $i=1,2$ are entire functions of $\lambda\in\mathbb{C}$. Hence, there exists a number $\delta>0$ such that $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are linearly independent solutions of $(l-\lambda I)y=0$ for $|\lambda-\lambda_0|<\delta$.

Step 2. Based on Step 1, when $|\lambda-\lambda_0|<\delta$,

$$\begin{pmatrix}y_1(x,\lambda)\\ y_2(x,\lambda)\end{pmatrix}=\begin{pmatrix}b_{11}(\lambda) & b_{12}(\lambda)\\ b_{21}(\lambda) & b_{22}(\lambda)\end{pmatrix}\begin{pmatrix}\phi_1(x,\lambda)\\ \phi_2(x,\lambda)\end{pmatrix},\quad \det\begin{pmatrix}b_{11}(\lambda) & b_{12}(\lambda)\\ b_{21}(\lambda) & b_{22}(\lambda)\end{pmatrix}\neq 0. \tag{3.1}$$

From (3.1) and the definitions of ϕ_1, ϕ_2 together with the fact that $\xi_{k,1}\in\operatorname{dom}(L-\lambda_0 I)$ for $k=0,1,\ldots,m_0-1$, it follows that

$$\begin{aligned}\Delta(\lambda)&=\det\begin{pmatrix}\int_0^1 y_1(x,\lambda)\,d\nu_0(x)-y_1(0,\lambda) & \int_0^1 y_2(x,\lambda)\,d\nu_0(x)-y_2(0,\lambda)\\ \int_0^1 y_1(x,\lambda)\,d\nu_1(x)-y_1(1,\lambda) & \int_0^1 y_2(x,\lambda)\,d\nu_1(x)-y_2(1,\lambda)\end{pmatrix}\\ &=(\lambda-\lambda_0)^{m_0}\det(g_{i,j}(\lambda))\det\begin{pmatrix}b_{11}(\lambda) & b_{21}(\lambda)\\ b_{12}(\lambda) & b_{22}(\lambda)\end{pmatrix},\end{aligned}$$

where $|\lambda-\lambda_0|<\delta$, and, for $i=1,2$,

$$\begin{aligned}g_{i,1}(\lambda)&=\int_0^1(R_\lambda(\widetilde{L}_0)\eta)(x)\,d\nu_{i-1}(x)-(R_\lambda(\widetilde{L}_0)\eta)(i-1),\\ g_{i,2}(\lambda)&=(\lambda-\lambda_0)\Big[\int_0^1(R_\lambda(\widetilde{L}_0)\xi_{0,2})(x)\,d\nu_{i-1}(x)-(R_\lambda(\widetilde{L}_0)\xi_{0,2})(i-1)\Big]\\ &\qquad+\int_0^1\xi_{0,2}(x)\,d\nu_{i-1}(x)-\xi_{0,2}(i-1).\end{aligned}$$

Recall that in this case $m_0=\chi(\lambda_0)$. Thus, in order to show that the order of λ_0 as a zero of $\Delta(\lambda)$ is equal to $\chi(\lambda_0)$, it is sufficient to prove that $\det(g_{i,j}(\lambda_0))\neq 0$, since $g_{i,j}(\lambda)$ are entire functions of $\lambda\in\mathbb{C}$. Otherwise, there exists a constant c such that

$$\begin{pmatrix}\int_0^1(R_{\lambda_0}(\widetilde{L}_0)\eta)(x)\,d\nu_0(x)-(R_{\lambda_0}(\widetilde{L}_0)\eta)(0)\\ \int_0^1(R_{\lambda_0}(\widetilde{L}_0)\eta)(x)\,d\nu_1(x)-(R_{\lambda_0}(\widetilde{L}_0)\eta)(1)\end{pmatrix}=c\begin{pmatrix}\int_0^1\xi_{0,2}(x)\,d\nu_0(x)-\xi_{0,2}(0)\\ \int_0^1\xi_{0,2}(x)\,d\nu_1(x)-\xi_{0,2}(1)\end{pmatrix}.$$

Put $u(x)=(R_{\lambda_0}(\widetilde{L}_0)\eta)(x)-c\xi_{0,2}(x)$. Then the above equation implies that

$$\int_0^1 u(x)\,d\nu_0(x)=u(0)\quad\text{and}\quad\int_0^1 u(x)\,d\nu_1(x)=u(1).$$

Therefore,

$$(l-\lambda_0 I)u=\eta\in\ker((L-\lambda_0 I)^{m_0}) \tag{3.2}$$

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and hence

$$u \in \ker((L-\lambda_0 I)^{m_0+1}) = \ker((L-\lambda_0 I)^{m_0}).$$

This implies that there exist constants α_i such that $u = \sum_{i=0}^{m_0-1} \alpha_i F^i \eta$. Thus,

$$(l-\lambda_0 I)u = \sum_{i=0}^{m_0-1} \alpha_i (l-\lambda_0 I) F^i \eta = \sum_{i=0}^{m_0-2} \alpha_i F^{i+1}\eta = \sum_{i=1}^{m_0-1} \alpha_{i-1} F^i \eta.$$

This, together with (3.2), yields $\eta = \sum_{i=1}^{m_0-1} \alpha_{i-1} F^i \eta$, which contradicts the linear independence of $\eta, F\eta, \ldots, F^{m_0-1}\eta$. Thus, $\det(g_{i,j}(\lambda_0)) \neq 0$ and the statement of Theorem 1.1 in the case of $m = 1$ is proved.

Now we turn to the case $m = 2$. We only need to make slight modifications to the solutions ϕ_1, ϕ_2 and $(g_{i,j}(\lambda))$. Note that it follows from [8, Ch. 57, Theorem 2] that there exist functions $\eta_1, \eta_2 \in \ker((L-\lambda_0 I)^{m_0})$ such that

$$\eta_1, F\eta_1, \ldots, F^{q_1-1}\eta_1, \eta_2, F\eta_2, \ldots, F^{q_2-1}\eta_2$$

form a basis of the generalised space $\ker((L-\lambda_0 I)^{m_0})$, where q_1, q_2 are such that $q_1 + q_2 = \chi(\lambda_0)$, $m_0 = q_1 \geq q_2 > 0$ and $F^{q_1}\eta_1 = F^{q_2}\eta_2 = 0$. In this case, denote

$$\xi_{0,i} := F^{q_i-1}\eta_i, \quad \xi_{1,i} := F^{q_i-2}\eta_i, \ldots, \xi_{m_0-1,i} := \eta_i \quad \text{for } i = 1, 2.$$

Hence, $\xi_{0,1}$ and $\xi_{0,2}$ are fundamental solutions of $(l-\lambda_0 I)y = 0$.

For $\lambda \in \mathbb{C}$, define

$$\phi_i(x,\lambda) = \sum_{k=0}^{q_i-1} (\lambda-\lambda_0)^k \xi_{k,i}(x) + (\lambda-\lambda_0)^{m_0} (R_\lambda(\widetilde{L}_0)\eta_i)(x) \quad \text{for } i = 1, 2.$$

Note that $\phi_i(x,\lambda_0) = \xi_{0,i}(x)$ for $i = 1, 2$. By a process similar to that in the case $m = 1$,

$$g_{i,j}(\lambda) = \int_0^1 (R_\lambda(\widetilde{L}_0)\eta_j)(x)\, d\nu_{i-1}(x) - (R_\lambda(\widetilde{L}_0)\eta_j)(i-1).$$

Similarly, $\det(g_{i,j}(\lambda_0)) \neq 0$. Otherwise, there exists a constant c such that

$$\begin{pmatrix} \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(x)\, d\nu_0(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(0) \\ \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(x)\, d\nu_1(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(1) \end{pmatrix} = c \begin{pmatrix} \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(x)\, d\nu_0(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(0) \\ \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(x)\, d\nu_1(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(1) \end{pmatrix}.$$

If we put $u(x) = (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(x) - c(R_{\lambda_0}(\widetilde{L}_0)\eta_2)(x)$, then $\int_0^1 u(x)\, d\nu_0(x) = u(0)$ and $\int_0^1 u(x)\, d\nu_1(x) = u(1)$. Hence,

$$(l-\lambda_0 I)u = \eta_1 - c\eta_2 \in \ker((L-\lambda_0 I)^{m_0})$$

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and

$$u \in \ker((L-\lambda_0 I)^{m_0+1}) = \ker((L-\lambda_0 I)^{m_0}).$$

This implies that there exist constants α_i, β_i such that $u = \sum_{i=0}^{q_1-1} \alpha_i F^i \eta_1 + \sum_{i=0}^{q_2-1} \beta_i F^i \eta_2$. Hence,

$$\eta_1 - c\eta_2 = (l-\lambda_0 I)u = \sum_{i=1}^{q_1-1} \alpha_{i-1} F^i \eta_1 + \sum_{i=1}^{q_2-1} \beta_{i-1} F^i \eta_2,$$

which contradicts the linear independence of $\eta_i, F\eta_i, \dots, F^{q_i-1}\eta_i$ for $i = 1, 2$. This completes the proof of Theorem 1.1. □

REMARK 3.1. From the above proof, one can see that Theorem 1.1 also holds even if $1/b_0, b_1/b_0 \in L^1(J, \mathbb{C})$.

Based on Theorem 1.1, we conclude this paper with three remarks on two concrete eigenvalue problems which have been treated in [2, 10, 12].

REMARK 3.2. Consider the eigenvalue problem with coefficients $b_0 \equiv -1$, $b_1 \equiv 0$ and $\nu_0 = \nu_1 = \delta_a$, $a \in (0, 1)$, that is,

$$\begin{cases} -y''(x) = \lambda y(x), & x \in (0,1), \\ y(0) = y(a) = y(1), & a \in (0,1). \end{cases} \tag{3.3}$$

In [10, Theorem 1], Kolb and Krejčiřík showed that all the eigenvalues of the problem (3.3) are algebraically simple if and only if $a \notin \mathbb{Q}$. Based on Theorem 1.1, this interesting result can be obtained from a different perspective.

In fact, from (1.2), λ is an eigenvalue of the problem (3.3) if and only if

$$\Delta(\lambda) = -\frac{4}{\sqrt{\lambda}} \sin\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} = 0.$$

Furthermore,

$$\begin{aligned}\Delta'(\lambda) = \frac{2}{\lambda^{3/2}} \sin\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} - \frac{1-a}{\lambda} \cos\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} \\ - \frac{a}{\lambda} \sin\frac{\sqrt{\lambda}(1-a)}{2} \cos\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} - \frac{1}{\lambda} \sin\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \cos\frac{\sqrt{\lambda}}{2}.\end{aligned}$$

Suppose that $\widehat{\lambda} \neq 0$. It is easy to see that $\Delta(\widehat{\lambda}) = \Delta'(\widehat{\lambda}) = 0$ if and only if

$$\sin\frac{\sqrt{\widehat{\lambda}}(1-a)}{2} = \sin\frac{\sqrt{\widehat{\lambda}}a}{2} = \sin\frac{\sqrt{\widehat{\lambda}}}{2} = 0,$$

that is,

$$\widehat{\lambda} = (2m\pi)^2 = \left(\frac{2n\pi}{a}\right)^2 = \left(\frac{2l\pi}{1-a}\right)^2 \quad \text{for } m, n, l \in \mathbb{N} := \{1, 2, \dots\}. \tag{3.4}$$

Hence, for each $\widehat{\lambda} \neq 0$ which satisfies $\Delta(\widehat{\lambda}) = \Delta'(\widehat{\lambda}) = 0$, direct calculation yields $\Delta''(\widehat{\lambda}) = 0$ and $\Delta'''(\widehat{\lambda}) = (3a^2 - 3a)/8\widehat{\lambda}^2 \neq 0$. Obviously, $\Delta'(0) = a(a-1)/2 \neq 0$. Thus, it follows from Theorem 1.1 that the algebraic multiplicity of each eigenvalue of the problem (3.3) is either one or three. Moreover, since (3.4) implies that $a = n/m = 1 - l/m \in \mathbb{Q}$, we can easily conclude that all the eigenvalues of the problem (3.3) are algebraically simple if and only if $a \notin \mathbb{Q}$.

REMARK 3.3. Consider the eigenvalue problem with constant coefficients $b_0 < 0$, $b_1 \in \mathbb{R}$ and $\nu_0 = \nu_1 = \delta_{1/2}$, that is,

$$\begin{cases} b_0 y''(x) + b_1 y'(x) = \lambda y(x), & x \in (0,1), \\ y(0) = y(\tfrac{1}{2}) = y(1). \end{cases} \tag{3.5}$$

Assume that $b_1 \neq 0$. It follows from Theorem 1.1 that each eigenvalue of the problem (3.5) is algebraically and geometrically simple. This partially answers the last open question posed by Kolb and Krejčiřík [10, Section 8].

In fact, under the transformation $v(x) = \exp(b_1 x/2b_0) y(x)$, the problem (3.5) is equivalent to the eigenvalue problem

$$\begin{cases} -v''(x) + qv(x) = -\lambda v(x)/b_0, & x \in (0,1), \\ v(0) = Av(\tfrac{1}{2}) = A^2 v(1), \end{cases} \tag{3.6}$$

where $q = \frac{1}{4}(b_1/b_0)^2$ and $A = \exp(-b_1/4b_0)$. Let $v_1(x,\lambda)$ and $v_2(x,\lambda)$ be the fundamental solutions of the differential equation in (3.6) with the initial conditions

$$v_1(0,\lambda) = v_2'(0,\lambda) = 1, \quad v_2(0,\lambda) = v_1'(0,\lambda) = 0, \quad \lambda \in \mathbb{C}.$$

Then

$$v_1(x,\lambda) = \cos\Big(\sqrt{-\frac{\lambda}{b_0} - q}\,x\Big), \quad v_2(x,\lambda) = \frac{\sin\Big(\sqrt{-\frac{\lambda}{b_0} - q}\,x\Big)}{\sqrt{-\frac{\lambda}{b_0} - q}}.$$

It follows that λ is an eigenvalue of the problem (3.5) or (3.6) if and only if

$$\Delta_1(\lambda) = \det\begin{pmatrix} Av_1(\tfrac{1}{2},\lambda) - 1 & Av_2(\tfrac{1}{2},\lambda) \\ 1 - A^2 v_1(1,\lambda) & -A^2 v_2(1,\lambda) \end{pmatrix} = 0.$$

For simplicity, write $u = -(\lambda/b_0) - q$ and

$$\widetilde{\Delta}_1(u) := \Delta_1(-b_0(u+q)) = -2A^2 \frac{\sin\frac{1}{2}\sqrt{u}}{\sqrt{u}}\Big(\frac{A^2+1}{2A} - \cos\frac{\sqrt{u}}{2}\Big).$$

When $b_1 \neq 0$, it is obvious that $(A^2+1)/2A > 1$. Let u_n be the zeros of $\Delta_1(u)$. By direct calculation, $\{u_n\} = \{u_{n,1}\} \cup \{u_{n,2}\} \cup \{u_{n,3}\}$, where

$$u_{n,1} = (2n\pi)^2, \quad u_{n,2} = (4n\pi - 2ir)^2 \quad \text{for } n \in \mathbb{N},$$
$$u_{n,3} = (4n\pi + 2ir)^2 \quad \text{for } n \in \mathbb{N}_0 := \{0,1,2,\ldots\}.$$

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Here $r > 0$ and $\cosh r = (A^2 + 1)/2A$, that is, $r = -b_1/4b_0$. Hence, the eigenvalues λ_n of the problem (3.5) or (3.6) are $\{\lambda_n\} = \{\lambda_{n,1}\} \cup \{\lambda_{n,2}\} \cup \{\lambda_{n,3}\}$, where

$$\lambda_{n,1} = -4b_0 n^2\pi^2 - \frac{b_1^2}{4b_0}, \quad \lambda_{n,2} = -16b_0 n^2\pi^2 - 2b_1 n\pi i \quad \text{for } n \in \mathbb{N}, \tag{3.7}$$

$$\lambda_{n,3} = -16b_0 n^2\pi^2 + 2b_1 n\pi i \quad \text{for } n \in \mathbb{N}_0. \tag{3.8}$$

For each eigenvalue λ_n of the problem (3.5), one can easily obtain $\Delta_1'(\lambda_n) \neq 0$. In order to use Theorem 1.1 to show that each eigenvalue of the problem (3.5) is algebraically simple, it is sufficient to show that $\Delta_1(\lambda) \equiv \Delta(\lambda)$. Note that $\Delta(\lambda)$ is the characteristic function defined in (1.2). Write

$$\widetilde{y}_1(x,\lambda) := \exp\Big(-\frac{b_1 x}{2b_0}\Big)v_1(x,\lambda), \quad \widetilde{y}_2(x,\lambda) := \exp\Big(-\frac{b_1 x}{2b_0}\Big)v_2(x,\lambda).$$

Then $\widetilde{y}_1(x,\lambda)$ and $\widetilde{y}_2(x,\lambda)$ are solutions of the differential equation in (3.5) determined by the initial conditions

$$\widetilde{y}_1(0,\lambda) = 1, \quad \widetilde{y}_1'(0,\lambda) = -\frac{b_1}{2b_0}, \quad \widetilde{y}_2(0,\lambda) = 0, \quad \widetilde{y}_2'(0,\lambda) = 1, \quad \lambda \in \mathbb{C}.$$

Thus, $\widetilde{y}_1(x,\lambda) = y_1(x,\lambda) - (b_1/2b_0)y_2(x,\lambda)$, $\widetilde{y}_2(x,\lambda) = y_2(x,\lambda)$ and

$$\Delta_1(\lambda) = \det\begin{pmatrix} \widetilde{y}_1(\frac{1}{2},\lambda) - 1 & \widetilde{y}_2(\frac{1}{2},\lambda) \\ 1 - \widetilde{y}_1(1,\lambda) & -\widetilde{y}_2(1,\lambda) \end{pmatrix} = \det\begin{pmatrix} y_1(\frac{1}{2},\lambda) - 1 & y_2(\frac{1}{2},\lambda) \\ 1 - y_1(1,\lambda) & -y_2(1,\lambda) \end{pmatrix} = \Delta(\lambda).$$

Therefore, each eigenvalue of the problem (3.5) is algebraically and thus geometrically simple.

REMARK 3.4. Denote the spectral gap of the problem (3.5) by $\gamma_1(\delta_{1/2})$, that is,

$$\gamma_1(\delta_{1/2}) := \inf\{\operatorname{Re}\lambda \mid \lambda \text{ is an eigenvalue of the problem (3.5) and } \lambda \neq 0\}.$$

If $b_1 = 0$, then $\lambda_n = -4b_0 n^2\pi^2$ for $n \in \mathbb{N}_0$. This, together with (3.7) and (3.8), yields

$$\gamma_1(\delta_{1/2}) = \begin{cases} -4b_0\pi^2 - b_1^2/4b_0 & \text{when } |b_1| \leq -4\sqrt{3}b_0\pi, \\ -16b_0\pi^2 & \text{when } |b_1| > -4\sqrt{3}b_0\pi, \end{cases}$$

which is already derived in [2, 12] by different approaches.

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