SELF-INJECTIVE RINGS

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Historically, the first example of a ring of quotients was the quotient field of an integral domain. Later on, conditions were found under which a noncommutative integral domain has a quotient division ring. More recently, R.E. Johnson [4], Y. Utumi [5], and G.D. Findlay and J. Lambek [3] have discussed the existence and structure of a maximal ring of quotients of any ring.

The present paper uses the methods of Findlay and Lambek to recast the results of Johnson on the quotient ring of a ring with zero singular ideal. It is also shown that such a ring has a unique left-right maximal ring of quotients.

For simplicity, each ring R considered is assumed to have a multiplicative identity element and each R-module is assumed to be unitary. If C and M are right R-modules, then M is called an essential extension of C if $M \supset C$ and $xR \cap C \neq 0$ for every nonzero x in M. If $M \supset C$, then M is called an injective extension of C if every homomorphism of any module A into M can be extended to a homomorphism of B into M where B is any module containing A. It is known [1] that every R-module C has a unique (up to isomorphism over C) maximal essential extension E that at the same time is the unique minimal injective extension of C.

DEFINITION 1. The module M is called a rational extension of C if: i. $M \supset C$ as right R-modules, and ii. If $M \supset B \supset C$ and $f \in \mathrm{Hom}_R(B, M)$, with $f(C) = 0$ then $f = 0$.

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An equivalent definition of a rational extension is that each $f \in \mathrm{Hom}_R(B,M)$, where $M \supset B \supset C$, can be extended to a unique irreducible partial homomorphism of M into M [3]. It is clear that a rational extension of C is an essential extension.

THEOREM 1[1]. The intersection of all kernels of endomorphisms of E which contain C is the unique (up to isomorphism) maximal rational extension of C.

Proof. Let M be a rational extension of C. The identity mapping of C into E can be extended to an R-homomorphism of M into E. This homomorphism is an isomorphism since its kernel must have zero intersection with C and M is an essential extension of C. Therefore M can be considered as a submodule of E. If $f \in \mathrm{Hom}_R(E,E)$, where the kernel of f contains C, let $K = \{ x \in M \mid f(x) \in C \}$. If $f(M) \neq 0$ then $f(M) \cap C \neq 0$ and hence $f(K) \neq 0$. But $f(K) = 0$ since M is rational over C. Therefore $f(M) = 0$, that is M is contained in the kernel of f.

Let $\overline{M}$ be the intersection of all kernels of endomorphisms of E which contain C. If $\overline{M}$ is not a rational extension of C then there exists a submodule B, $C \subset B \subset \overline{M}$; and $f \in \mathrm{Hom}_R(B,\overline{M})$, $f(B) \neq 0$ but $f(C) = 0$. Since E is R-injective, f can be extended to an endomorphism of E with kernel not containing M. This is a contradiction. Thus $\overline{M}$ is the maximal rational extension of C.

DEFINITION 2. A submodule N of a right R-module M is called large if M is an essential extension of N. A right ideal A of a ring R is called large if A is large considered as a right submodule of R.

The proofs of the following two lemmas are either similar to or can be found in the proofs of Theorem 1 and Theorem 2 in [4]. Hence we state them here without proof.

LEMMA 1.

i. If N_1 and N_2 are large submodules of M then so is $N_1 \cap N_2$.

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[1] It is stated without proof in [3, 2.1].

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ii. If N is a large submodule then $T_m = \{r \in R \mid mr \in N\}$ is a large right ideal of R for any $m \in M$.

iii. If A is any submodule of M then there exists a submodule B such that $A \cap B = 0$ and $A + B$ is large.

An element x of C is called a singular element if the annihilator N_x of x in R is a large right ideal.

LEMMA 2. The union $J_R(C)$ of all singular elements of C is a submodule of C.

It is an immediate consequence from the definitions that $J_R(M) = 0$ if and only if $J_R(C) = 0$ where M is any essential extension of C. If we consider C as an abelian group and R as the ring of integers then $J_R(C)$ is the torsion subgroup of C.

THEOREM 2. $J_R(C) = 0$ if and only if each $f \in Hom_R(L, C)$ has a unique irreducible extension in R where L is any large right ideal of R.

Proof. Let f_1, f_2 be irreducible extensions of $f \in Hom_R(L, C)$. Suppose $(f_1 - f_2)x \neq 0$, x being in the intersection of the domains of f_1 and f_2. Then $(f_1 - f_2)x = c$, $c \in C$, $c \neq 0$. Since the intersection of the domains of f_1 and f_2 contains L and hence is large, by lemma 1 there exists a large ideal K of R such that $xK \subset$ domain of f. If $J_R(C) = 0$ then $cK \neq 0$. But $cK = [(f_1 - f_2)x]K = (f_1 - f_2)(xK) = 0$. Therefore $(f_1 - f_2)x = 0$ for all x in the intersection of the domains of f_1 and f_2. By the irreducibility of f_1 and f_2, $f_1 = f_2$ [3].

Conversely, suppose every $f \in Hom_R(L, C)$ has a unique irreducible extension for every large right ideal L of R. Given any element x of C, x can be considered as an irreducible homomorphism of R into C, therefore, if $xL = 0$, where L is a large right ideal of R, then $x = 0$. Hence $J_R(C) = 0$.

If we consider the additive group R^+ of R as a right R-module, then $J_R(R^+)$ is an ideal of R, called the (right) singular ideal in [4] and [5]. It may be shown that $J_R(R^+) = 0$ if and only if R is a rational extension of each large right ideal of R (considered as an R-module).

THEOREM 3. If $J_R(C) = 0$ then every essential extension of C is a rational extension of C.

Proof. Let M be an essential extension of C and $f \in Hom_R(B, M)$, where $M \supset B \supset C$. If $f(C) = 0$ but $f(B) \neq 0$, then there exists some $b \in B$ such that $f(b) \in C$ and $f(b) \neq 0$. The set $T_b = \{r \in R \mid br \in C\}$ is a large right ideal of R such that $f(b)T_b = f(bT_b) = 0$. This contradicts the assumption that $J_R(C) = 0$, and therefore $f = 0$ whenever $f(C) = 0$. This proves the theorem.

LEMMA 3. If C is a rational extension of C_1 and C_2, then:

i. C is a rational extension of $C_1 \cap C_2$.

ii. If $f \in Hom_R(C_2, C)$ and $D = \{x \in C_2 \mid f(x) \in C_1\}$, then C is a rational extension of D.

Proof. See Propositions 1.1 and 1.2 in [3].

Let M be the unique maximal rational extension of C and let S be the set of all irreducible fractional homomorphisms of M into M [3]. Thus, $f \in S$ if M is rational over the domain of f.

THEOREM 4. S is a ring with an identity element and it contains a subring isomorphic with the ring $Hom_R(C, C)$.

Proof. According to [3; Section 4], $S = Hom_R(M, M)$ and therefore S is a ring with an identity element. Since each $f \in Hom_R(C, C)$ has a unique extension in S, $Hom_R(C, C)$ is isomorphic to a subring of S.

COROLLARY. If $J_R(C) = 0$ then $S = Hom_R(E, E)$ where E is the maximal essential extension of C.

Proof. If $J_R(C) = 0$ then $M = E$.

THEOREM 5. If $J_R(C) = 0$ then S is a regular ring [7] and self-injective (considered as a right S-module).

Proof. The zero singular submodule of C implies $J_R(M) = 0$. Therefore M is a rational extension of any large submodule of M. So if $x \in S$, let N be a submodule of M such that $N \cap N_x = 0$ and $N + N_x$ is large where $N_x = \{m \in M \mid xm = 0\}$. Now x is an isomorphism on N, so there exists a mapping y such that $y(xa) = a$ for all a in N. Thus y is an R-homomorphism of xN into M. By the injectivity of M, y can be extended to an element $\bar{y}$ in S. Since $\bar{y}(xt) = 0$ for all t in N_x, $(x\bar{y}x - x)w = 0$ for all w in $N + N_x$. Since $N + N_x$ is large, therefore $x\bar{y}x = x$. Thus S is a regular ring.

To prove S is self-injective, we must show that for each right ideal A of S and $\varphi \in \mathrm{Hom}_S(A,S)$ there exists $z \in S$ such that $\varphi a = za$ for all a in A [2, Theorem 3.2]. If $AM = \{\sum_i a_i x_i \mid a_i \in A,\ x_i \in M\}$, define the mapping η of AM into M as follows:

$$\eta(\sum_i a_i x_i) = \sum_i (\varphi a_i) x_i .$$

This mapping is well defined, for if $\sum_i a_i x_i = 0$ choose e as the idempotent generator of the ideal $a_1 S + a_2 S + \ldots\ldots + a_n S$. This can be done since S is a regular ring [7, lemma 15]. Then $ea_i = a_i$, $e \in A$ and $\sum_i (\varphi a_i) x_i = \sum_i (\varphi e) a_i x_i = 0$. Clearly η is an R-homomorphism of AM into M. Hence η can be extended to an element z of S and $(\varphi(a))x = \eta(ax) = z(ax) = (za)x$, for all x in M. Thus $\varphi a = za$ for all a in A. This completes the proof.

Suppose C is a ring and C is a left C and right R bimodule. Then C can be considered as a subring of S if for all $c \in C$, $cC = 0$ implies $c = 0$ (C is left-faithful).

THEOREM 6. If C is a left-faithful ring and a left C right R bimodule then S is a rational extension of C as a right C-module.

Proof. Let $f \in \mathrm{Hom}_C(B,S)$ where $C \subset B \subset S$ and $f(C) = 0$.

Suppose $f \neq 0$, then there exists $b \in B$ such that $f(b) \neq 0$. M is rational over the intersection of $b^{-1}C$ and C and hence is rational over T where $T = \{x$ is in the intersection of $b^{-1}C$ and $C \mid bx \in C\}$. Hence $f(b)T \neq 0$. However $f(b)T = f(bT) = 0$. Therefore $f = 0$ and S is a rational extension of C as a right C-module.

A ring Q is called a right ring of quotients of a ring C if $Q \supset C$ and Q is rational over C as a right C-module [3]. This definition coincides with R.E. Johnson's definition [4] in the case when $J_C(C) = 0$.

LEMMA 4. Suppose C is a ring and C is rational over R as a right R-module and $J_R(C) = 0$, then any rational extension of C as C-module is a rational extension of C as R-module.

Proof. Let K be a rational extension of C as a right C-module. K is an R-extension of C. Since $J_R(C) = 0$, we only have to show that K is an R-essential extension of C. For any $k \in K$, $k \neq 0$, there exists c, c' in C, $c' \neq 0$, and $kc = c'$.

Consider the large right ideal T of R, where $T = \{r \in R \mid cr \in R\}$, then $c'T \neq 0$. In other words $kR \cap C \neq 0$, thus K is an essential extension of C as a right R-module.

THEOREM 7. If C is a ring and C is rational over R as a right R-module and $J_R(C) = 0$, then C can be imbedded into a right self-injective ring S where S is a regular ring and the right maximal ring of quotients of C.

Proof. Any ring of quotients of C can be considered as a rational extension of C as a right R-module by the above lemma. We already showed that $S = \mathrm{Hom}_R(E, E)$ is a right ring of quotients of C. The rest of the proof follow from Theorem 5.

Let R be a ring with an identity element. Consider the additive group R^+ of R as a right R-module.

LEMMA 5. If M is an essential extension of R^+ (as a right R-module) then M is rational over R^+ if and only if R^+ is rational over T_m for all m in M, where $T_m = \{r \in R \mid mr \in R^+\}$.

Proof. Suppose R^+ is rational over T_m for all m in M. Let $f \in \mathrm{Hom}_R(B, M)$, $M \supset B \supset R^+$ and $f(R^+) = 0$. If $f \neq 0$ then $f(B) \cap R^+ \neq 0$. Hence there exists $b \in B$, $r \in R$, such that $r \neq 0$ and $f(b) = r$. Now $f(b)T_b = rT_b \neq 0$, for R is rational over T_b. But $bT_b \subset R^+$ and $f(b)T_b = f(bT_b) = 0$. Therefore $f = 0$ and M is rational over R^+.

Conversely, let M be a rational extension of R^+ and let $f \in \mathrm{Hom}_R(B, R^+)$ where $R^+ \supset B \supset T_m$ for any m in M and $f(T_m) = 0$. Consider the submodule $R^+ + mB$. Define $\varphi(r + mb) = f(b)$. If $r + mb = 0$, $b \in T_m$ and hence $f(b) = 0$. Thus the mapping φ is well defined. But $\varphi(R^+) = 0$, therefore $\varphi = 0$ and $f = 0$. Thus R^+ is rational over T_m for all m in M.

Let L be the left maximal ring of quotients of R and Q be the right maximal ring of quotients of R.

Consider L as a right R-module and let T be the set of all elements x in L for which there exists a right ideal A of R such that $xA \subset R$ and R is rational over A.

THEOREM 8. T is a subring of L and T is the unique (up to isomorphism over R) left and right maximal ring of quotients of R.

Proof. If x and y are in T, let A_x and A_y be the right ideals of R such that $xA_x \subset R$, $yA_y \subset R$ and R is rational over A_x and A_y. Since R is rational over $A_x \cap A_y$ and $(x - y)(A_x \cap A_y) \subset R$, x - y is in T. Also R is rational over D, where $D = \{r \in A_y \mid yr \in A_x\}$ and $xyD \subset R$. Therefore xy is in T. This proves T is a subring of L. It is clear that T contains the identity element of L. If $x \in T$, $A_x \subset T_x$. Thus R is rational over T_x for all x in T. By lemma 5, T is rational over R and hence a right ring of quotients of R.

To show T is the left and right maximal ring of quotients of R, let K be a left and right ring of quotients of R. Then $K \subset L$ and also $K \subset T$, since for every x of K there exists a right ideal T_x of R such that $xT_x \subset R$ and R is rational over T_x.

If R is a commutative ring, any left R-module can be considered as a right R-module. In this case, the left maximal ring of quotients of R coincides with the right maximal ring of quotients of R.

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