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A Note on Algebras that are Sums of Two Subalgebras

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Abstract. We study an associative algebra A over an arbitrary field that is a sum of two subalgebras B and C (*i.e.*, $A = B + C$). We show that if B is a right or left Artinian *PI* algebra and C is a *PI* algebra, then A is a *PI* algebra. Additionally, we generalize this result for semiprime algebras A. Consider the class of all semisimple finite dimensional algebras $A = B + C$ for some subalgebras B and C that satisfy given polynomial identities $f = 0$ and $g = 0$, respectively. We prove that all algebras in this class satisfy a common polynomial identity.

1 Introduction

Let R be an associative ring and let R_1, R_2 be subrings such that $R = R_1 + R_2$ (we keep this notation throughout the paper). In [3], K. I. Beidar and A. V. Mikhalev stated the following problem: if the subrings R_i satisfy polynomial identities (in short, *PI* rings), is it true that also R is a *PI* ring? The problem is still open. A positive answer in particular cases can be found in many papers (*cf.* [2–5, 9–12]). In [11] it was shown that if R_1 is a nil ring of bounded index (*i.e.*, satisfies identity $x^{n_i} = 0$) and R_2 is a *PI* ring then R is a *PI* ring. Beidar and Mikhalev proved in [3] that if R_1 satisfies an identity $[x_1, x_2] \cdots [x_{2m-1}, x_{2m}] = 0$ and R_2 satisfies an identity $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] = 0$ for some $m, n \geq 2$, then R is a *PI* ring. We say that R satisfies (f, g) if R_1 and R_2 satisfy polynomial identities $f = 0$ and $g = 0$, respectively. Consider the class (f, g) of all rings R that satisfy (f, g) for fixed f and g. Denote this class by $\mathcal{R}(f, g)$. Since $\mathcal{R}(f, g)$ is closed under direct products, it follows that if a Beidar–Mikhalev problem has a positive solution, then all rings in $\mathcal{R}(f, g)$ satisfy a common polynomial identity. Similarly, by the results cited above, all rings in $\mathcal{R}(x^n, g)$ and correspondingly in $\mathcal{R}([x_1, x_2] \cdots [x_{2m-1}, x_{2m}], [x_1, x_2] \cdots [x_{2n-1}, x_{2n}])$ satisfy a common polynomial identity.

Let K be a field. Assume that $A = B + C$ is a K-algebra and B, C are subalgebras of A. Some results concerning algebras in the context of the Beidar–Mikhalev problem can be found in [6–8]. Let $f = 0$ and $g = 0$ be given polynomial identities and let $\mathcal{M}$ be the class of all semisimple finite-dimensional K-algebras A, where B satisfies $f = 0$ and C satisfies $g = 0$. Note that $\mathcal{M}$ is not closed under direct products. In this paper we show that all algebras in $\mathcal{M}$ satisfy a common polynomial identity.

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We say that an algebra H almost satisfies a certain property w if it has an ideal of finite codimension in H that satisfies w. Suppose that $f = 0$ and $g = 0$ are polynomial identities such that every ring in $\mathcal{R}(f,g)$ is a *PI* ring. In [6] it was proved that if B almost satisfies $f = 0$ and C almost satisfies $g = 0$, then A is a *PI* algebra. This observation was used in [6] to extend the aforementioned two results (from [3, 11]) to algebras. In this paper we present further results of this type. We prove that if B is a right (or left) Artinian *PI* algebra and C is a *PI* algebra, then A is a *PI* algebra. Moreover, we show that if B is an almost nil *PI* algebra, C is a *PI* algebra and A is semiprime, then A is a *PI* algebra.

2 Preliminary Material

We consider associative algebras over a fixed field K that are not assumed to have an identity. Suppose that A is an algebra and B and C are subalgebras of A such that $A = B + C$ (we keep this notation throughout the paper). For a K-linear space H, we write $\dim H$ instead of $\dim_K H$. To denote that I is an ideal (right ideal) of an algebra H, we write $I \lhd H$ ($I <_r H$). We denote the degree of a polynomial f by $\deg f$.

The following theorem is the basic fact that we shall use in this note.

Theorem 2.1 ([12, Theorem 3]) *Suppose $\mathfrak{F}$ is a homomorphically closed class of rings that is closed under direct powers. If every nonzero prime ring in $\mathfrak{F}$ contains a nonzero one sided PI ideal, then $\mathfrak{F}$ consists of PI rings.*

We will also use the following two results from [6].

Lemma 2.2 ([6, Lemma 4]) *Let H be a K-algebra and let S, T be finite-dimensional K-subspaces of H. If M and P are K-subspaces of H such that* $\dim(SMT + P)/P < \infty$, *then* $\dim M/N < \infty$, *where* $N = \{v \in M \mid SvT \subseteq P\}$.

Remark 2.3 It is not hard to check that

$$\dim M/N \le (\dim S)(\dim T)\big(\dim(SMT + P)/P\big).$$

Let $A = B + C$, $B_0 \lhd B$, and $C_0 \lhd C$, where $\dim B/B_0 < \infty$ and $\dim C/C_0 < \infty$.

Lemma 2.4 ([6, Lemma 6]) *Suppose that for the algebra A, C_0 is a PI ideal of C. If I is an ideal of the product $\prod A$ such that $\prod B_0 \subseteq I$, then $(\prod A)/I$ is a PI algebra.*

Remark 2.5 Suppose that C_0 satisfies a polynomial identity of degree d. Using Remark 2.3 and the proof of [6, Lemma 6], we can check that $(\prod A)/I$ satisfies a polynomial identity whose degree depends only on d and $\dim A/(B_0 + C_0)$.

Below we prove a certain modification of [1, Theorem 1], which will be used in further considerations.

Lemma 2.6 *Let P be a semiprime ring and let S be a PI subring of P that satisfies a polynomial identity of degree d. Moreover, let J be a nilpotent subring of P and let n be*

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a positive integer such that $J^n = 0$ and $J^{n-1} \neq 0$. If $A_i = J^{n-i}PJ^i \subseteq S$ for all $1 \leq i \leq n-1$, then $n \leq d$.

Proof Suppose that $n > d$. The subring S satisfies a polynomial identity of degree d, so it satisfies the identity

$$x_1x_2\cdots x_d = \sum_{\mathrm{id}\neq\pi\in S_d} \alpha_\pi x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(d)},$$

where S_d is the set of permutations of the set $\{1,2,\ldots,d\}$ and α_π are some integers. Therefore

$$(J^{n-1}P)^d J^d = A_1A_2\cdots A_d = \sum_{\mathrm{id}\neq\pi\in S_d} \alpha_\pi A_{\pi(1)}A_{\pi(2)}\cdots A_{\pi(d)} = 0,$$

so $(J^{n-1}P)^{d+1} = 0$. Since P is semiprime, it follows that $J^{n-1} = 0$, a contradiction. Thus, $n \leq d$. ∎

3 New Results

Let $f = 0$ and $g = 0$ be given polynomial identities and let $\mathcal{S}$ and $\mathcal{T}$ be the classes of all K-algebras satisfying identity $f = 0$ and $g = 0$, respectively.

Let H be a K-algebra. Denote by $\mathcal{H}(H)$ the minimal homomorphically closed class of algebras that is closed under direct powers such that $H \in \mathcal{H}(H)$.

Theorem 3.1 *Let $\mathcal{M}$ be the class of all semisimple finite dimensional K-algebras of the form $A = B + C$, where $B \in \mathcal{S}$ and $C \in \mathcal{T}$. Then all algebras in $\mathcal{M}$ satisfy a common polynomial identity.*

Proof Suppose that $A \in \mathcal{M}$. Let us note that since A is a finite dimensional semisimple algebra, we have $A = A_1 \times A_2 \times \cdots \times A_k$, where the $A_i = B_i + C_i$ are simple and $B_i \in \mathcal{S}$ and $C_i \in \mathcal{T}$. Consequently, without loss of generality, we can assume that the class $\mathcal{M}$ consists of simple algebras. Since $A \otimes_K \overline{K} = B \otimes_K \overline{K} + C \otimes_K \overline{K}$ is a simple finite-dimensional $\overline{K}$-algebra and $B \otimes_K \overline{K}$, $C \otimes_K \overline{K}$ are its PI subalgebras, where $\overline{K}$ is the algebraic closure of K, we can assume additionally that K is an algebraically closed field. Therefore, A is isomorphic to K_n, the ring of $n \times n$ matrices over K. Assume that B satisfies an identity $f = 0$ of degree d. Since B is finite dimensional, the Jacobson radical $J(B)$ of B is nilpotent. Note that the index of nilpotency of $J(B)$, *i.e.*, the smallest positive integer s such that $J(B)^s = 0$, is bounded by d. Furthermore, $B/J(B)$ is a finite direct product of K_{n_i}, where $i = 1,2,\ldots,m$, for some positive integer m and $n_i \leq d$. The subalgebra C has similar properties. Let $\mathcal{M} = \{A_\alpha = B_\alpha + C_\alpha \mid B_\alpha \in \mathcal{S}, C_\alpha \in \mathcal{T} \text{ and } \alpha \in T\}$ for a set T. It is enough to prove that $\overline{A} = \prod_{\alpha\in T} A_\alpha$ is a PI algebra. Since B_α is a finite dimensional K-algebra, we can assume that for all α we have $B_\alpha = D_{\alpha,1} \oplus D_{\alpha,2} \oplus \cdots \oplus D_{\alpha,n_\alpha} \oplus J(B_\alpha)$ for some positive integer n_α. Additionally, $D_{\alpha,l}D_{\alpha,t} \subseteq J(B_\alpha)$, where $l,t \in \{1,2,\ldots,n_\alpha\}$ and $l \neq t$. Let s_α be the index of nilpotency of $J(B_\alpha)$. We follow the convention that $J(B_\alpha)^0 = 1$. If $B_\alpha J(B_\alpha)^{s_\alpha-1} \neq 0$, without loss of generality, we can assume that $D_{\alpha,1}J(B_\alpha)^{s_\alpha-1} \neq 0$. We have also that $(D_{\alpha,2} \oplus \cdots \oplus D_{\alpha,n_\alpha} \oplus J(B_\alpha))D_{\alpha,1}J(B_\alpha)^{s_\alpha-1} = 0$. Let $G_\alpha = B_\alpha + D_{\alpha,1}J(B_\alpha)^{s_\alpha-1}A_\alpha$ if $B_\alpha J(B_\alpha)^{s_\alpha-1} \neq 0$, and $G_\alpha = B_\alpha + J(B_\alpha)^{s_\alpha-1}A_\alpha$

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if $B_\alpha J(B_\alpha)^{s_\alpha-1} = 0$. Clearly, G_α is a subalgebra of A_α and since $B_\alpha \subseteq G_\alpha$, we have $G_\alpha = G_\alpha \cap (B_\alpha + C_\alpha) = B_\alpha + (G_\alpha \cap C_\alpha)$. Let $D_\alpha = D_{\alpha,2} \oplus \cdots \oplus D_{\alpha,n_\alpha} \oplus J(B_\alpha)$ if $B_\alpha J(B_\alpha)^{s_\alpha-1} \neq 0$, and $D_\alpha = B_\alpha$ otherwise. Clearly, $D_\alpha \triangleleft B_\alpha$. Since $D_\alpha G_\alpha \subseteq D_\alpha$, we obtain that D_α is a right *PI* ideal of G_α that satisfies the identity $f = 0$. Consider $\overline{G} = \prod_{\alpha\in T} G_\alpha$, $\overline{B} = \prod_{\alpha\in T} B_\alpha$, $\overline{E} = \prod_{\alpha\in T}(G_\alpha \cap C_\alpha)$ and $\overline{D} = \prod_{\alpha\in T} D_\alpha$. Clearly, $\overline{G} = \overline{B}+\overline{E}$, $\overline{D}$ is a right ideal of $\overline{G}$ and $\overline{D}$ satisfies the identity $f = 0$. Suppose that $D_\alpha = 0$ for all $\alpha \in T$. Based on Lemma 2.4 and Remark 2.5 it is not hard to see that there exists a polynomial identity common for all G_α. Hence by Lemma 2.4, if $\prod \overline{D} \subseteq I$ for some ideal I of $\prod \overline{G}$, then $(\prod \overline{D})/I$ is a *PI* algebra. Therefore, every nonzero algebra from $\mathcal{H}(\overline{G})$ contains a nonzero *PI* ideal. Applying Theorem 2.1 we obtain that $\overline{G}$ is a *PI* algebra. Summing up, we have that for every $\alpha \in T$ there exists a nonzero ideal I_α of B_α such that $\prod_{\alpha\in T} I_\alpha A_\alpha$ is a right *PI* ideal of $\overline{A}$. Analogously, we can prove that for every $\alpha \in T$ there exists a nonzero ideal J_α of C_α such that $\prod_{\alpha\in T} A_\alpha J_\alpha$ is a left *PI* ideal of $\overline{A}$. Let us note that $J_\alpha A_\alpha I_\alpha \subseteq J_\alpha I_\alpha$, so $J_\alpha I_\alpha \neq 0$, since A_α being simple is a prime algebra. Thus, $A_\alpha J_\alpha I_\alpha A_\alpha = A_\alpha$. This implies that every nonzero algebra in $\mathcal{H}(\overline{A})$ contains a nonzero one sided *PI* ideal. Using Theorem 2.1 we get that $\overline{A}$ is a *PI* algebra, which completes the proof. ∎

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2 *Suppose that $\mathcal{X}$ is the class of all finite dimensional K-algebras of the form $A = B + C$, where $B \in \mathcal{S}$ and $C \in \mathcal{T}$. Then there exists a polynomial identity $h = 0$ such that every $A \in \mathcal{X}$ satisfies the identity $h^n = 0$ for some positive integer n.*

We now prove the following theorem.

Theorem 3.3 *Assume $A = B + C$. If B is a right Artinian PI algebra and C is a PI algebra, then A is a PI algebra.*

Proof Adjoining an identity if necessary, we can assume that $K \subseteq A$, $K \subseteq B$ and $K \subseteq C$. Since B is right Artinian, we have that for some positive integer k, $B/J(B) = B_1 \times B_2 \times \cdots \times B_k$, where the B_i are simple algebras and $I = J(B)$ is nilpotent. Additionally, since B is a *PI* algebra, each B_i is finite dimensional over its center. Passing if necessary to $A \otimes_K \overline{K}$, where $\overline{K}$ is the algebraic closure of K, we can assume that $K = \overline{K}$ and all the B_i are finite dimensional over K.

Consequently, B/I is a finite dimensional algebra over K and I is nilpotent. Let $G = B + IA$. Clearly G is a subalgebra of A and $G = B + (C \cap G)$. We show that G is a *PI* algebra. We proceed by induction with respect to n, where n is the index of nilpotency of I. If $n = 1$, the assertion follows from Lemma 2.4. Suppose that $n > 1$ and the result holds for smaller integers. Since $I^n = 0$, it follows that $I^{n-1} <_r G$. Consequently, there exists a two sided nilpotent ideal J of G such that $I^{n-1} \subseteq J$. Note that $G/J = (B + J)/J + ((C \cap G) + J)/J$ and $((B + J)/J)^{n-1} = 0$. Thus, G/J is a *PI* algebra by the induction hypothesis. Since J is nilpotent, we have that G is a *PI* algebra. Consider $L = I + IA$. Obviously, $L <_r A$ and $L \subseteq G$. It follows that L is a *PI* algebra. By Lemma 2.4, if I is an ideal of the product $\prod A$ such that $\prod L \subseteq I$, then

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$(\prod A)/I$ is a *PI* algebra. This implies that every nonzero algebra from $\mathcal{H}(A)$ contains a nonzero right *PI* ideal. Applying Theorem 2.1 we obtain that A is a *PI* algebra. ∎

Remark 3.4 It is easy to see that Theorem 3.3 remains true if we replace "right Artinian" by "left Artinian".

It is well known that if P is a nil *PI* ring, then $P^n \subseteq W(P)$ for some positive integer n, where $W(P)$ denotes the sum of all nilpotent ideals of the ring P. Suppose that R is a ring and R_1, R_2 are its subrings such that $R = R_1 + R_2$. In [12, Theorem 8] it was proved that if R_1 is a nil *PI* ring that satisfies a polynomial identity of degree d and R_2 is a *PI* ring, then $R_1^{d-1} \subseteq \beta(R)$, where $\beta(R)$ denotes the prime radical of R. As a consequence of the result, it was shown in [12, Corollary 9] that if R_1 is a nil *PI* ring, R_2 is a *PI* ring, and R is semiprime, then R is a *PI* ring.

Below we extend [12, Corollary 9] for algebras.

Theorem 3.5 *Assume $A = B + C$. If B is an almost nil PI algebra, C is a PI algebra and A is semiprime, then A is a PI algebra.*

Proof Let $I \triangleleft B$ be a nil *PI* ideal and $\dim B/I = s$. We proceed by induction with respect to s. For $s = 0$ the assertion follows from [12, Corollary 9]. Assume that $s > 0$, and the assertion holds for smaller integers. Assume that I satisfies a polynomial identity $f = 0$ of degree k. Therefore, B satisfies identity

$$g = \big[f(x_{1,1}, x_{2,1}, \dots x_{k,1}), \dots, f(x_{1,2s}, x_{2,2s}, \dots x_{k,2s}) \big] = 0$$

of degree $d = 2sk$.

We show that I is nilpotent. Suppose that J is a nilpotent ideal of I and m is the index of nilpotency of J. Consider $\overline{B} = B + JAJ^{m-1} + J^2AJ^{m-2} + \dots + J^{m-1}AJ$. Clearly, $\overline{B}$ is a subalgebra of A and $\overline{B} = B + (\overline{B} \cap C)$. Let $\overline{I} = I + JAJ^{m-1} + J^2AJ^{m-2} + \dots + J^{m-1}AJ$. It is not hard to check that $\overline{I}$ is a nil ideal of $\overline{B}$. Moreover, B is a homomorphic image of $\overline{B}/(JAJ^{m-1} + J^2AJ^{m-2} + \dots + J^{m-1}AJ)$, so $\overline{B}$ is a *PI* algebra. Let $t \in (\overline{I} \cap (\overline{B} \cap C))$. Since $tA = t\overline{B} + tC$ and $t\overline{B}$ is a nil *PI* algebra and tC is a *PI* algebra, [12, Theorem 8] implies $(t\overline{B})^{d-1} \subseteq (tA)$. However, since A is semiprime and $tA <_r A$, $(\beta(tA))^2 = 0$. Hence for every $t \in (\overline{I} \cap (\overline{B} \cap C))$, $t^{4(d-1)} = 0$. Since $\overline{I}$ is a nil *PI* ideal of $\overline{B}$ and $\dim \overline{B}/\overline{I} \le \dim B/I$, by induction hypothesis we can assume that $\dim \overline{B}/\overline{I} = s$. It follows that $\overline{I} \subseteq I + (\overline{I} \cap (\overline{B} \cap C))$. Applying [8, Lemma 5], we obtain that $\overline{I} \subseteq Q_1 + Q_2$, where Q_1 and Q_2 are subalgebras of A such that $Q_1 \subseteq I$, $Q_2 \subseteq (\overline{I} \cap (\overline{B} \cap C))$ and $Q_1 + Q_2$ is a subalgebra of $\overline{B}$. So $\overline{I}$ satisfies a common polynomial identity of all rings in the class $\mathcal{R}(g, x^{4(d-1)})$ say $h = 0$. By Lemma 2.6, we have that $m < \deg h$. Hence we proved that the index of nilpotency of any nilpotent ideal J of I does not exceed $\deg h$. So I is a nilpotent ideal of B and $\dim B/I < \infty$. Now using a similar argument as in the second paragraph of the proof of Theorem 3.3, we obtain the result. ∎

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