THE CONVEX FUNCTION DETERMINED BY A MULTIFUNCTION

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We shall show how each multifunction on a Banach space determines a convex function that gives a considerable amount of information about the structure of the multifunction. Using standard results on convex functions and a standard minimax theorem, we strengthen known results on the local boundedness of a monotone operator, and the convexity of the interior and closure of the domain of a maximal monotone operator. In addition, we prove that any point surrounded by (in a sense made precise) the convex hull of the domain of a maximal monotone operator is automatically in the interior of the domain, thus settling an open problem.

INTRODUCTION

We shall assume throughout this paper that $E$ is a nontrivial Banach space. We shall show how each multifunction $S : E \to 2^{E^*}$ with $D(S) \neq \emptyset$ determines a convex function $\chi_S : E \to \mathbb{R} \cup \{\infty\}$, and we shall also show that $\chi_S$ gives a considerable amount of information about the structure of $S$.

We define $\chi_S$ in Definition 2. Lemma 3 contains a technical result which will be useful later in the paper, and Lemma 4 is our main result about $\chi_S$. Our first application of Lemma 4 is in Theorem 6, in which we give a sufficient condition for $S$ to be locally bounded at a point of $E$.

We next discuss the concept of an element $x$ of $E$ being "surrounded" by a subset $A$ of $E$. This concept is related to $x$ being an "absorbing point" of $A$, but differs in that we do not require that $x \in A$ (see [5, Definition 2.27(b), p.28]). Among other things, this difference will enable us to strengthen (in Theorem 12(b)) the result of Borwein and Fitzpatrick (see [1]) on the local boundedness of monotone operators.

Lemma 13(b) contains a result on the existence of elements of $E^*$, which we apply to maximal monotone operators in Theorem 14. Rockafellar proved in [7, Theorem 1, p.398] (see also [6, Theorem 1.9, p.6] that if $S$ is maximal monotone and $\text{int} (\text{co} D(S)) \neq \emptyset$, then $\text{int} D(S)$ and $\overline{D(S)}$ are both convex. (As usual, “co” stands for “convex hull of”.) In Theorem 14, we give more explicit results and prove that, in fact,

$$\text{int} D(S) = \text{int} (\text{dom} \chi_S) \quad \text{and} \quad \overline{D(S)} = \overline{\text{dom} \chi_S}.$$
The first of these results is true even if \( \text{int} (\co D(S)) = 0 \).

The equation "\( \text{int} D(S) = \text{sur} (\co D(S)) \)" in Theorem 14 means the following: if \( x \in E \) and,

for all \( w \in E \setminus \{0\} \), there exists \( \delta > 0 \) such that \( x + \delta w \in \co D(S) \)

then \( x \in \text{int} D(S) \). This result answers in the affirmative a question raised by Phelps (see [5, p.29] and [6, p.8]).

The analysis in this paper gives insight into the "relative difficulty" of the results on the convexity of \( \text{int} D(S) \) and \( \text{dom} \chi_S \) on the one hand, and the results on local boundedness on the other. The former use Lemma 4 in full generality, while the latter use Lemma 4 only for \( m = 1 \).

A word about tools. In Lemma 4 we use the standard result that a proper convex lower semicontinuous function on \( E \) is continuous on the interior of its domain. In Lemma 13(b), we use a minimax theorem. In fact, we could have used the Hahn-Banach theorem or a sandwich theorem instead, but a minimax theorem gives the fastest proof. We use the following classical minimax theorem, which can be deduced from more general results of Fan (see [2]) or Sion (see [9]). Fan’s proof used a separation theorem for sets in finite dimensional spaces, and Sion’s proof used the KKM theorem, but Theorem 1 can easily be proved without any functional analysis or fixed-point related concepts. See, for instance, the proof of Sion’s theorem given by Komiya in [4].

**Theorem 1.** Let \( X \) and \( Y \) be nonempty compact convex subsets of topological vector spaces. Let \( f : X \times Y \to \mathbb{R} \) be (separately) concave and upper semicontinuous on \( X \) and convex and lower semicontinuous on \( Y \). Then

\[
\max_X \min_Y f = \min_Y \max_X f.
\]

**The convex function determined by a multifunction**

**Definition 2:** If \( m \geq 1 \), let

\[
\sigma_m := \{a = (a_1, \ldots, a_m): a_1, \ldots, a_m \geq 0, a_1 + \cdots + a_m = 1\} \subset \mathbb{R}^m.
\]

If \( S : E \to 2^{E^*} \) and \( D(S) \neq \emptyset \), we define \( \chi_S : E \to \mathbb{R} \cup \{\infty\} \) by

\[
\chi_S(w) := \sup_{m \geq 1, (y_1, y_1^*), \ldots, (y_m, y_m^*) \in G(S), a \in \sigma_m} \frac{\sum a_i \langle w - y_i, y_i^* \rangle}{1 + \sum a_i y_i}.
\]

\( \chi_S \) is clearly convex and lower semicontinuous. (Here \( G(S) \) stands for the graph of \( S \).)
In [8], a function \( \psi_S : E \to \mathbb{R} \cup \{\infty\} \) was defined by the formula

\[
\psi_S(w) := \sup_{(y,y^*) \in G(S)} \frac{\langle w - y, y^* \rangle}{1 + \|y\|},
\]

which is the \( m = 1 \) version of the formula used to define \( \chi_S \). This was adequate for proving the convexity of \( \text{int} D(S) \) and \( \overline{D(S)} \) in the reflexive case, but it seems that the more complicated function \( \chi_S \) is required in the general case.

The results on translation contained in Lemma 3 will enable us to simplify the computations in Theorems 6 and 14 considerably.

**Lemma 3.** Let \( S : E \to 2^{E^*} \) with \( D(S) \neq \emptyset \), and \( z \in E \). Define \( T : E \to 2^{E^*} \) by

\[ Tx := S(z + z). \]

Then:

(a) For all \( w \in E \),

\[
\frac{\chi_T(w)}{1 + \|z\|} \leq \chi_S(w + z) \leq (1 + \|z\|)\chi_T(w).
\]

(b) \( \text{Dom} \chi_T = \text{dom} \chi_S - z \).

(c) \( D(T) = D(S) - z \).

(d) If \( S \) is monotone or maximal monotone then so is \( T \).

**Proof:** In (a), we shall prove the second inequality — the first inequality follows by replacing \( z \) by \( -z \) and interchanging the roles of \( S \) and \( T \).

Let \( m \geq 1 \), \((y_1, y_1^*), \ldots, (y_m, y_m^*) \in G(S)\) and \( a \in a_m \). Then

\[(y_1 - z, y_1^*), \ldots, (y_m - z, y_m^*) \in G(T).\]

Thus, using the definition of \( \chi_T(w) \),

\[
\sum_i a_i \langle (w + z) - y_i, y_i^* \rangle = \sum_i a_i \langle w - (y_i - z), y_i^* \rangle \\
\leq (1 + \|\sum_i a_i (y_i - z)\|)\chi_T(w) \\
= (1 + \|\sum_i a_i y_i - z\|)\chi_T(w) \\
\leq (1 + \|\sum_i a_i y_i\| + \|z\|)\chi_T(w) \\
\leq (1 + \|\sum_i a_i y_i\|)(1 + \|z\|)\chi_T(w).
\]
We obtain (a) by dividing by \((1 + \|\sum \alpha_i y_i\|)\), taking the supremum over \(m, (y_i, y_i^*)\) and \(\alpha\), and using the definition of \(\chi_S(w + z)\).

(b) follows from (a), and (c) and (d) are immediate.

**Lemma 4.** Let \(S : E \to 2^B^*\) with \(D(S) \neq \emptyset\), and \(0 \in \text{int}(\text{dom } \chi_S)\). Then there exist \(\eta \in (0, 1]\) and \(P > 0\) such that

\[
\sum a_i(y_i, y_i^*) \geq \eta\|\sum a_i y_i^*\| - P(1 + \|\sum a_i y_i\|).
\]

**Proof:** From [5, Proposition 3.3, p.39], there exist \(\eta \in (0, 1]\) and \(P > 0\) such that

\[
w \in E \text{ and } \|w\| \leq \eta \implies \chi_S(w) \leq P.
\]

Thus,

\[
w \in E, \|w\| \leq \eta, m \geq 1, (y_1, y_1^*), \ldots, (y_m, y_m^*) \in G(S) \text{ and } \alpha \in \sigma_m
\]

imply that

\[
\sum a_i(w - y_i, y_i^*) \leq P(1 + \|\sum a_i y_i\|),
\]

that is to say,

\[
\sum a_i(y_i, y_i^*) \geq \sum a_i(w, y_i^*) - P(1 + \|\sum a_i y_i\|) = \langle w, \sum a_i y_i^* \rangle - P(1 + \|\sum a_i y_i\|).
\]

We complete the proof of Lemma 4 by taking the supremum of the right hand expression over all \(w \in E\) such that \(\|w\| \leq \eta\).

**Definition 5:** Let \(S : E \to 2^B^*\) with \(D(S) \neq \emptyset\), and \(x \in E\). Following [6, Definition 1.8, p.5] we say that \(S\) is locally bounded at \(x\) if there exist \(\delta, Q > 0\) such that

\[
(y, y^*) \in G(S) \text{ and } \|y - x\| < \delta \implies \|y^*\| \leq Q.
\]

Note that this definition does not require that \(x \in D(S)\).

**Theorem 6.** Let \(S : E \to 2^B^*\) with \(D(S) \neq \emptyset\). Then \(S\) is locally bounded at each point of \(\text{int}(\text{dom } \chi_S)\).

**Proof:** From the results on translation in Lemma 3, it suffices to prove that

\[
0 \in \text{int}(\text{dom } \chi_S) \implies S \text{ is locally bounded at } 0.
\]
So suppose that $0 \in \text{int}(\text{dom}\chi_S)$. Let $\eta$ and $P$ be as in Lemma 4. From Lemma 4 with $m = 1$,

$$(y, y^*) \in G(S) \implies \eta \|y^*\| \leq \langle y, y^* \rangle + P(1 + \|y\|).$$

So

$$\frac{\eta}{2} \|y\| \leq \frac{\eta}{2} \|y^*\| + P\left(\frac{\eta}{2}\right) \implies \frac{\|y\|}{\eta} \leq \frac{3P}{\eta}.$$

Thus Definition 5 is satisfied with $\delta := \eta/2$ and $Q := 3P/\eta$.

Since Theorem 6 only uses the $m = 1$ version of Lemma 4, it could in fact be strengthened to give the result that $S$ is locally bounded at each point of $\text{int}(\text{dom}\psi_S)$ — see the comment following Definition 2.

**Surrounded points and surrounding sets**

**Definition 7:** Let $x \in E$ and $A \subset E$. We say that $A$ surrounds $x$ if, for each $w \in E \setminus \{0\}$, there exists $\delta > 0$ such that $x + \delta w \in A$. Furthermore, we define

$$\text{sur} A := \{x : x \in E, A \text{ surrounds } x\}.$$  

We note that, in general, $\text{sur} A \not\subset A$. (Consider, for example, the case where $A$ is the circumference of a circle in the plane and $x$ is the centre of $A$.)

Lemma 8 provides some general culture concerning surrounding sets.

**Lemma 8.** Suppose that $C$ is a nonempty, convex subset of $E$. Then:

(a) $\text{sur} C$ is convex.

(b) $\text{sur} C \subset C$.

(c) $x \in \text{sur} C$ if and only if, for each $w \in E$ there exists $\delta > 0$ such that $x + [-\delta, \delta]w \subset C$, that is to say, $x$ is a radial point of $C$, (see [3, p.14]).

(d) If $\text{sur} C \neq \emptyset$ then $\overline{C} = \text{sur} C$.

**Proof:** (a) Suppose that $x, y \in \text{sur} C$ and $\theta \in [0,1]$. Let $w \in E \setminus \{0\}$, and pick $\delta_1, \delta_2 > 0$ such that $x + \delta_1 w \in C$ and $y + \delta_2 w \in C$. Define $\delta := (1 - \theta)\delta_1 + \theta \delta_2$. Then, from the convexity of $C$,

$$[(1 - \theta)x + \theta y] + \delta w = (1 - \theta)(x + \delta_1 w) + \theta(y + \delta_2 w) \in C.$$

Since this holds for all $w \in E \setminus \{0\}$, $(1 - \theta)x + \theta y \in \text{sur} C$, as required.
(b) Suppose that \( x \in \text{sur} C \). Let \( w \in E \setminus \{0\} \) and pick \( \delta_1, \delta_2 > 0 \) such that
\[
x + \delta_1 w \in C \quad \text{and} \quad x - \delta_2 w \in C.
\]
Since \( C \) is convex, \( [x + \delta_1 w, x - \delta_2 w] \subset C \). In particular, \( x \in C \).

(c) Suppose that \( x \in \text{sur} C \). Let \( w \in E \). If \( w = 0 \) then, by (b), \( x + [-1,1]w = \{x\} \subset C \). If \( w \neq 0 \), pick \( \delta_1, \delta_2 > 0 \) such that
\[
x + \delta_1 w \in C \quad \text{and} \quad x - \delta_2 w \in C.
\]
Let \( \delta = \min\{\delta_1, \delta_2\} \). Since \( C \) is convex,
\[
x + [-\delta, \delta]w = [x - \delta w, x + \delta w] \subset [x + \delta_1 w, x - \delta_2 w] \subset C.
\]
The converse is immediate.

(d) Suppose that \( x \in C \). Let \( y \in \text{sur} C \). We claim that
\[
\theta \in (0,1] \quad \implies \quad (1 - \theta)x + \theta y \in \text{sur} C.
\]
So let \( \theta \in (0,1] \). Let \( w \in E \setminus \{0\} \), and pick \( \rho > 0 \) such that \( y + \rho w \in C \). Define \( \delta := \rho \theta \). Then, from the convexity of \( C \),
\[
[(1 - \theta)x + \theta y] + \delta w = (1 - \theta)x + \theta(y + \rho w) \in C.
\]
Since this holds for all \( w \in E \setminus \{0\} \), \( (1 - \theta)x + \theta y \in \text{sur} C \), as required. It now follows by letting \( \theta \to 0^+ \) that \( x \in \text{sur} C \). So we have proved that \( C \subset \text{sur} \overline{C} \), from which it follows immediately that \( \overline{C} \subset \text{sur} \overline{C} \). The reverse inclusion follows from (b), and this completes the proof of (d).

Let \( E \) be infinite dimensional. Then there exists a discontinuous linear functional \( L : E \to \mathbb{R} \). Let \( C := \{x \in E : |Lx| \leq 1\} \). Then \( C \) is convex and \( 0 \in \text{sur} C \), but \( 0 \notin \text{int} C \). The point of this simple example is to contrast the situation for general convex sets with that exhibited in Theorem 9.

**Theorem 9.** Let \( 0 \neq C \subset E \). Suppose that \( \{F_n\} \) is an increasing sequence of closed convex sets such that \( C = \bigcup_{n \geq 1} F_n \). Then \( \text{sur} C = \text{int} C \).

**Proof:** It suffices from a translation argument to show that
\[
0 \in \text{sur} C \implies 0 \in \text{int} C.
\]
Since \( 0 \in \text{sur} C \), \( E = \bigcup_{k \geq 1} kC \). So \( E = \bigcup_{k, n \geq 1} kF_n \). By the Baire category theorem, there exist \( n, k \geq 1 \) such that \( \text{int} kF_n \neq \emptyset \), from which \( \text{int} F_n \neq \emptyset \). Choose \( x \in \text{int} F_n \). If
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If \( x \neq 0 \) then, since \( 0 \in \text{sur} \ C \), there exists \( p > 0 \) such that \( -x \in pC \), from which there exists \( m \geq 1 \) such that \( -x \in pF_m \). Let \( q = m \lor n \). Then

\[
x \in \text{int} F_q \quad \text{and} \quad \frac{-x}{p} \in F_q.
\]

Using [3, 13.1(i), p.110], the convexity of \( F_q \) implies \( 0 \in \text{int} F_q \subset \text{int} C \). This completes the proof of Theorem 9.

**Corollary 10.** Let \( f : E \to \mathbb{R} \cup \{\infty\} \) be proper, convex and lower semicontinuous. Then \( \text{sur} (\text{dom} f) = \text{int} (\text{dom} f) \).

**Proof:** This follows from Theorem 9, with \( F_n := E \{f \leq n\} \).

**Results for monotone operators**

**Lemma 11.** Let \( S : E \to 2^{E'} \) be monotone, with \( D(S) \neq \emptyset \). Then:

(a) \( D(S) \subset \text{co} D(S) \subset \text{dom} \chi_S \).

(b) Let \( m \geq 1 \), \( \{(y_1, y_1^*), \cdots, (y_m, y_m^*)\} \subset G(S) \) and \( a \in \sigma_m \). Then

\[
\sum_i a_i \langle y_i, y_i^* \rangle \geq \langle \sum_i a_i y_i, \sum_j a_j y_j^* \rangle.
\]

**Proof:** (a) Since \( \text{dom} \chi_S \) is convex, it suffices to prove that

\[
D(S) \subset \text{dom} \chi_S
\]

To this end, let \( w \in D(S) \). Pick \( w^* \in Sw \), and define \( \beta := \langle w, w^* \rangle \lor \|w^*\| \). Let \( m \geq 1 \), \( (y_1, y_1^*), \cdots, (y_m, y_m^*) \in G(S) \), and \( a \in \sigma_m \). Then, since \( S \) is monotone,

\[
\sum_i a_i \langle w - y_i, y_i^* \rangle \leq \sum_i a_i \langle w - y_i, w^* \rangle
\]

\[
= \langle w, w^* \rangle - \langle \sum_i a_i y_i, w^* \rangle
\]

\[
\leq \langle w, w^* \rangle + \left\| \sum_i a_i y_i \right\| \|w^*\|
\]

\[
\leq \beta (1 + \left\| \sum_i a_i y_i \right\|).
\]

Dividing by \( 1 + \left\| \sum_i a_i y_i \right\| \), we obtain

\[
\frac{\sum_i a_i \langle \|w - y_i, y_i^* \rangle}{1 + \left\| \sum_i a_i y_i \right\|} \leq \beta.
\]
Taking the supremum over \( m \geq 1 \), \((y_1, y^*_1), \ldots, (y_m, y^*_m) \in G(S)\) and \( a \in \sigma_m \) we see that \( \chi_S(w) \leq \beta \), which implies that \( w \in \text{dom } \chi_S \). This completes the proof of (11.1), and hence that of Lemma 11(a).

(b) follows from the following relations:

\[
\sum_i a_i \langle y_i, y^*_i \rangle - \langle \sum_i a_i y_i, \sum_j a_j y^*_j \rangle = \sum_{i,j} a_i a_j \langle y_i, y^*_i - y^*_j \rangle \\
= \sum_{i,j} a_i a_j \langle y_i, y^*_i - y^*_j \rangle + \sum_{j<i} a_i a_j \langle y_i, y^*_i - y^*_j \rangle \\
= \sum_{i<j} a_i a_j \langle y_i, y^*_i - y^*_j \rangle + \sum_{i<j} a_i a_j \langle y_j, y^*_j - y^*_i \rangle \\
= \sum_{i<j} a_i a_j \langle y_i - y_j, y^*_i - y^*_j \rangle \geq 0.
\]

\[\square\]

THEOREM 12. Let \( S : E \to 2^{E^*} \) be monotone, with \( D(S) \neq \emptyset \). Then:

(a) \( \text{sur } D(S) \subset \text{sur } (\text{co } D(S)) \subset \text{sur } (\text{dom } \chi_S) \)
\[= \text{int } (\text{dom } \chi_S) \supset \text{int } (\text{co } D(S)) \supset \text{int } D(S).\]

(b) \( S \) is locally bounded at each point of \( \text{sur } (\text{co } D(S)) \).

PROOF: (a) It follows from Lemma 11(a) that \( \text{sur } D(S) \subset \text{sur } (\text{co } D(S)) \subset \text{sur } (\text{dom } \chi_S) \) and \( \text{int } (\text{dom } \chi_S) \supset \text{int } (\text{co } D(S)) \supset \text{int } D(S) \). Since \( D(S) \neq \emptyset \), \( \chi_S \) is proper so, from Corollary 10, \( \text{sur } (\text{dom } \chi_S) = \text{int } (\text{dom } \chi_S) \).

(b) This is immediate from (a) and Theorem 6. \[\square\]

LEMMA 13. Let \( S : E \to 2^{E^*} \) be monotone with \( D(S) \neq \emptyset \), \( 0 \in \text{int } (\text{dom } \chi_S) \), and \( \eta \) and \( P \) be as in Lemma 4. Define \( M := P/\eta \). Now let \( m \geq 1 \) and \((y_1, y^*_1), \ldots, (y_m, y^*_m) \in G(S)\). Then:

(a) For all \( a \in \sigma_m \),

\[(13.1) \sum_i a_i \langle y_i, y^*_i \rangle + M \| \sum_i a_i y_i \| \geq 0.\]

(b) There exists \( z^* \in E^* \) such that

\[\| z^* \| \leq M \text{ and, for all } i = 1, \ldots, m, \langle y_i, y^*_i - z^* \rangle \geq 0. \]
PROOF: (a) Let \( a \in \sigma_m \). If \( \| \sum_i a_i y_i^* \| > M \) then, since \( M = P/\eta \geq P \),
\[
\sum_i a_i \langle y_i, y_i^* \rangle + M \| \sum_i a_i y_i \| \geq \sum_i a_i \langle y_i, y_i^* \rangle + P \| \sum_i a_i y_i \|,
\]
from Lemma 4,
\[
\geq \eta \| \sum_i a_i y_i^* \| - P.
\]
and (13.1) follows. If, on the other hand, \( \| \sum_i a_i y_i^* \| \leq M \) then, from Lemma 11(b),
\[
\sum_i a_i \langle y_i, y_i^* \rangle + M \| \sum_i a_i y_i \| \geq ( \sum_i a_i y_i, \sum_i a_i y_i^* ) + M \| \sum_i a_i y_i \|
\geq M \| \sum_i a_i y_i \| - \| \sum_i a_i y_i \| \| \sum_i a_i y_i^* \|,
\]
and (13.1) follows again. This completes the proof of Lemma 13(a).

(b) From Theorem 1,
\[
\max_{\| z^* \| \leq M} \min_{a \in \sigma_m} \left[ \sum_i a_i \langle y_i, y_i^* - z^* \rangle \right] = \min_{a \in \sigma_m} \max_{\| z^* \| \leq M} \left[ \sum_i a_i \langle y_i, y_i^* - z^* \rangle \right]
\leq \min_{a \in \sigma_m} \max_{\| z^* \| \leq M} \left[ \sum_i a_i \langle y_i, y_i^* \rangle - \langle \sum_i a_i y_i, z^* \rangle \right]
\leq \min_{a \in \sigma_m} \left[ \sum_i a_i \langle y_i, y_i^* \rangle + M \| \sum_i a_i y_i \| \right] \geq 0,
\]
using (a). Thus there exists \( z^* \in E^* \) such that \( \| z^* \| \leq M \) and
\[
\text{for all } a \in \sigma_m, \sum_i a_i \langle y_i, y_i^* - z^* \rangle \geq 0.
\]
We complete the proof of Lemma 13(b) by letting \( a \) run through the vertices of \( \sigma_m \).

THEOREM 14. Let \( S : E \to 2^{E^*} \) be maximal monotone. Then:

(a) \( \text{Sur } D(S) = \text{sur}(\text{co } D(S)) = \text{sur}(\text{dom } \chi_S) \)
\[= \text{int}(\text{dom } \chi_S) = \text{int}(\text{co } D(S)) = \text{int } D(S). \]
(b) If \( \text{sur} (\text{co} D(S)) \neq \emptyset \) then

\[
\overline{D(S)} = \text{co} D(S) = \text{dom} \chi_S = \overline{\text{sur} D(S)} = \overline{\text{sur} (\text{co} D(S))} \\
= \text{sur} (\text{dom} \chi_S) = \text{int} (\text{dom} \chi_S) = \text{int} (\text{co} D(S)) = \text{int} D(S).
\]

**Proof:** (a) We first prove that

\[
(14.1) \quad \text{int} (\text{dom} \chi_S) \subseteq D(S).
\]

We can suppose that \( \text{int} (\text{dom} \chi_S) \neq \emptyset \), for otherwise there is nothing to prove. From the results on translation in Lemma 3, it suffices to prove that

\[
(14.2) \quad 0 \in \text{int} (\text{dom} \chi_S) \implies 0 \in D(S).
\]

So suppose that \( 0 \in \text{int} (\text{dom} \chi_S) \). Let \( M \) be as in Lemma 13. Then, for each finite subset \( F \) of \( G(S) \), the set

\[
\bigcap_{(y,y^*) \in F} \{ z^* : z^* \in E^*, \|z^*\| \leq M, \langle y, y^* - z^* \rangle \geq 0 \}
\]

is nonempty. As \( F \) runs, these sets are \( \omega(E^*,E) \)-compact and directed downwards, hence their intersection is nonempty. It follows that there exists \( z^* \in E^* \) such that

\[
\text{for all} \ (y,y^*) \in G(S), \quad \langle y, y^* - z^* \rangle \geq 0.
\]

Since \( S \) is maximal monotone, this implies that \( z^* \in S0 \), from which \( 0 \in D(S) \). This establishes (14.2), and hence (14.1). From (14.1), \( \text{int} (\text{dom} \chi_S) \subseteq \text{int} D(S) \subseteq \text{sur} D(S) \).

The result follows from Theorem 12(a).

(b) From Lemma 11(a), \( \overline{D(S)} \subseteq \text{co} D(S) \subseteq \text{dom} \chi_S \). From (a), \( \text{int} (\text{dom} \chi_S) \neq \emptyset \). Thus, from [3, 13.1(i)] again, with \( C := \text{dom} \chi_S \), and a second application of (a),

\[
\text{dom} \chi_S = \text{int} (\text{dom} \chi_S) = \text{int} D(S) \subseteq D(S).
\]

Thus we have proved that

\[
D(S) = \text{co} D(S) = \text{dom} \chi_S = \text{int} D(S).
\]

The result now follows from a third application of (a).
REFERENCES


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