Can. J. Math., Vol. XXXVII, No. 5, 1985, pp. 893-907

A NOTE ON THE MOD 2 COHOMOLOGY OF $B\hat{SO}_n\langle 16\rangle$

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1. Introduction. Recall BSO_n is the classifying space for the special orthogonal group of rank n. It is well known that the mod 2 cohomology ring of BSO_n is given as follows:

$$H^*(BSO_n; \mathbf{Z}_2) \cong \mathbf{Z}_2[w_2, \dots, w_n]$$

where w_i is the i-th mod 2 Stiefel-Whitney class. For the 3-connective cover of BSO_n, $B\mathrm{Spin}_n$, Quillen in **[4]** has determined $H^*(B\mathrm{Spin}_n)$ completely for all n. Let $B\hat{SO}_n\langle 8\rangle$, $B\hat{SO}_n\langle 16\rangle$ be the classifying spaces for n-plane spin bundle ξ satisfying $w_4(\xi) = 0$ and $w_4(\xi) = w_8(\xi) = 0$ respectively. This note follows the method of A. Borel **[2]** and gives the mod 2 cohomology ring of $B\hat{SO}_n\langle 8\rangle$ and $B\hat{SO}_n\langle 16\rangle$ for small n. In particular we answer the question "when is $H^*(B\hat{SO}_n\langle 8\rangle; \mathbf{Z}_2)$ or $H^*(B\hat{SO}_n\langle 16\rangle; \mathbf{Z}_2)$ a polynomial algebra?"

We shall regard $B\hat{SO}_n\langle 8\rangle$ as the principal fibration over $B\mathrm{Spin}_n$ with k-invariant $w_4 \in H^*(B\mathrm{Spin}_n; \mathbf{Z}_2)$ and $B\hat{SO}_n\langle 16\rangle$ as the principal fibration over $B\hat{SO}_n\langle 8\rangle$ with k-invariant $w_8 \in H^*(B\hat{SO}_n\langle 8\rangle; \mathbf{Z}_2)$.

Through the paper cohomology means mod 2 cohomology and $\mathfrak{A}$ denotes the mod 2 Steenrod algebra.

2. The Leray-Serre spectral sequence for $B\hat{SO}_n\langle 8\rangle \to B\mathrm{Spin}_n$.

2.1. We shall use the following formula frequently:

$$(2.1.1) \qquad Sq^a w_b = \sum_{i=0}^{a} \binom{b-a+i-1}{i} w_{a-i} w_{b+i}.$$

Let $\{E_r^{p,q}, d_r\}$ be the Leray-Serre spectral sequence for the fibration

$$B\hat{SO}_n\langle 8\rangle \to B\mathrm{Spin}_n.$$

Then

$$E_2^{p,q} \approx H^p(B\mathrm{Spin}_n) \otimes H^q(K_3),$$

where K_j for any integer $j \geqq 1$ is the Eilenberg-MacLane space of type $(\mathbf{Z}_2, j)$.

Received May 30, 1983.

We shall give a description of $E_r^{p,q}$ for some large r, depending on n. We shall work formally in $H^*(BSO)$ and then reduce the calculation to the unstable case.

Definition 2.2. We redefine generators for $H^*(BSO)$ as follows:

$$v_i = \begin{cases} Sq^{2^r} \dots Sq^{2^2} Sq^2 w_4 & \text{if } i = 2^{r+1} + 2,\ r \geqq 0 \\ Sq^{2^j(2^{t+1}+1)} \dots Sq^{2^{t+1}+1} Sq^{2^t} \dots Sq^2 w_4 & \text{if } i = 2^{t+j+2} + 2^{j+1} + 1,\ t \geqq 1, j \geqq 0 \\ Sq^{2^j(2+1)} Sq^{2^{j-1}(2+1)} \dots Sq^{(2+1)} w_4 & \text{if } i = 2^{j+2} + 2^{j+1} + 1,\ j \geqq 0 \\ Sq^{2^{r-1}} \dots Sq^2 Sq^1 w_2 & \text{if } i = 2^r + 1,\ r \geqq 1 \\ w_i & \text{otherwise.} \end{cases}$$

Definition 2.3. Let $I(Sq^1 \iota_3)$ be the $\mathfrak{A}(H^*(K_3))$-module generated by $Sq^1 \iota_3$, where ι_3 is the fundamental class of K_3. We give a simple system of generators for $H^*(K_3)/I(Sq^1 \iota_3)$:

$$M_0 = \iota_3$$

$$M_k = Sq^{2^k} \dots Sq^{2^2} Sq^{2^1} \iota_3, \quad k \geqq 7.$$

We use the convention: $M_j = 0$ for $j < 0$.

We have the following easy lemma.

LEMMA 2.4. *Formally, the differential d_r on M_k is given as follows*:

(i) $\quad d_{2^{k+t+1}+2^t+1}(M_k)^{2^t} = v_{2^{k+t+1}+2^t+1}, \quad t, k \geqq 0,$

(ii) $\quad d_5(Sq^1 M_0) = 0 \quad$ *in* $H^*(B\mathrm{Spin}_n)$.

2.5. $H^*(B\mathrm{Spin}_n)$.

THEOREM [**4**].

$$H^*(B\mathrm{Spin}_n) \cong H^*(BSO_n)/J \otimes \mathbf{Z}_2[\eta_{2^h}]$$

where J is the ideal generated by the elements $\{v_2, v_{2+1}, \dots, v_{2^{h-1}+1}\}$,

$$i^*(\eta_{2^h}) = \iota^{2^h},$$

$$i: K(\mathbf{Z}_2, 1) \to B\mathrm{Spin}_n$$

is the inclusion of the fibre and the integer h is given by the following table

n	$8l + 1$	$8l + 2$	$8l + 3$	$8l + 4$	$8l + 5$	$8l + 6$	$8l + 7$	$8l + 8$
h	$4l$	$4l + 1$	$4l + 2$	$4l + 2$	$4l + 3$	$4l + 3$	$4l + 3$	$4l + 3$

2.6. $H^*(B\hat{S}O_n\langle 8\rangle)$.

LEMMA 2.6.1. *Let j be of the form $2^k + 2^l + 1$, $k \geqq l \geqq 0$ or $j = 2$. Then*

(i) *For* $j \geqq 2$, $Sq^{j-1}v_j = v_{2j-1}$;

(ii) *For* $j \geqq 3$, $Sq^{j-2}v_j = \begin{cases} v_{2j-2} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd;} \end{cases}$

(iii) *For* $j \geqq 4$, $Sq^{j-3}v_j = \begin{cases} v_{2j-3} & \text{if } j \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$

(iv) *For* $j \geqq 9$, $Sq^{j-4}v_j = \begin{cases} v_{j-2}^2 & \text{if } j \equiv 5 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$

Proof. We will prove only part (iv). The others are similar. The proof makes use of the following Adém relations:

(2.6.2) $\quad Sq^{2n-3}Sq^n = Sq^{2n-1}Sq^{n-2}$

(2.6.3) $\quad Sq^{2n-2}Sq^n = Sq^{2n-1}Sq^{n-1}$

(2.6.4) $\quad Sq^{2n-1}Sq^n = 0.$

If $j \geqq 9$ is of the form $2^r + 1$ then

$$v_{2^r+1} = Sq^{2^{r-1}}Sq^{2^{r-2}}\dots Sq^2Sq^1w_2,\ r \geqq 3.$$

Thus

$$\begin{aligned} Sq^{2^r-3}v_{2^r+1} &= Sq^{2^r-3}Sq^{2^{r-1}}\dots Sq^2Sq^1w_2 \\ &= Sq^{2^r-1}Sq^{2^{r-1}-2}Sq^{2^{r-2}}\dots Sq^2Sq^1w_2, \quad r \geqq 3 \\ &\hspace{20em}\text{by (2.6.2)} \\ &= 0 \quad \text{by (2.6.4) and (2.6.3).} \end{aligned}$$

Similarly if j is of the form $2^r + 2$ and $r \geqq 3$, then $Sq^{j-4}v_j = 0$.

If $j = 2^{t+r} + 2^r + 1 > 9$, then

$$\begin{aligned} Sq^{j-4}v_j &= Sq^{2^{t+r}+2^r-3}v_{2^{t+r}+2^r+1} \\ &= Sq^{2^{t+r}+2^r-3}Sq^{2^{r-1}(2^t+1)}\dots Sq^{(2^t+1)}Sq^{2^{t-1}} \\ &\hspace{18em}\dots Sq^4Sq^2w_4 \\ &= \begin{cases} Sq^{2^{t+r}+2^r-1}Sq^{2^{r-1}(2^t+1)-2}Sq^{2^{r-2}(2^t+1)} \\ \qquad \dots Sq^{2^{t-1}}\dots Sq^4Sq^2w_4 \quad r \geqq 2 \text{ by (2.6.3)} \\ Sq^{2^{t+1}-1}Sq^{2^t+1}Sq^{2^{t-1}}\dots Sq^4Sq^2w_4 \quad r = 1. \end{cases} \end{aligned}$$

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$$= \begin{cases} 0 & \Big| r \geqq 3 \text{ or } r = 1, t \geqq 2 \\ (v_{2^t+2}{}_{+2+1})^2 & \Big| r = 2. \end{cases}$$

This completes the proof of part (iv).

Let $I(j)$ be the ideal generated by $\{v_2, \dots, v_j\}$. Suppose $x \in I(j)$ and $\dim x = a$. We have

LEMMA 2.6.5. $Sq^{a-i}x \in I(2j - 1)$, $i = 1, 2, 3$ *and* 4.

Proof. Note that for $i \geqq j \geqq 0$,

$$v_{2^i+2^j+1} \equiv w_{2^i+2^j+1} \quad \text{modulo decomposables.}$$

Therefore

$$H^*(BSO) \cong \mathbf{Z}_2[v_i;\ i \geqq 2].$$

Now for j not of the form $2^i + 2^k + 1$, $i \geqq k \geqq 0$, $v_j = w_j$. Let $\alpha(n)$, for any non-negative integer n, be the number of ones in the dyadic expansion of n. Then $\alpha(j - 1) \geqq 3$ if and only if j is not of the form $2^i + 2^k + 1$, $i \geqq k$. Therefore

$$\alpha(2j - 2) \geqq 3.$$

Hence by the Wu formula,

$$Sq^{j-i}v_j \equiv v_{2j-i} \text{ or } 0 \bmod I(j - 1), \quad i = 1, 2, 3, 4.$$

Now suppose $x \in I(j)$, $\dim x = a$, then x is a sum of terms of the form $v_l \cdot b$, where b is of dimension $a - l$ and $l \leqq j$. Therefore to prove Lemma 2.6.5 it is sufficient to show that

$$Sq^{a-i}(v_l \cdot b) \in I(2j - 1) \quad \text{for } i = 1, 2, 3 \text{ or } 4.$$

By the Cartan formula

$$\begin{aligned} Sq^{a-1}(v_l \cdot b) &= v_l^2 \cdot Sq^{a-l-1}b + Sq^{l-1}v_l \cdot b^2 \\ &\equiv v_l^2 \cdot Sq^{a-l-1}b + v_{2l-1}b^2 \bmod I(l - 1) \\ &\in I(2j - 1). \end{aligned}$$

Similarly, we have

$$Sq^{a-i}(v_l \cdot b) \in I(2j - 1), \quad i = 2, 3 \text{ and } 4.$$

Hence we have completed the proof of Lemma 2.6.5.

Definition 2.6.6. For integers $i \geqq j \geqq 0$ we define the group $L(i, j)$ to be the ideal generated by

$$\begin{cases} v_{2^r+2^s+1} & i > r \geqq s \geqq 0, \\ v_{2^i+2^s+1} & j \geqq s \geqq 0 \text{ and } v_2. \end{cases}$$

By convention, $L(i, -1) = L(i - 1, i - 1)$ for $i \geqq 1$ and $L(0, -1)$ is the ideal generated by v_2.

COROLLARY 2.6.7. *Suppose* $x \in L(i, j)$, $i \geqq j$ *and* dim $x = a$. *Then*

$$Sq^{a-k}x \in L(i + 1, j + 1), \text{ for } k = 1, 2, 3 \text{ or } 4.$$

2.6.8. Since for $k \geqq 2$,

$$v_{2^k+2} = Sq^{2^{k-1}} v_{2^{k-1}+2},$$

by induction on k using 2.1.1 and Lemma 2.6.5, we have

$$v_{2^k+2} \equiv w_{2^k+2} + w_{2^{k-1}} \cdot w_{2^{k-1}+2} \bmod I(2^{k-1} - 1).$$

Therefore, using 2.1.1 and Lemma 2.6.5 again, we have for $k \geqq 4$

$$(2.6.9) \quad v_{2^k+2} \equiv w_{2^k+2} + w_{2^{k-1}} \cdot w_{2^{k-1}+2} + w_{2^{k-1}-1} \cdot w_{2^{k-1}+3} + w_{2^{k-1}-2} \cdot w_{2^{k-1}+4} \bmod I(2^{k-1} - 3).$$

Similarly for $k \geqq j$ we have

$$v_{2^k+2^j+1} \equiv w_{2^k+2^j+1} + w_{2^{k-1}+2^{j-1}} \cdot w_{2^{k-1}+2^{j-1}+1} \bmod I(2^{k-1} + 2^{j-1} - 1)$$

and therefore for $k \geqq j \geqq 3$

$$(2.6.10) \quad v_{2^k+2^j+1} \equiv w_{2^k+2^j+1} + w_{2^{k-1}+2^{j-1}} \cdot w_{2^{k-1}+2^{j-1}+1} + w_{2^{k-1}+2^{j-1}-1} \cdot w_{2^{k-1}+2^{j-1}+2} + w_{2^{k-1}+2^{j-1}-2} \cdot w_{2^{k-1}+2^{j-1}+3} \bmod I(2^{k-1} + 2^{j-1} - 3).$$

Given a positive integer $n \geqq 2$ we define $s(n) > \rho(n) \geqq 0$ to be the integers such that

$$2^{s(n)-1} < n \leqq 2^{s(n)}$$

and

$$2^{s(n)-1} + 2^{\rho(n)-1} < n \leqq 2^{s(n)-1} + 2^{\rho(n)}.$$

By inspection of (2.6.9) with $k = s(n)$, we see that the right hand side contains a decomposable term lying outside

$$L = \bigcup_{t,r} L(t, r)$$

when $s(n) \geqq 5$ and $\rho(n) \geqq 3$ or $n = 20$. Similarly the right hand side of (2.6.10) with $k = s(n) - 1$ and $j = 4$ contains a decomposable term lying outside L when $s(n) \geqq 6$ and $\rho(n) \leqq 4$. Notice that

$$2^{s(n)} + 2 > n \text{ and } 2^{s(n)-1} + 2^4 + 1 > n \text{ when } \rho(n) \leqq 4.$$

Therefore the killing of $v_{2^{s(n)}+2}$ by the differential gives a non-trivial relation in $H^*(B\hat{SO}_n\langle 8\rangle)$ if $s(n) \geqq 5$ and $\rho(n) \geqq 3$ or $n = 20$ and $v_{2^{s(n)-1}+2^4+1}$ gives a non-trivial relation if $s(n) \geqq 6$ and $\rho(n) \leqq 4$. Therefore if $H^*(B\hat{SO}_n\langle 8\rangle)$ is a polynomial algebra then $n < 2^4 + 2^2$.

Now it is easily seen that we have the following congruences:

$$v_i \equiv w_i \bmod L \text{ for } 2 \leqq i \leqq 25 \text{ and } i \neq 8, 12, 14, 15, 16, 20, 22, 23 \text{ and } 24;$$

$$v_{33} \equiv w_{33} \bmod L;\ v_{34} \equiv w_{34} + w_{14} \cdot w_{20} + w_{12} \cdot w_{22} + w_8 \cdot w_{26} \bmod L;$$

$$v_{35} \equiv w_{35} + w_{15} \cdot w_{20} + w_{12} \cdot w_{23} + w_8 \cdot w_{27} \bmod L \quad \text{and}$$

$$v_{37} \equiv w_{37} + w_{15} \cdot w_{22} + w_{14} + w_{23} + w_8 \cdot w_{29} \bmod L.$$

Thus if $n < 2^4 + 2^2$, v_{21}, v_{25}, v_{33}, v_{34} and v_{37} are zero modulo L. Hence there are no non-trivial relations in $H^*(B\hat{SO}_n\langle 8\rangle)$ and so $H^*(B\hat{SO}_n\langle 8\rangle)$ must be a polynomial algebra. We have thus proved:

THEOREM 2.6.11. *$H^*(B\hat{SO}_n\langle 8\rangle)$ is a polynomial algebra if and only if $n \leqq 19$.*

Thus for $n \leqq 19$,

$$v_{2^{s(n)}+1}\{v_{2^{s(n)-1}+2^j+1}\}_{\rho(n)\leqq j\leqq s(n)-2} \quad \text{and}$$

$$\{v_{2^{s(n)}+2^k+1}\}_{0\leqq k\leqq \rho(n)}$$

are all zero modulo $L(s(n) - 1, \rho(n) - 1)$. Moreover, if $20 \leqq n \leqq 32$,

$$v_{2^{s(n)}+1} \equiv 0 \bmod L(s(n) - 1, \rho(n) - 1)$$

but

$$v_{2^{s(n)+1}+1} \not\equiv 0 \bmod L(s(n) - 1, \rho(n) - 1).$$

If $n \geqq 33$,

$$v_{2^{s(n)}+1} \not\equiv 0 \bmod L(s(n) - 1, \rho(n) - 1).$$

Therefore by successively inspecting the images of the differentials in the spectral sequence for $B\hat{SO}_n\langle 8\rangle \to B\mathrm{Spin}_n$ and Theorem 2.5 we have

THEOREM 2.6.12. *Additively there is the following isomorphism for the Leray-Serre spectral sequence for the fibration $B\hat{SO}_n\langle 8\rangle \to B\mathrm{Spin}_n$.*

$$E_{[2^{s(n)-1}+2^{\rho(n)-1}+2]} \cong E_{2^{s(n)-1}+2^{\rho(n)}+1}$$
$$\cong H^*(B\mathrm{Spin}_n)/L(s(n) - 1, \rho(n) - 1) \otimes A$$
$$\otimes \mathbf{Z}_2 [\{Sq^I M_0;\ I = (i_1, \ldots, i_r) \text{ admissible sequence with } i_r = 1 \text{ and } e(I) < 3\}]$$

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$$\otimes \mathbf{Z}_2[\{M_{s(n)-\rho(n)-i}^{2^{\rho(n)-2+i}}\} \quad i = 2, \ldots, s(n) - \rho(n)]$$

$$\otimes \mathbf{Z}_2[\{M_{s(n)-1-i}^{2^i}\} \quad i = 1, \ldots, \rho(n) - 1, \rho(n)]$$

$$\otimes \mathbf{Z}_2[\{M_i\}_{i \geqq s(n)-1}],$$

where

$$A \approx \begin{cases} \mathbf{Z}_2[\Theta_{2^{s(n)}}, \ldots, \Theta_{2^{h-1}}] & \text{if } 19 \geqq n \geqq 10, \\ \mathbf{Z}_2[\Theta_{2^5}] & \text{if } 20 \leqq n \leqq 32, \\ \mathbf{Z}_2 & \text{if } n \leqq 9 \text{ or } n \geqq 33, \end{cases}$$

the integer h is given by Theorem 2.5 and Θ_{2^j} corresponds to the killing of the relation v_{2^j+1} in $H^*(B\mathrm{Spin}_n)$.

Moreover A and $\{Sq^I M_0;\ I = (i_1 r, \ldots, i_r)$ admissible sequence with $i_r = 1$ and $e(I) < 3\}$ are permanent cycles; that is they are included in E^∞.

COROLLARY 2.6.13. *Additively,*

$$H^*(B\hat{S}O_{19}\langle 8\rangle) \cong \mathbf{Z}_2[w_8, w_{12}, w_{14}, w_{15}, w_{16}] \otimes \mathbf{Z}_2[\eta_{2^{10}}]$$

$$\otimes \mathbf{Z}_2[\Theta_{2^5}, \Theta_{2^6}, \Theta_{2^7}, \Theta_{2^8}, \Theta_{2^9}] \otimes \mathbf{Z}_2[Sq^I M_0;\ I = (i_1, \ldots, i_r)$$

admissible sequence with $i_r = 1$ *and* $e(I) < 3]$

$$\otimes \mathbf{Z}_2[\zeta_{20}, \zeta_{24}, \zeta_{34}, \zeta_{36}] \otimes \mathbf{Z}_2[Sq^I \zeta_{33};\ I = \emptyset \text{ or}$$

I admissible sequence of excess < 33 and of the form $(2^k, 2^{k-1}, \ldots, 2^5)\,]$,

where $\zeta_{20}, \zeta_{24}, \zeta_{34}, \zeta_{36}$ *correspond to the vanishing of* $M_1^{2^2}, M_0^{2^3}, M_3^2, M_2^4$ *under the differentials and* $\{Sq^I \zeta_{33},\ I = (2^k, \ldots, 2^5)\}$ *correspond to the vanishing of* $\{M_i\}_{i \geqq 4}$ *under the differentials.*

Proof. This is an immediate consequence of Theorem 2.5 and Theorem 2.6.12.

3. The Leray Serre spectral sequence for $B\hat{S}O_n\langle 16\rangle \to B\hat{S}O_n\langle 8\rangle$. We will choose formally generators for $H^*(BSO)$.

Definition 3.1. Suppose $\alpha(i - 1) \leqq 2$ then v_i is as defined in Definition 2.2. If $\alpha(i - 1) = 3$ then v_i is defined as follows:

$$v_{2^{k+1}+2^2} = Sq^{2^k} Sq^{2^{k-1}} \ldots sq^{2^2} w_8,\ k \geqq 2;$$

$$v_{2^{k+2+j}+2^{2+j}+2} = Sq^{2^j(2^{k+1}+2)} \ldots Sq^{2^0(2^{k+1}+2)} Sq^{2^k} \ldots Sq^{2^2} w_8,$$

$$k \geqq 2, j \geqq 0;$$

$$v_{2^{3+j}+2^{2+j}+2} = Sq^{2^j(2^2+2)} \dots Sq^{2^0(2^2+2)} w_8, j \geqq 0;$$

$$v_{2^{t+k+j+3}+2^{3+j+t}+2^{t+1}+1} = Sq^{2^t(2^{k+j+2}+2^{2+j}+1)} \dots Sq^{2^0(2^{k+j+2}+2^{2+j}+1)}$$

$$\cdot Sq^{2^j(2^{k+1}+2)} Sq^{2^{j-1}(2^{k+1}+2)} \dots Sq^{2^0(2^{k+1}+2)} Sq^{2^k} \dots Sq^{2^2} w_8$$

$k \geqq 2, j \geqq 0, t \geqq 0;$

$$v_{2^{t+k+2}+2^{t+2}+2^{t+1}+1} = Sq^{2^t(2^{k+1}+2+1)} \dots Sq^{2^0(2^{k+1}+2+1)} Sq^{2^k} Sq^{2^{k-1}} \dots Sq^{2^2} w_8$$

$k \geqq 2, t \geqq 0;$

$$v_{2^{t+3}+2^{t+2}+2^{t+1}+1} = Sq^{2^t(2^2+2+1)} \dots Sq^{2^0(2^2+2+1)} w_8, t \geqq 0;$$

and

$$v_{2^{t+j+4}+2^{3+j+t}+2^{t+1}+1} = Sq^{2^t(2^{j+3}+2^{2+j}+1)} \dots Sq^{2^0(2^{j+3}+2^{2+j}+1)} \cdot Sq^{2^j(2^2+2)} \dots Sq^{2^0(2^2+2)} w_8, \quad j \geqq 0, t \geqq 0.$$

If $\alpha(i - 1) \geqq 4$ then $v_i = w_i$.

Thus

$$H^*(BSO) \approx \mathbf{Z}_2[v_i, i \geqq 2].$$

Definition 3.2. $B\hat{S}O_n\langle 16\rangle$ fibres over $B\hat{S}O_n\langle 8\rangle$ with k-invariant

$$w_8 : B\hat{S}O_n\langle 8\rangle \rightarrow K_8$$

and so the fibre of the principal fibration is $K(\mathbf{Z}_2, 7)$. Let the fundamental class be denoted by ι_7. Let $I(Sq^1\iota_7, Sq^2\iota_7)$ be the $\mathfrak{A}(H^*(K_7))$-sub-module generated by $\{Sq^1\iota_7, Sq^2\iota_7\}$. We give a simple system of generators for $H^*(K_7)/I(Sq^1\iota_7, Sq^2\iota_7)$ as follows:

$$X_k = Sq^{2^k} Sq^{2^{k-1}} \dots Sq^{2^2} \iota_7, \quad k \geqq 2;$$

$$Y_k = \begin{cases} Sq^{2^k+2^1} Sq^{2^{k-1}} \dots Sq^{2^2} \iota_7, & k \geqq 3, \\ Sq^{2^2+2} \iota_7, & k = 2; \end{cases}$$

$$Y_{k,j} = \begin{cases} Sq^{2^j}(2^k + 2) \dots Sq^{2^0}(2^k + 2) Sq^{2^{k-1}} \dots Sq^{2^2} \iota_7, & k \geqq 3, j \geqq 0, \\ Sq^{2^j(2^2+2)} \dots (Sq^{2^0(2^2+2)} \iota_7, & k = 2, j \geqq 0. \end{cases}$$

We shall adopt the following convention

$$Y_{k,-1} = X_{k-1} \quad k \geqq 2$$

$$Y_{k,0} = Y_k \quad k \geqq 2$$

$$X_1 = \iota_7 \text{ and } Y_{i,j} = 0 \; i < 2 \text{ or } j < -1.$$

3.3. *Images of differentials.* Let $\{E_r^{p,q}, d_r\}$ be the Leray-Serre cohomology spectral sequence for the fibration

$$\pi: B\hat{S}O_n\langle 16\rangle \to B\hat{S}O_n\langle 8\rangle.$$

Then

$$E_2^{p,q} \approx H^p(B\hat{S}O_n\langle 8\rangle) \otimes H^q(K_7).$$

The images of $\{Y_{k,j}^{2^t}\}$ under the differentials are given below formally by

(3.3.2) $$d_{2^{k+1+j+t}+2^{2+j+t}+2^t+1}(Y_{k,j}^{2^t}) = v_{2^{t+k+j+1}+2^{2+j+t}+2^t+1},$$
$$k \geqq 2, j \geqq -1, t \geqq 0.$$

3.4. Using the Adém relations;

(3.4.1) $$Sq^{2n-4}Sq^n = Sq^{2n-2}Sq^{n-2} + Sq^{2n-1}Sq^{n-3}$$

(3.4.2) $$Sq^{2n-6}Sq^n = Sq^{2n-1}Sq^{n-5} + Sq^{2n-2}Sq^{n-4} + Sq^{2n-3}Sq^{n-3}$$

and (2.6.2) - (2.6.4) we can prove the following

LEMMA 3.4.3. *Suppose* $\alpha(n - 1) = 3$.

(i) $$Sq^{n-1}v_n = v_{2n-1}.$$

(ii) $$Sq^{n-2}v_n = \begin{cases} v_{2n-2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

(iii) *If n is not of the form* $2^k + 2^2$, $k \geqq 3$ *then*

$$Sq^{n-3}v_n = v_{2n-2} \quad or \quad 0;$$
$$Sq^{2^k+1}v_{2^k+2^2} = 0 \text{ in } H^*(B\hat{S}O\langle 8\rangle).$$

(iv) *If n is not of the form* $2^k + 2^2 + 2$, $2^{k+1} + 2^2 + 2 + 1$, $2^k + 2^2$, $k \geqq 3$ *then*

$$Sq^{n-4}v_n = v_{n-2}^2 \text{ or } 0.$$

(v) *For* $k \geqq 3$

$$Sq^{2^k}v_{2^k+2^2} = v_{2^{k+1}+2^2},$$
$$Sq^{2^k+2}v_{2^k+2^2+2} = 0 \text{ in } H^*(B\hat{S}O\langle 8\rangle)$$

and

$$Sq^{2^{k+1}+2+1}v_{2^{k+1}+2^2+2+1} = 0 \text{ in } H^*(B\hat{S}O\langle 8\rangle).$$

Similarly we have:

LEMMA 3.4.4. *Suppose* $\alpha(n - 1) \leqq 2$.
(i) *If* $n \geqq 18$ *and n even then* $Sq^{n-5}v_n = 0$.
(ii) *If n is not of the form* $2^k + 2^2 + 1$, $k \geqq 3$ *and* $n \geqq 11$ *is odd*

then

$$Sq^{n-5}v_n = 0.$$

(iii) *If n is of the form $2^k + 2^2 + 1$, $k \geqq 3$ then*

$$Sq^{n-5}v_n = v_{2n-5}.$$

From Theorem 2.5, Lemma 2.6.1 and Lemma 3.4.4, we deduce:

LEMMA 3.4.5. *The ideal over the Steenrod algebra generated by $(Sq^1\iota_7, Sq^2\iota_7)$, $\tilde{I}(Sq^1\iota_7, Sq^2\iota_7)$, are permanent cycles in E_∞.*

Definition 3.4.6. We shall associate to each triple $(t, j, k) \in \mathbf{Z}^3$ with $t \geqq j \geqq k \geqq 0$ an ideal $L(t, j, k)$ in $H^*(BSO)$.

For $t > j \geqq k \geqq 0$, $L(t, j, k)$ is the ideal generated by

$$\begin{cases} v_{2^r+2^s+2^l+1}, \ t-1 > r \geqq s \geqq l \geqq 0 \text{ or} \\ \qquad\qquad\qquad\qquad t-1 = r > s \geqq l \geqq 0; \\ v_{2^t+2^s+2^l+1}, \ j > s \geqq l \geqq 0; \\ v_{2^t+2^j+2^l+1}, \ k \geqq l \geqq 0, \ v_2 \text{ and } v_3. \end{cases}$$

For $t > j + 1 > 1$, $L(t, j, j)$ is the ideal generated by

$$\begin{cases} v_{2^r+2^s+2^l+1}, \ t-1 > r \geqq s \geqq l \geqq 0 \text{ or} \\ \qquad\qquad\qquad\qquad t-1 = r > s \geqq l \geqq 0; \\ v_{2^t+2^s+2^l+1}, \ j \geqq s \geqq l \geqq 0, \ v_2 \text{ and } v_3. \end{cases}$$

For $j \geqq 1$, $L(j + 1, j, j)$ is the ideal generated by

$$\{v_{2^r+2^s+2^l+1}, \ j+1 > r \geqq s \geqq l \geqq 0 \text{ or } j+1 = r > s \geqq l \geqq 0, \ v_2 \text{ and } v_3\}.$$

For $t > j > 1$,

$$L(t, t, j) = L(t + 1, j - 1, j - 1).$$

For $t \geqq 1$, $L(t, t, 0)$ is the ideal generated by

$$\{v_{2^r+2^s+2^l+1}, \ t > r \geqq s \geqq l \geqq 0 \text{ or } t = r > s \geqq l \geqq 0; \ v_{2^{t+1}+2}, v_2, v_3\}.$$

For $t \geqq 1$, $L(t, 0, 0)$ is the ideal generated by

$$\{v_{2^r+2^s+2^l+1}, \ t-1 > r \geqq s \geqq l \geqq 0 \text{ or } t-1 = r > s \geqq l \geqq 0; \ v_{2^t+2}; \ v_{2^t+2+1}, v_2, v_3\}.$$

For $t \geqq 1$, $L(t, t, t)$ is defined to be $L(t + 1, t - 1, t - 1)$ and $L(0, 0, 0)$ is the ideal generated by $\{v_2, v_3, v_4\}$.

By convention

$$L(t, j, -1) = L(t, j - 1, j - 1) \text{ for } t \geqq j \geqq 1;$$

$$L(t, 0, -1) = L(t - 1, t - 1, 0) \text{ for } t \geqq 1;$$

$$L(t, -1, -2) = L(t - 1, t - 2, t - 2) \text{ for } t \geqq 2;$$

$$L(1, -1, -2) = L(0, 0);\ L(0, -1, -2) = L(0, -1).$$

We can easily derive the following:

LEMMA 3.4.7. *Suppose* $x \in L(i, j, k)$, $i \geqq j \geqq k$ *and* dim $x = a$. *Then*

$$Sq^{a-p}x \in L(i + 1, j + 1, k + 1) \text{ for } p = 1, 2, 3 \text{ or } 4.$$

Using Cartan formula, Wu's formula and Lemma 3.4.3, we have

COROLLARY 3.4.8. *Suppose* $x \in I(j)$ *then* $Sq^{a-i}x \in I(2j - 1)$, $i = 1, 2, 3$ *and* 4.

3.5. Let n be any integer $\geqq 2$. Recall that $s(n)$ and $\rho(n)$ are integers $\geqq 0$ such that

$$2^{s(n)-1} + 2^{\rho(n)-1} < n \leqq 2^{s(n)-1} + 2^{\rho(n)}$$

$$2^{s(n)-1} < n \leqq 2^{s(n)}.$$

We define $q(n)$ to be an integer $\geqq -1$ such that

$$2^{s(n)-1} + 2^{\rho(n)-1} + 2^{q(n)-1} < n \leqq 2^{s(n)-1} + 2^{\rho(n)-1} + 2^{q(n)}.$$

Notice that $\rho(n) - q(n) \geqq 1$ and $s(n) - \rho(n) \geqq 1$.

Let

$$f(n) = \rho(n) - q(n) \text{ and } d(n) = s(n) - \rho(n).$$

Let $\{E_r^{**}, d_r\}$ be the mod 2 cohomology spectral sequence for the fibration

$$B\hat{S}O_n\langle 16\rangle \to B\hat{S}O_n\langle 8\rangle.$$

An inspection of the differentials of the spectral sequence gives us the following

THEOREM 3.5.1. *Additively there is the following isomorphism:*

$$E_{[2^{s(n)-1}+2^{\rho(n)-1}+2^{q(n)-1}+2]} \cong E_{[2^{s(n)-1}+2^{\rho(n)\ \ 1}+2^{q(n)-1}+1]}$$

$$\cong H^*(B\hat{S}O_n\langle 8\rangle)/_{L(s(n)-1,\rho(n)-1,q(n)-1)} \otimes A$$

$$\otimes \tilde{I}(Sq^1\iota_7, Sq^2\iota_7)$$

$$\otimes \mathbf{Z}_2[\ \{\ \{Y_{s(n)-\rho(n)+1,l-3}^{2^{q(n)+f(n)-l}}\}\quad l = 2, \ldots, f(n) = \rho(n) - q(n)\,]$$

$$\otimes \mathbf{Z}_2[\ \{Y_{s(n)-\rho(n)-i,\rho(n)+i-t-2}^{2^t}\},$$

$$i = 0, \ldots, d(n) - 2, t = 0, \ldots, \rho(n) - 1 + i\}]$$

$$\otimes \mathbf{Z}_2[\{Y^{2^t}_{s(n)-\rho(n)+1,\rho(n)-2-t}\}, t = 0, \ldots, q(n)]$$

$$\otimes \mathbf{Z}_2[\{Y^{2^t}_{s(n)+1-l,l-2-t}\}, l = 1, \ldots, \rho(n) - 1, t = 0, \ldots, l - 1]$$

$$\otimes \mathbf{Z}_2[\{Y_{l+1-m,m-2}\}, l \geqq s(n) + 1, l > m \geqq 1]$$

$$\otimes \mathbf{Z}_2[\{Y_{l+1,s(n)-l-2}\}, l = 1, \ldots, d(n) - 1],$$

where A is given by

$$A = \begin{cases} \mathbf{Z}_2 \text{ if } n \geqq 2^6 + 1, \\ \mathbf{Z}_2[\theta_{2^6}] \text{ if } 40 \leqq n \leqq 2^6, \\ \text{polynomial algebra on an undetermined number of} \\ \text{generators corresponding to the vanishing of the basic} \\ \text{set of relations in } H^*(B\hat{SO}_n\langle 8\rangle); \text{ if } n \leqq 39, \end{cases}$$

θ_{2^6} is the permanent cycle given by the vanishing of the relation v_{2^6+1} under the differentials in the spectral sequence and $\tilde{I}(Sq^1\iota_7, Sq^2\iota_7)$ are permanent cycles in the spectral sequence.

THEOREM 3.5.2. *$H^*(B\hat{SO}_n\langle 16\rangle)$ is a polynomial algebra if and only if $n \leqq 39$.*

Proof. First notice that by 2.1.1 and Corollary 3.4.8, we have for $k \geqq 6$,

$$(3.5.3) \quad v_{2^k+2^2} \equiv w_{2^k+2^2} + \sum_{0 \leqq i \leqq 6} w_{2^{k-1}-i} \cdot w_{2^{k-1}+2^2+i} \mod I(2^{k-1} - 7)$$

$$(3.5.4) \quad v_{2^k+2^3+2} \equiv w_{2^{k-1}+2^3+2} + \sum_{0 \leqq i \leqq 6, i \neq 3} w_{2^{k-1}+2^2-i} \cdot w_{2^{k-1}+2^2+2+i} \mod I(2^{k-1} - 3)$$

With $k = s(n)$ the right hand side of (3.5.3) contains a decomposable term lying outside

$$L' = \bigcup_{i \geqq j \geqq k} L(i, j, k)$$

when $s(n) \geqq 6$ and $\rho(n) \geqq 4$ or $n = 40$. Therefore the killing of $v_{2^{s(n)}+2^2}$ gives a non-trivial relation in $H^*(B\hat{SO}_n\langle 16\rangle)$ when $s(n) \geqq 6$, $\rho(n) \geqq 4$ or $n = 40$. Similarly using (3.5.4) with $k = s(n) - 1$ when $s(n) \geqq 7$ and $\rho(n) < 4$ the killing of $v_{2^{s(n)-1}+2^3+2}$ gives a non-trivial relation in $H^*(B\hat{SO}_n\langle 16\rangle)$. Therefore if $H^*(B\hat{SO}_n\langle 16\rangle)$ is a polynomial algebra, then $n \leqq 39$.

Now by Wu's formula and Lemma 3.4.7 it can be shown that for $i \leqq 57$,

$$v_i \equiv w_i \text{ modulo } L'.$$

Thus if $n \leqq 39$,

$$v_{2^5+2^j+2^i+1} \equiv 0 \text{ modulo } L'$$

for $4 \geqq j \geqq 3, j > i \geqq 0$ or $3 \geqq j = i \geqq 2$. Furthermore if $n \leqq 39$, $v_{2^6+2^2}$, $v_{2^6+2^2+2}$, $v_{2^6+2^2+2+1}$, $v_{2^6+2^3+2}$, $v_{2^6+2^3+2+1}$, $v_{2^6+2^3+2^2+1}$, v_{2^6+2}, v_{2^6+2+1}, $v_{2^6+2^2+1}$, and v_{2^6+1} are all congruent to 0 modulo L'.

Thus by Theorem 3.5.1 and Lemma 3.4.7 together with the preceding discussion, there are no non-trivial relations given by $v_{2^i+2^j+2^k+1}$, $i \geqq j \geqq k \geqq 0$, $i \geqq 6$, if $n \leqq 39$. So in E_∞ there are no non-trivial relations if $n \leqq 39$. That is if $n \leqq 39$, $H^*(B\hat{SO}_n\langle 16\rangle)$ is a polynomial algebra. This completes the proof of Theorem 3.5.2.

3.5.5. [c.f. **[5]**]. Recall that an admissible sequence

$$I = (2^{s_q}k_q, \dots, k_q, 2^{s_{q-1}}k_{q-1}, \dots, k_{q-1}, \dots, k_2, 2^{s_1}k_1, \dots, k_1)$$

with

$$k_i > 2^{s_{i-1}+1}k_{i-1}$$

will be called a Θ^m-sequence if

(a) I is empty or
(b) $\exists$ integers $0 \leqq r_q < \dots < r_1 < m$ such that

$$k_1 = 2^m - 2^{r_1} \quad \text{and} \quad k_i = 2^{r_{i-1}} - 2^{r_i} + 2^{s_{i-1}+1}k_{i-1}.$$

COROLLARY. *Let $i:K_7 \to B\hat{SO}_j\langle 16\rangle$ be the inclusion of the fibre. Then for $j \geqq 16$ there exist $\theta_8, \theta_9 \in H^*(B\hat{SO}_j\langle 16\rangle)$, with*

$$i^*\theta_9 = Sq^2\iota_7, \quad i^*\theta_8 = Sq^1\iota_7$$

such that

(1) $$w_{16} = (Sq^4Sq^2Sq^1 + Sq^7)\Theta_9 + (Sq^8 + Sq^6Sq^2)\Theta_8.$$

More generally for any Θ^4-sequence I of degree $i - 2^4$ for $i \leqq j$ and $\alpha(i - 1) = 4$

$$Sq^I(Sq^4Sq^2Sq^1 + Sq^7)\Theta_9 + Sq^I(Sq^8 + Sq^6Sq^2)\Theta_8 = w_i$$

modulo decomposables;
(2) *For $i > j$ and $\alpha(i - 1) = 4$ and I a Θ^4-sequence of degree $i - 2^4$,*

$$Sq^I(Sq^4Sq^2Sq^1 + Sq^7)\Theta_9 + Sq^I(Sq^8 + Sq^6Sq^2)\Theta_8$$

is decomposable.

Proof. Adams in [1] gave a decomposition of Sq^{2^i}, $i \geqq 4$ by stable secondary cohomology operations: In particular,

$$(3.5.6) \quad (Sq^7 + Sq^4Sq^2Sq^1)\phi_{1,3} + (Sq^8 + Sq^6Sq^2)\phi_{0,3} + Sq^1\phi_{3,3} + (Sq^6Sq^3 + Sq^8Sq^1)\phi_{2,2} + (Sq^{12} + Sq^6Sq^6)\phi_{0,2} + (Sq^{13} + Sq^{12}Sq^1)\phi_{1,1} + (Sq^{10}Sq^5 + Sq^{15})\phi_{0,0} = Sq^{16}$$

where $\phi_{i,j}$ $i \leqq j$ are the Adams basic stable secondary cohomology operations. Apply (3.5.6) to the Thom class, U, of the j-plane bundle over $B\hat{S}O_j\langle 16\rangle$ induced from the universal orientable j-plane bundle over BSO_j.

By Proposition 3.4 of [3] and 2.6.12, treating $B\hat{S}O_j\langle 16\rangle$ as a principal fibration over BSO_j, $\phi_{1,3}(U) = U \cdot \theta_9$, $\phi_{0,3}(U) = U \cdot \theta_8$, $\phi_{3,3}(U) = 0$, $\phi_{2,2}(U) = 0$, $\phi_{1,1}(U) = 0$, $\phi_{0,0}(U) = 0$ and $\phi_{0,2}(U) = U \cdot \Theta_4$ where $\Theta_4 \in H^*(B\hat{S}O_j\langle 16\rangle)$, $\Theta_8 \in H^*(B\hat{S}O_j\langle 16\rangle)$ and $\Theta_9 \in H^9(B\hat{S}O_j\langle 16\rangle)$ are such that $i^*\Theta_9 = Sq^2\iota_7$, $i^*\Theta_8 = Sq^1\iota_7$, $\Theta_4 = \pi^*\zeta_4$, $\zeta_4 \in H^4(B\hat{S}O_j\langle 8\rangle)$ is a generator.

But

$$(Sq^{12} + Sq^6Sq^6)\phi_{0,2}(U) = (Sq^{12} + Sq^6Sq^6)(U \cdot \Theta_4) = U\{(Sq^{12} + Sq^6Sq^6)\Theta_4\} = 0.$$

Thus (3.5.6) implies that

$$(Sq^4Sq^2Sq^1 + Sq^7)\phi_{1,3}(U) + (Sq^8 + Sq^6Sq^2)\phi_{0,3}(U) = Sq^{16}(U);$$

That is

$$(Sq^4Sq^2Sq^1 + Sq^7)\Theta_9 + (Sq^8 + Sq^6Sq^2)\Theta_8 = w_{16}.$$

The other assertions are trivial and are left to the reader.

An immediate consequence of Theorem 2.5, Theorem 2.6.12 and Theorem 3.5.1 is the following.

COROLLARY 3.5.7. (a) *There is the following additive isomorphism*:

$$H^*(B\hat{S}O_{22}\langle 8\rangle) \approx \mathbf{Z}_2[w_8, w_{12}, w_{14}, w_{15}, w_{16}, w_{20}, w_{22}]/J \otimes B,$$

where

$$B = \mathbf{Z}_2[\eta_{2^{11}}] \otimes \mathbf{Z}_2[\Theta_{2^5}, \Theta_{2^6}, \dots, \Theta_{2^{10}}] \otimes \mathbf{Z}_2[Sq^I\iota_3;\ I = (i_1, \dots, i_r)$$

is an admissible sequence with $i_r = 1$ *and* $e(I) < 3]$

$$\otimes \mathbf{Z}_2[Sq^{2^k}Sq^{2^{k-1}} \dots Sq^2\iota_3;\ k \geqq 6]$$

$$\otimes \mathbf{Z}_2[\zeta_{2^4+2^3}, \zeta_{2^5+2^3}, \zeta_{2^6+2}, \zeta_{2^6+2^2}, \zeta_{2^6+2^3}, \zeta_{2^7+2}],$$

and J is the ideal generated by

$$\{(w_{14}\cdot w_{20}+w_{12}\cdot w_{22}),\ w_{15}w_{20},\ w_{15}w_{22},\ (w_{22}^3+w_{22}\cdot w_{16}w_{14}^2+w_{22}^2w_8w_{14})\},$$

Θ_{2^i}, $i=5,6,\dots,10$ *correspond to the vanishing of* v_{2^i+1} *under the differentials,*

$$\{\zeta_{2^4+2^3},\ \zeta_{2^5+2^3},\ \zeta_{2^6+2},\ \zeta_{2^6+2^2},\ \zeta_{2^6+2^3},\ \zeta_{2^7+2}\}$$

correspond to the vanishing of

$$\{v_{2^4+2^3+1},\ v_{2^5+2^3+1},\ v_{2^6+2+1},\ v_{2^6+2^2+1},\ v_{2^6+2^3+1},\ v_{2^7+2+1}\}$$

under the differentials in the spectral sequence and $\eta_{2^{11}}$ *comes from* $H^*(B\mathrm{Spin}_{22})$ *by Theorem* 2.5.

(b) *Additively,*

$$\begin{aligned} &H^*(B\hat{SO}_{22}\langle 16\rangle) \approx \mathbf{Z}_2[w_{16}]\otimes B\\ &\otimes \tilde{I}(Sq^1\iota_7, Sq^2\iota_7)\\ &\otimes \mathbf{Z}_2[\xi_{33}, \xi_{34}, \xi_{36}, \xi_{65}]\\ &\otimes \mathbf{Z}_2[\{Y_{3,-1}^2, Y_{2,1}, Y_{2,0}^2, Y_{2,-1}^4, Y_{3,1}, Y_{3,0}^2, Y_{5,-1}, Y_{4,0}, Y_{4,-1}^2, Y_{2,2}\}]\\ &\otimes \mathbf{Z}_2[\{Y_{l+1-m,m-2}\}_{l\geqq 6}, l>m\geqq 1], \end{aligned}$$

where $\{\xi_{33}, \xi_{34}, \xi_{36}, \xi_{65}\}$ corresponds to the vanishing of the following relations under the differentials in the spectral sequence:

$$\{(w_{14}\cdot w_{20}+w_{12}\cdot w_{22}),\ w_{15}w_{20},\ w_{15}w_{22},\ (w_{22}^3+w_{22}w_{16}\cdot w_{14}^2+w_{22}^2\cdot w_8\cdot w_{14})\}.$$

References

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. *72* (1960), 20-104.
2. A. Borel, *Sur la cohomologie des espaces fibres principau et des espaces homogeneous de groupes de Lie compactes*, Ann. of Maths *57* (1953), 115-207.
3. Tze-Beng Ng, *The existence of 7-fields and 8-fields on manifolds*, Quart. J. Math. Oxford *30* (1979), 197-221.
4. D. Quillen, *The* mod 2 *cohomology rings of extra-special 2-groups and the spinor groups*, Math. Ann. *194* (1971), 197-212.
5. R. E. Strong, *Determination of* $H^*(BO(k,\dots,\infty),\mathbf{Z}_2)$ *and* $H^*(BU(k,\dots,\infty),\mathbf{Z}_2)$, Trans. Amer. Math. Soc. *107* (1963), 526-544.
6. W. T. Wu, *Les i-carrês dans une variete grassmannienne*, C. R. Acad. Sci. Paris *230* (1950), 918-920.

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