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Asymptotic K-Theory for Groups Acting on $\tilde{A}_2$ Buildings

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Abstract. Let Γ be a torsion free lattice in $G = \mathrm{PGL}(3, \mathbb{F})$ where $\mathbb{F}$ is a nonarchimedean local field. Then Γ acts freely on the affine Bruhat-Tits building $\mathcal{B}$ of G and there is an induced action on the boundary Ω of $\mathcal{B}$. The crossed product C^*-algebra $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$ depends only on Γ and is classified by its K-theory. This article shows how to compute the K-theory of $\mathcal{A}(\Gamma)$ and of the larger class of rank two Cuntz-Krieger algebras.

1 Introduction

Let $\mathbb{F}$ be a nonarchimedean local field with residue field of order q. The Bruhat-Tits building $\mathcal{B}$ of $G = \mathrm{PGL}(n+1, \mathbb{F})$ is a building of type $\tilde{A}_n$ and there is a natural action of G on $\mathcal{B}$. The vertex set of $\mathcal{B}$ may be identified with the homogeneous space G/K, where K is an open maximal compact subgroup of G. The boundary Ω of $\mathcal{B}$ is the homogeneous space G/B, where B is the Borel subgroup of upper triangular matrices in G.

Let Γ be a torsion free lattice in $G = \mathrm{PGL}(n + 1, \mathbb{F})$. Then Γ is automatically cocompact in G [Ser, Chapitre II.1.5, p. 116] and acts freely on $\mathcal{B}$. If $n = 1$, then Γ is a finitely generated free group [Ser], $\mathcal{B}$ is a homogeneous tree, and the boundary Ω is the projective line $\mathbb{P}_1(\mathbb{F})$. If $n \geq 2$ then the group Γ and its action on Ω are not so well understood. In contrast to the rank one case, Γ has Kazhdan's property (T) and by the Strong Rigidity Theorem of Margulis [Mar, Theorem VII.7.1], the lattice Γ determines the ambient Lie group G. Since the Borel subgroup B of G is unique, up to conjugacy, it follows that the action of Γ on Ω is also unique, up to conjugacy. This action may be studied by means of the crossed product C^*-algebra $C(\Omega) \rtimes \Gamma$, which depends only on Γ and may conveniently be denoted by $\mathcal{A}(\Gamma)$.

Geometrically, a locally finite $\tilde{A}_n$ building $\mathcal{B}$ is an n-dimensional contractible simplicial complex in which each codimension one simplex lies on $q + 1$ maximal simplices, where $q \geq 2$. If $n \geq 2$ then the number q is necessarily a prime power and is referred to as the order of the building. The building is the union of a distinguished family of n-dimensional subcomplexes, called *apartments*, and each apartment is a Coxeter complex of type $\tilde{A}_n$. If $\mathcal{B}$ is a locally finite building of type $\tilde{A}_n$, where $n \geq 3$, then $\mathcal{B}$ is the building of $\mathrm{PGL}(n + 1, \mathbb{F})$ for some (possibly non commutative) local field $\mathbb{F}$ [Ron, p. 137]. The case of $\tilde{A}_2$ buildings is somewhat different, because such a building might not be the Bruhat-Tits building of a linear group. In fact this is

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the case for the $\tilde{A}_2$ buildings of many of the groups constructed in [CMSZ]. The boundary Ω of $\mathcal{B}$ is the set of chambers of the spherical building at infinity [Ron, Chapter 9], endowed with a natural totally disconnected compact Hausdorff topology [CMS], [Ca, Section 4].

Given an $\tilde{A}_n$ building $\mathcal{B}$ with vertex set $\mathcal{B}^0$, there is a type map $\tau\colon \mathcal{B}^0 \to \mathbb{Z}/(n+1)\mathbb{Z}$ such that each maximal simplex (chamber) has exactly one vertex of each type. An automorphism α of Δ is said to be *type-rotating* if there exists $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ such that $\tau(\alpha v) = \tau(v) + i$ for all vertices $v \in \mathcal{B}^0$. If $\mathcal{B}$ is the Bruhat-Tits building of $G = \mathrm{PGL}(n+1, \mathbb{F})$ then the action of G on $\mathcal{B}$ is type rotating [St].

Now let Γ be a group of type rotating automorphisms of an $\tilde{A}_n$ building $\mathcal{B}$ and suppose that Γ acts freely on the vertex set $\mathcal{B}^0$ with finitely many orbits. Then Γ acts on the boundary Ω and the rigidity results of [KL] imply that, as in the linear case above, the action is unique up to conjugacy and the crossed product C^*-algebra $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$ depends only on the group Γ.

The purpose of this paper is to compute the K-theory of the algebras $\mathcal{A}(\Gamma)$ in the case $n = 2$. This is done by using the fact that the algebras are higher rank Cuntz-Krieger algebras, whose structure theory was developed in [RS2]. In particular they are purely infinite, simple and nuclear. It was proved in [RS2] that a higher rank Cuntz-Krieger algebra is stably isomorphic to a crossed product of an AF algebra by a free abelian group. The computation of the K-groups is therefore in principle completely routine: no new K-theoretic or geometric ideas are needed. Actually organizing and performing the computations is another matter. This paper does this in the case $n = 2$. The most precise results are obtained in Section 7 for the algebra $\mathcal{A}(\Gamma)$ where Γ is an $\tilde{A}_2$ group; that is Γ acts freely *and* transitively on the vertices of an $\tilde{A}_2$ building. Such groups have been studied intensively in [CMSZ].

The detailed numerical results of our computations are available elsewhere, but we do present, in Example 7.3, the K-theory of $\mathcal{A}(\Gamma)$ for two torsion free lattices Γ in $\mathrm{PGL}(3, \mathbb{Q}_2)$. The non isomorphism of these two groups is seen in the K-theory of $\mathcal{A}(\Gamma)$ but not in the K-theory of the reduced group C^*-algebra $C_r^*(\Gamma)$.

The article concludes with some results on the order of the class of the identity in $K_0\big(\mathcal{A}(\Gamma)\big)$.

2 Groups Acting on $\tilde{A}_2$ Buildings: Statement of the Main Result

Let $\mathcal{B}$ be a finite dimensional simplicial complex, whose maximal simplices we shall call chambers. All chambers are assumed to have the same dimension and adjacent chambers have a common codimension one face. A gallery is a sequence of adjacent chambers. $\mathcal{B}$ is a chamber complex if any two chambers can be connected by a gallery. $\mathcal{B}$ is said to be *thin* if every codimension one simplex is a face of precisely two chambers. $\mathcal{B}$ is said to be *thick* if every codimension one simplex is a face of at least three chambers. A chamber complex $\mathcal{B}$ is called a *building* if it is the union of a family of subcomplexes, called *apartments*, satisfying the following axioms [Br3].

(B0) Each apartment Σ is a thin chamber complex with $\dim \Sigma = \dim \mathcal{B}$.
(B1) Any two simplices lie in an apartment.

(B2) Given apartments Σ, Σ' there exists an isomorphism $\Sigma \to \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.
(B3) $\mathcal{B}$ is thick.

A building of type $\tilde{A}_2$ has apartments which are all Coxeter complexes of type $\tilde{A}_2$. Such a building is therefore a union of two dimensional apartments, each of which may be realized as a tiling of the Euclidean plane by equilateral triangles.

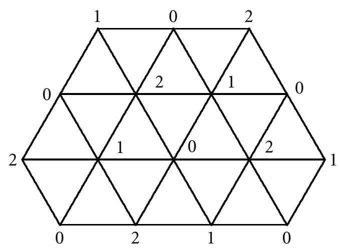

Figure 1: Part of an apartment in an $\tilde{A}_2$ building, showing vertex types.

From now on we shall consider only locally finite buildings of type $\tilde{A}_2$. Each vertex v of $\mathcal{B}$ is labeled with a type $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber has exactly one vertex of each type. An automorphism α of $\mathcal{B}$ is said to be *type rotating* if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau\big(\alpha(v)\big) = \tau(v) + i$ for all vertices $v \in \mathcal{B}$.

A sector is a $\frac{\pi}{3}$-angled sector made up of chambers in some apartment (Figure 2). Two sectors are equivalent if their intersection contains a sector.

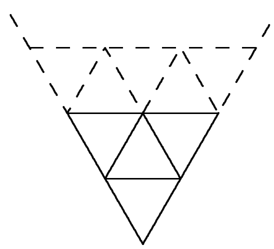

Figure 2: A sector in a building $\mathcal{B}$ of type $\tilde{A}_2$.

The boundary Ω of $\mathcal{B}$ is defined to be the set of equivalence classes of sectors in $\mathcal{B}$. In $\mathcal{B}$ fix some vertex O. For any $\omega \in \Omega$ there is a unique sector $[O, \omega)$ in the class ω having base vertex O [Ron, Theorem 9.6]. The boundary Ω is a totally disconnected compact Hausdorff space with a base for the topology given by sets of the form

$$\Omega(v) = \{\omega \in \Omega \colon [O, \omega) \text{ contains } v\}$$

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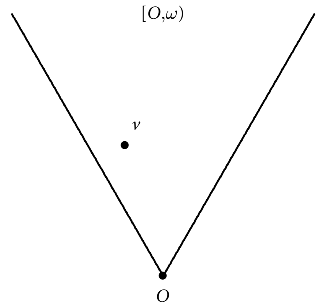

Figure 3: The sector $[O, \omega)$, where $\omega \in \Omega(v)$.

where v is a vertex of $\mathcal{B}$ [CMS, Section 2]. If $\mathcal{B}$ is the (type $\tilde{A}_2$) Bruhat-Tits building of $\mathrm{PGL}(3, \mathbb{F})$ where $\mathbb{F}$ is a nonarchimedean local field then this definition of the boundary coincides with that given in the introduction [St].

Let $\mathcal{B}$ be a locally finite affine building of type $\tilde{A}_2$. Let Γ be a group of type rotating automorphisms of $\mathcal{B}$ that acts freely on the vertex set with finitely many orbits. Let $\mathfrak{t}$ be a model tile for $\mathcal{B}$ consisting of two chambers with a common edge and with vertices coordinatized as shown in Figure 4. For definiteness, assume that the vertex (j, k) has type $\tau(j, k) = j - k \pmod 2 \in \{0, 1, 2\}$. Let $\mathfrak{T}$ denote the set of type rotating isometries $i \colon \mathfrak{t} \to \mathcal{B}$, and let $A = \Gamma \setminus \mathfrak{T}$. Take A as an alphabet. Informally we think of elements of A as labeling the tiles of the building according to Γ-orbits.

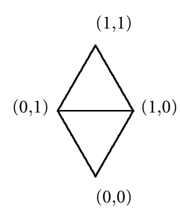

Figure 4: The model tile $\mathfrak{t}$

Two matrices M_1, M_2 with entries in $\{0, 1\}$ are defined as follows. If $a, b \in A$, say that $M_1(b, a) = 1$ if and only if there are representative isometries i_a, i_b in $\mathfrak{T}$ whose ranges lie as shown in the diagram on the right of Figure 5. In that diagram, the tiles $i_a(\mathfrak{t})$, $i_b(\mathfrak{t})$ have base vertices $i_a(0, 0)$, $i_b(0, 0)$ respectively and $i_b(0, 0) = i_a(1, 0)$. A similar definition applies for $M_2(c, a) = 1$.

In Section 5 below the following result is proved. It expresses the K-theory of $\mathcal{A}(\Gamma)$ in terms of the cokernel of the homomorphism $\mathbb{Z}^A \oplus \mathbb{Z}^A \to \mathbb{Z}^A$ defined by $\left(I - M_1, \ I - M_2 \right)$.

Theorem 2.1 *Let Γ be a group of type rotating automorphisms of a building $\mathcal{B}$ of type $\tilde{A}_2$ which acts freely on the set of vertices of $\mathcal{B}$ with finitely many orbits. Let Ω be the boundary of the building and let $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$. Denote by M_1, M_2 the*

$M_2(c, a) = 1$ $\qquad\qquad$ $M_1(b, a) = 1$

Figure 5: Definition of the transition matrices.

associated transition matrices, as defined above. Let r be the rank, and T the torsion part, of the finitely generated abelian group coker $\left(I-M_1,\ I-M_2 \right)$. *Thus* coker $\left(I-M_1,\ I-M_2 \right) = \mathbb{Z}^r \oplus T$. *Then*

$$K_0\big(\mathcal{A}(\Gamma)\big) = K_1\big(\mathcal{A}(\Gamma)\big) = \mathbb{Z}^{2r} \oplus T$$

Remark 2.2 The group Γ in Theorem 2.1 need not necessarily be torsion free. It may have 3-torsion and stabilize a chamber of the building.

3 Rank 2 Cuntz-Krieger Algebras

In [RS2], the authors introduced a class of C^*-algebras which are higher rank analogues of the Cuntz-Krieger algebras [CK]. We shall refer to the original algebras of [CK] as rank one Cuntz-Krieger algebras. The rank 2 case includes the algebras $\mathcal{A}(\Gamma)$ arising from discrete group actions on the boundary of an $\tilde{A}_2$ building as described in Section 2. In this section we shall compute the K-theory of a general rank 2 Cuntz-Krieger algebra $\mathcal{A}$.

We first outline how the algebra $\mathcal{A}$ is defined. For our present investigation of the rank two case the assumptions in [RS2] can be somewhat simplified. Fix a finite set A, which is the "alphabet". Start with a pair of nonzero matrices M_1, M_2 with entries $M_j(b, a) \in \{0, 1\}$ for $a, b \in A$. For an algebra of the form $\mathcal{A}(\Gamma)$, the alphabet A and the matrices M_1, M_2 are defined in Section 2 above.

Let $[m, n]$ denote $\{m, m+1, \dots, n\}$, where $m \le n$ are integers. If $m, n \in \mathbb{Z}^2$, say that $m \le n$ if $m_j \le n_j$ for $j = 1, 2$, and when $m \le n$, let $[m, n] = [m_1, n_1] \times [m_2, n_2]$. In $\mathbb{Z}^2$, let 0 denote the zero vector and let e_j denote the j-th standard unit basis vector. If $m \in \mathbb{Z}^2_+ = \{m \in \mathbb{Z}^2 \,;\, m \ge 0\}$, let

$$W_m = \Big\{ w \colon [0, m] \to A \,;\, M_j\big(w(l + e_j), w(l)\big) = 1 \text{ whenever } l, l + e_j \in [0, m] \Big\}$$

and call the elements of W_m words. Let $W = \bigcup_{m \in \mathbb{Z}^2_+} W_m$. We say that a word $w \in W_m$ has shape $\sigma(w) = m$, and we identify W_0 with A in the natural way via the map $w \mapsto w(0)$. Define the initial and final maps $o \colon W_m \to A$ and $t \colon W_m \to A$ by $o(w) = w(0)$ and $t(w) = w(m)$. We assume that the matrices M_1, M_2 satisfy the following conditions.

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(H0) Each M_i is a nonzero $\{0, 1\}$-matrix.
(H1a) $M_1M_2 = M_2M_1$.
(H1b) M_1M_2 is a $\{0, 1\}$-matrix.
(H2) The directed graph with vertices $a \in A$ and directed edges (a, b) whenever $M_i(b, a) = 1$ for some i, is irreducible.
(H3) For any nonzero $p \in \mathbb{Z}^2$, there exists a word $w \in W$ which is not p-*periodic*, *i.e.* there exists l so that $w(l)$ and $w(l + p)$ are both defined but not equal.

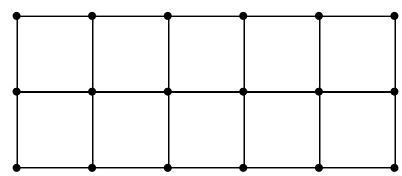

Figure 6: Representation of a two dimensional word v of shape $m = (5, 2)$.

If $v \in W_m$ and $w \in W_{e_j}$ with $t(v) = o(w)$ then there exists a unique word $vw \in W_{m+e_j}$ such that $vw|_{[0,m]} = v$ and $t(vw) = t(w)$ [RS2, Lemma 1.2]. The word vw is called the *product* of v and w.

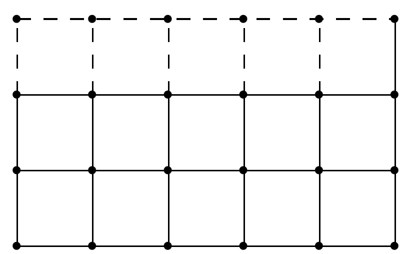

Figure 7: The product word vw, where $w \in W_{e_2}$.

The C^*-algebra $\mathcal{A}$ is the universal C^*-algebra generated by a family of partial isometries $\{s_{u,v}\ ;\ u, v \in W \text{ and } t(u) = t(v)\}$ and satisfying the relations

(3.1a) $$s_{u,v}{}^* = s_{v,u}$$

(3.1b) $$s_{u,v}s_{v,w} = s_{u,w}$$

(3.1c) $$s_{u,v} = \sum_{\substack{w\in W;\sigma(w)=e_j,\\ o(w)=t(u)=t(v)}} s_{uw,vw}$$

(3.1d) $$s_{u,u}s_{v,v} = 0 \quad \text{for } u, v \in W_0,\ u \neq v.$$

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It was shown in [RS2] that $\mathcal{A}$ is simple, unital, nuclear and purely infinite, and that it is therefore classified by its K-theory. Moreover the algebra $\mathcal{A}(\Gamma)$ arising from a discrete group action on the boundary of an $\tilde{A}_2$ building is stably isomorphic to the corresponding algebra $\mathcal{A}$. See Section 5.

By [RS2, Section 6], the stabilized algebra $\mathcal{A} \otimes \mathcal{K}$ can be constructed as a crossed product by $\mathbb{Z}^2$. The details are as follows. Let $\mathcal{C} = \bigoplus_{a \in A} \mathcal{K}(\mathcal{H}_a)$, where $\mathcal{H}_a$ is a separable infinite dimensional Hilbert space. For each $l \in \mathbb{Z}_+^2$ there is an endomorphism $\alpha_l \colon \mathcal{C} \to \mathcal{C}$ defined by the equation

$$\alpha_l(x) = \sum_{w \in W_l} v_w x v_w^* \tag{3.2}$$

where v_w is a partial isometry with initial space $\mathcal{H}_{o(w)}$ and final space lying inside $\mathcal{H}_{t(w)}$. For the precise definition we refer to [RS2], where v_w is denoted $\psi(s'_{w,o(w)})$. For each $m \in \mathbb{Z}^2$ let $\mathcal{C}^{(m)}$ be an isomorphic copy of $\mathcal{C}$, and for each $l \in \mathbb{Z}_+^2$, let

$$\alpha_l^{(m)} \colon \mathcal{C}^{(m)} \to \mathcal{C}^{(m+l)}$$

be a copy of α_l. Let $\mathcal{F} = \varinjlim \mathcal{C}^{(m)}$ be the direct limit of the category of C^*-algebras with objects $\mathcal{C}^{(m)}$ and morphisms α_l. Then $\mathcal{F}$ is an AF-algebra. Since α_l is an endomorphism, we may identify $\mathcal{C}^{(m)}$ with its image in $\mathcal{F}$ ([KR, Proposition 11.4.1]). If $x \in \mathcal{C}$, let $x^{(m)}$ be the corresponding element of $\mathcal{C}^{(m)}$. Now $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{F} \rtimes \mathbb{Z}^2$, so that $K_*(\mathcal{A}) = K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$. The action of $\mathbb{Z}^2$ on $\mathcal{F}$ is defined by two commuting generators T_1, T_2, where $T_j(x^{(m)}) = x^{(m-e_j)}$, $j = 1, 2$. We have $K_0(\mathcal{F}) = \varinjlim K_0(\mathcal{C}^{(m)})$. The maps T_j induce maps $(T_j)_* \colon K_0(\mathcal{F}) \to K_0(\mathcal{F})$, $j = 1, 2$.

Remark 3.1 Note that in [RS2] we considered a more general algebra denoted $\mathcal{A}_D$, where D is a nonempty countable set (called the set of "decorations") and there is an associated map $\delta \colon D \to A$. The algebra $\mathcal{A}$ described above is simply the algebra $\mathcal{A}_A$ where $D = A$ and δ is the identity map on A. It was shown in [RS2, Lemma 4.13, Corollary 4.16] that for any set D of decorations there exists an isomorphism $\mathcal{A}_D \otimes \mathcal{K} \cong \mathcal{A} \otimes \mathcal{K}$. The algebra $\mathcal{A}_D$ therefore has the same K-theory as $\mathcal{A}$, namely the K-theory of the algebra $\mathcal{F} \rtimes \mathbb{Z}^2$.

4 K-Theory for Rank 2 Cuntz-Krieger Algebras

Consider the chain complex

$$0 \longleftarrow K_0(\mathcal{F}) \xleftarrow{(1-T_{2*},\ T_{1*}-1)} K_0(\mathcal{F}) \oplus K_0(\mathcal{F}) \xleftarrow{\begin{pmatrix} 1-T_{1*} \\ 1-T_{2*} \end{pmatrix}} K_0(\mathcal{F}) \longleftarrow 0 \tag{4.1}$$

For $j \in \{0, 1, 2\}$, denote by $\mathfrak{H}_j$ the j-th homology group of the complex (4.1). In particular $\mathfrak{H}_0 = \operatorname{coker}(1 - T_{2*}, T_{1*} - 1)$ and $\mathfrak{H}_2 = \ker \begin{pmatrix} 1-T_{1*} \\ 1-T_{2*} \end{pmatrix}$.

Proposition 4.1 *If the group $\mathfrak{H}_2$ is free abelian then*

$$K_0(\mathcal{A}) \cong \mathfrak{H}_0 \oplus \mathfrak{H}_2,$$
$$K_1(\mathcal{A}) \cong \mathfrak{H}_1.$$

Proof As observed in the introduction, $K_*(\mathcal{A}) = K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$. It is known that the Baum-Connes Conjecture with coefficients in an arbitrary C^*-algebra is true for the group $\mathbb{Z}^2$ (and much more generally: [BBV] [Tu] [BCH, Section 9]). This implies that $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ coincides with its "γ-part", and $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ may be computed as the limit of a Kasparov spectral sequence [Ka, p. 199, Theorem]. The initial terms of the spectral sequence are $E^2_{p,q} = H_p\big(\mathbb{Z}^2, K_q(\mathcal{F})\big)$, the p-th homology of the group $\mathbb{Z}^2$ with coefficients in the module $K_q(\mathcal{F})$. (See [W, Chapter 5] for an explanation of spectral sequences and their convergence.) Noting that $K_q(\mathcal{F}) = 0$ for q odd (since the algebra $\mathcal{F}$ is AF), it follows that $E^2_{p,q} = 0$ for q odd. Also $E^2_{p,q} = 0$ for $p \notin \{0, 1, 2\}$, and the differential d_2 is zero. Thus

$$E^\infty_{p,q} = E^2_{p,q} = \begin{cases} H_p\big(\mathbb{Z}^2, K_q(\mathcal{F})\big) & \text{if } p \in \{0,1,2\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

To clarify notation, write $G = \mathbb{Z}^2 = \langle s, t \mid st = ts \rangle$. We have a free resolution F of $\mathbb{Z}$ over $\mathbb{Z}G$ given by

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}G \xleftarrow{(1-t,\ s-1)} \mathbb{Z}G \oplus \mathbb{Z}G \xleftarrow{\left(\begin{smallmatrix} 1-s \\ 1-t \end{smallmatrix}\right)} \mathbb{Z}G \longleftarrow 0$$

It follows [Br1, Chapter III.1] that $H_*\big(G, K_0(\mathcal{F})\big) = H_*\big(F \otimes_G K_0(\mathcal{F})\big)$ is the homology of the complex (4.1). Therefore

$$E^\infty_{p,q} = \begin{cases} \mathfrak{H}_p & \text{if } p \in \{0,1,2\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Convergence of the spectral sequence to $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ (see [W, Section 5.2]) means that

$$K_1(\mathcal{F} \rtimes \mathbb{Z}^2) = \mathfrak{H}_1 \tag{4.2}$$

and that there is a short exact sequence

$$0 \longrightarrow \mathfrak{H}_0 \longrightarrow K_0(\mathcal{F} \rtimes \mathbb{Z}^2) \longrightarrow \mathfrak{H}_2 \longrightarrow 0 \tag{4.3}$$

The group $\mathfrak{H}_2$ is free abelian. Therefore the exact sequence (4.3) splits. This proves the result ■

Remark 4.2 Writing $\mathcal{A} \otimes \mathcal{K} = \mathcal{F} \rtimes \mathbb{Z}^2 = (\mathcal{F} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$ and applying the PV-sequence of M. Pimsner and D. Voiculescu one obtains (4.3) without using the Kasparov spectral sequence. See [WO, Remarks 9.9.3] for a description of the PV-sequence and [WO, Exercise 9.K] for an outline of the proof.

$$\begin{array}{ccc} K_0(\mathcal{F}) & \longrightarrow & K_0(\mathcal{F}\rtimes\mathbb{Z}^2) \\ \big\downarrow & \nearrow & \\ \mathfrak{H}_0 & & \end{array}$$

Figure 8

Remark 4.3 From (4.1) one notes that $\mathfrak{H}_0$ is none other than the $\mathbb{Z}^2$-coinvariants of $K_0(\mathcal{F})$. Hence the functorial map $K_0(\mathcal{F}) \to K_0(\mathcal{F}\rtimes\mathbb{Z}^2)$ factors through $\mathfrak{H}_0$ (Figure 8).

It follows from the double application of the PV-sequence (Remark 4.2) that the maps $\mathfrak{H}_0 \to K_0(\mathcal{F}\rtimes\mathbb{Z}^2)$ of equation (4.3) and Figure 8 coincide. In particular, the map in Figure 8 is injective.

Remark 4.4 Double application of the PV-sequence is not sufficient to prove the formula (4.2). However if one generalizes [WO, Exercise 9.K] from $\mathbb{Z}$ to $\mathbb{Z}^2$ one obtains a proof of (4.2) at a (relatively) low level of K-sophistication.

Choose for each $a \in A$, a minimal projection $P_a \in \mathcal{K}(\mathcal{H}_a)$, and let $[P_a]$ denote the corresponding class in $K_0\big(\mathcal{K}(\mathcal{H}_a)\big)$. Then $K_0\big(\mathcal{K}(\mathcal{H}_a)\big) \cong \mathbb{Z}$, with generator $[P_a]$. Identify $\mathbb{Z}^A$ with $K_0(\mathcal{C}) = \bigoplus_a K_0\big(\mathcal{K}(\mathcal{H}_a)\big)$ via the map $(n_a)_{a\in A} \mapsto \sum_{a\in A} n_a[P_a]$. The endomorphism α_l induces a map $(\alpha_l)_*$ on K_0. The following lemma is crucial for the calculations which follow.

Lemma 4.5 *The map $(\alpha_{e_j})_*\colon K_0(\mathcal{C}) \to K_0(\mathcal{C})$ is given by the matrix $M_j\colon \mathbb{Z}^A \mapsto \mathbb{Z}^A$, $j = 1, 2$.*

Proof Note that $(\alpha_{e_j})_*([P_a]) = [\alpha_{e_j}(P_a)]$. Now

$$\alpha_{e_j}(P_a) = \sum_{w\in W_{e_j}} v_w P_a v_w^* = \sum_{w\in W_{e_j};o(w)=a} v_w P_a v_w^*.$$

If $t(w) = b$ then $v_w P_a v_w*$ is a minimal projection in $\mathcal{K}(\mathcal{H}_b)$, and so its class in $K_0\big(\mathcal{K}(\mathcal{H}_b)\big)$ equals $[P_b]$. Therefore

$$(\alpha_{e_j})_*([P_a]) = \sum_{w\in W_{e_j};o(w)=a} [v_w P_a v_w^*] = \sum_{b;M_j(b,a)=1} [P_b].$$

Consequently

$$(\alpha_{e_j})_*\Big(\sum_a n_a[P_a]\Big) = \sum_a n_a \sum_b M_j(b,a)[P_b] = \sum_b \Big(\sum_a M_j(b,a)n_a\Big)[P_b].$$

This proves the result. ∎

Recall that M_1 and M_2 commute. If $l = (l_1, l_2) \in \mathbb{Z}_+^2$ then we write $(M_1, M_2)^l = M_1^{l_1} M_2^{l_2}$.

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Remark 4.6 The direct limit $K_0(\mathcal{F})$ may be constructed explicitly as follows. Consider the set $\mathcal{S}$ consisting of all elements $(s_n)_{n\in\mathbb{Z}^2} \in \bigoplus_n K_0(\mathcal{C}^{(n)})$ such that there exists $l \in \mathbb{Z}^2$, for which $s_{n+e_j} = M_j s_n$ for all $n \geq l$, $j = 1,2$. Say that two elements $(s_n)_{n\in\mathbb{Z}^2}$, $(t_n)_{n\in\mathbb{Z}^2}$ of $\mathcal{S}$ are equivalent if there exists $l \in \mathbb{Z}^2$, for which $s_n = t_n$ for all $n \geq l$. Then $K_0(\mathcal{F}) = \varinjlim K_0(\mathcal{C}^{(n)})$ may be identified with $\mathcal{S}$ modulo this equivalence relation. We refer to [Fu, Section 11] for more information about direct limits.

For $c \in K_0(\mathcal{C}) = \mathbb{Z}^A$ let $c^{(m)} \in K_0(\mathcal{C}^{(m)})$ be the corresponding element in $K_0(\mathcal{F})$, defined as follows.

$$c^{(m)} = (s_n)_{n\in\mathbb{Z}^2} \quad \text{where } s_n = \begin{cases} (M_1, M_2)^{(n-m)} c & \text{if } n \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

In particular we identify $c^{(m)}$ with $\big((\alpha_l)_*(c)\big)^{(m+l)}$ for $l \in \mathbb{Z}^2_+$.

Define $\gamma_m \colon K_0(\mathcal{C}) \to K_0(\mathcal{F})$ by $\gamma_m(c) = c^{(m)} \in K_0(\mathcal{F})$.

Remark 4.7 It follows immediately from the definitions that

1. $T_{j*}\gamma_m = \gamma_{m-e_j}$.
2. $\gamma_m(c) = \gamma_{m+l}\big((M_1, M_2)^l c\big)$ for $l \in \mathbb{Z}^2_+$.

Lemma 4.8 *The following assertions hold.*

1. *Any element in $K_0(\mathcal{F})$ can be written as $\gamma_m(c)$ for some $m \in \mathbb{Z}^2_+$ and $c \in K_0(\mathcal{C})$.*
2. *If $c \in K_0(\mathcal{C})$ and $m \in \mathbb{Z}^2$ then $\gamma_m(c) = 0$ if and only if $(M_1, M_2)^l c = 0$ for some $l \in \mathbb{Z}^2_+$.*
3. *$T_{j*}\gamma_m(c) = \gamma_m(M_j c)$ for $j = 1, 2$.*

Proof Statements (1) and (2) follow from the definitions. To prove (3) note that $\big((T_j)_*\gamma_m\big)(c) = (T_j)_*(c^{(m)}) = c^{(m-e_j)} = (M_j c)^{(m)} = (\gamma_m M_j)(c)$. ∎

Lemma 4.9 *For $j = 1, 2$ the map induced on the following complex by M_j acts as the identity on the homology groups.*

$$0 \longleftarrow K_0(\mathcal{C}) \xleftarrow{(I-M_2,\ M_1-I)} K_0(\mathcal{C}) \oplus K_0(\mathcal{C}) \xleftarrow{\binom{I-M_1}{I-M_2}} K_0(\mathcal{C}) \longleftarrow 0 \tag{4.4}$$

Proof Denote by $[\,\cdot\,]$ the equivalence classes in the relevant homology groups.

0-***homology*** If $c \in K_0(\mathcal{C})$, then $[c] - [M_j c] = [(I - M_j)c] = 0$, the zero element in the 0-homology group.

1-***homology*** Let $c_1, c_2 \in K_0(\mathcal{C})$ with $(I - M_2)c_1 = (I - M_1)c_2$. Then $[(c_1, c_2)] - [(M_1c_1, M_1c_2)] = \big[\big((I-M_1)c_1, (I-M_2)c_1\big)\big] = 0$ in the 1-homology group. Likewise for M_2.

2-***homology*** Let $c \in K_0(\mathcal{C})$ with $c = M_1 c = M_2 c$. Then $[c] = [M_j c]$ in the 2-homology group. ∎

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Lemma 4.10

$$\varinjlim\big(K_0(\mathcal{C}^{(m)}) \xrightarrow{M_j} K_0(\mathcal{C}^{(m)})\big) = K_0(\mathcal{F}) \xrightarrow{T_{j*}} K_0(\mathcal{F})$$

Proof The direct limit of maps makes sense because the diagram

$$\begin{array}{ccc} K_0(\mathcal{C}^{(m)}) & \xrightarrow{(M_1,M_2)^l} & K_0(\mathcal{C}^{(m+l)}) \\ {\scriptstyle M_j}\uparrow & & \uparrow{\scriptstyle M_j} \\ K_0(\mathcal{C}^{(m)}) & \xrightarrow{(M_1,M_2)^l} & K_0(\mathcal{C}^{(m+l)}) \end{array}$$

commutes. The diagram

$$\begin{array}{ccc} K_0(\mathcal{C}^{(m)}) & \longrightarrow & K_0(\mathcal{F}) \\ {\scriptstyle M_j}\uparrow & & \uparrow{\scriptstyle T_{j*}} \\ K_0(\mathcal{C}^{(m)}) & \longrightarrow & K_0(\mathcal{F}) \end{array}$$

commutes by Lemma 4.8. Since $K_0(\mathcal{F}) = \varinjlim K_0(\mathcal{C}^{(m)})$ the result follows from the uniqueness assertion in the universal property of direct limits [Fu, Theorem 11.2]. ■

By Lemma 4.10 we have

$$\begin{aligned} 0 \longleftarrow K_0(\mathcal{F}) &\xleftarrow{(1-T_{2*},\ T_{1*}-1)} K_0(\mathcal{F}) \oplus K_0(\mathcal{F}) \xleftarrow{\binom{I-T_{1*}}{I-T_{2*}}} K_0(\mathcal{F}) \longleftarrow 0 \\ &= \varinjlim\big(0 \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \oplus K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow 0\big) \end{aligned}$$

where the map $K_0(\mathcal{C}^{(m)}) \to K_0(\mathcal{C}^{(m+l)})$ is given by $(M_1, M_2)^l$. Now homology is continuous with respect to direct limits [Sp, Theorems 5.19, 4.17]. Therefore it follows from Lemma 4.9 that

$$\begin{aligned} &\mathrm{Hom}\Big(\varinjlim\big(0 \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \oplus K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow 0\big)\Big) \\ &\quad = \varinjlim\Big(\mathrm{Hom}\big(0 \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \oplus K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow 0\big)\Big) \\ &\quad = \mathrm{Hom}\big(0 \longleftarrow K_0(\mathcal{C}) \longleftarrow K_0(\mathcal{C}) \oplus K_0(\mathcal{C}) \longleftarrow K_0(\mathcal{C}) \longleftarrow 0\big) \end{aligned}$$

where Hom denotes the homology of the complex. We have proved

Lemma 4.11 *The map of complexes in Figure 9 induces isomorphisms of the homology groups.*

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$$\begin{array}{ccccccccc}
0 & \longleftarrow & K_0(\mathcal{F}) & \xleftarrow{(\,1-T_{2*},\ T_{1*}-2)} & K_0(\mathcal{F}) \oplus K_0(\mathcal{F} & \xleftarrow{\left(\begin{smallmatrix} 1-T_{1*} \\ 1-T_{2*} \end{smallmatrix}\right)} & K_0(\mathcal{F}) & \longleftarrow & 0 \\
 & & \gamma_0 \Big\uparrow & & \left(\begin{smallmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) \Big\uparrow & & \gamma_0 \Big\uparrow & & \\
0 & \longleftarrow & K_0\mathcal{C} & \xleftarrow[(\,I-M_2,\ M_1-I\,)]{} & K_0\mathcal{C} \oplus K_0(\mathcal{C}) & \xleftarrow[\left(\begin{smallmatrix} I-M_1 \\ I-M_2 \end{smallmatrix}\right)]{} & K_0(\mathcal{C}) & \longleftarrow & 0
\end{array}$$

Figure 9

Recall that $\mathfrak{H}_j$ denotes the j-th homology group of the complex (4.1). Lemma 4.11 shows that $\mathfrak{H}_j$ is the j-th homology group of (4.4), *i.e.* the j-th homology group of the complex

$$0 \longleftarrow \mathbb{Z}^A \xleftarrow{(\,I-M_2,\ M_1-I\,)} \mathbb{Z}^A \oplus \mathbb{Z}^A \xleftarrow{\left(\begin{smallmatrix} I-M_1 \\ I-M_2 \end{smallmatrix}\right)} \mathbb{Z}^A \longleftarrow 0 \tag{4.5}$$

Remark 4.12 In particular, $\mathfrak{H}_2$ is a free abelian group, and so Proposition 4.1 applies.

Let $\operatorname{tor}(G)$ denote the torsion part of the finitely generated abelian group G, and let $\operatorname{rank}(G)$ denote the rank of G; that is the rank of the free abelian part of G (also sometimes called the torsion-free rank of G).

We have, by definition

1. $\mathfrak{H}_0 = \operatorname{coker}\,(\,I-M_2,\ M_1-I\,)$,
2. $\mathfrak{H}_2 = \ker \left(\begin{smallmatrix} I-M_1 \\ I-M_2 \end{smallmatrix}\right)$,
3. $\mathfrak{H}_1 = \ker\,(\,I-M_2,\ M_1-I\,) \,/\, \operatorname{im} \left(\begin{smallmatrix} I-M_1 \\ I-M_2 \end{smallmatrix}\right)$.

The next result determines the K-theory of the algebra $\mathcal{A}$ in terms of the matrices M_1 and M_2.

Proposition 4.13 *The following equalities hold.*

$$\begin{aligned}
\operatorname{rank}\big(K_0(\mathcal{A})\big) &= \operatorname{rank}\big(K_1(\mathcal{A})\big) \\
&= \operatorname{rank}\big(\operatorname{coker}\,(\,I-M_1,\ I-M_2\,)\big) + \operatorname{rank}\big(\operatorname{coker}\,(\,I-M_1^t,\ I-M_2^t\,)\big) \\
\operatorname{tor}\big(K_0(\mathcal{A})\big) &\cong \operatorname{tor}\big(\operatorname{coker}\,(\,I-M_1,\ I-M_2\,)\big) \\
\operatorname{tor}\big(K_1(\mathcal{A})\big) &\cong \operatorname{tor}\big(\operatorname{coker}\,(\,I-M_1^t,\ I-M_2^t\,)\big).
\end{aligned}$$

In particular $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ have the same torsion free parts.

Proof We have $\operatorname{rank}\ker \left(\begin{smallmatrix} I-M_1 \\ I-M_2 \end{smallmatrix}\right) = \operatorname{rank}\operatorname{coker}\,(\,I-M_1^t,\ I-M_2^t\,)$. Hence, by Proposition 4.1 (and Remark 4.12),

$$\begin{aligned}
\operatorname{rank}\big(K_0(\mathcal{A})\big) &= \operatorname{rank}(\mathfrak{H}_0) + \operatorname{rank}(\mathfrak{H}_2) \\
&= \operatorname{rank}\big(\operatorname{coker}\,(\,I-M_1,\ I-M_2\,)\big) + \operatorname{rank}\big(\operatorname{coker}\,(\,I-M_1^t,\ I-M_2^t\,)\big).
\end{aligned}$$

Also

$$\begin{aligned}\operatorname{rank}\big(K_1(\mathcal{A})\big) &= \operatorname{rank}(\mathfrak{H}_1)\\ &= \operatorname{rank}\big(\ker\left(\, I-M_2,\ M_1-I\,\right)\big) - \operatorname{rank}\Big(\operatorname{im}\left(\begin{smallmatrix} I-M_1\\ I-M_2\end{smallmatrix}\right)\Big)\\ &= 2n - \operatorname{rank}\big(\operatorname{im}\left(\, I-M_1,\ I-M_2\,\right)\big) - \operatorname{rank}\big(\operatorname{im}\left(\, I-M_1^t,\ I-M_2^t\,\right)\big)\\ &= \operatorname{rank}\big(\operatorname{coker}\left(\, I-M_1,\ I-M_2\,\right)\big) + \operatorname{rank}\big(\operatorname{coker}\left(\, I-M_1^t,\ I-M_2^t\,\right)\big).\end{aligned}$$

Since $\operatorname{tor}(\mathfrak{H}_0) = \operatorname{tor}\big(\operatorname{coker}\left(\, I-M_1,\ I-M_2\,\right)\big)$ and $\operatorname{tor}(\mathfrak{H}_2) = 0$, it follows that

$$\operatorname{tor}\big(K_0(\mathcal{A})\big) = \operatorname{tor}\big(\operatorname{coker}\left(\, I-M_1,\ I-M_2\,\right)\big).$$

Finally

$$\operatorname{tor}\big(K_1(\mathcal{A})\big) = \operatorname{tor}(\mathfrak{H}_1) = \operatorname{tor}\Big(\operatorname{coker}\left(\begin{smallmatrix} I-M_1\\ I-M_2\end{smallmatrix}\right)\Big) = \operatorname{tor}\big(\operatorname{coker}\left(\, I-M_1^t,\ I-M_2^t\,\right)\big)$$

where the last equality follows from the Smith normal form for integer matrices. ■

5 K-Theory for Boundary Algebras Associated With $\tilde{A}_2$ Buildings

Return now to the setup of Section 2. That is, let $\mathcal{B}$ be a locally finite affine building of type $\tilde{A}_2$. Let Γ be a group of type rotating automorphisms of $\mathcal{B}$ that acts freely on the vertex set with finitely many orbits. Let A denote the associated finite alphabet and let M_1, M_2 be the transition matrices with entries indexed by elements of A.

It was shown in [RS2] that the conditions (H0), (H1a), (H1b) and (H3) of Section 3 are satisfied by the matrices M_1, M_2. It was also proved in [RS2] that condition (H2) is satisfied if Γ is a lattice subgroup of $\mathrm{PGL}_3(\mathbb{F})$, where $\mathbb{F}$ is a local field of characteristic zero. The proof uses the Howe-Moore Ergodicity Theorem. In forthcoming work of T. Steger it is shown how to extend the methods of the proof of the Howe-Moore Theorem and so prove condition (H2) in the stated generality.

It follows from [RS2, Theorem 7.7] that the algebra $\mathcal{A}(\Gamma)$ is stably isomorphic to the algebra $\mathcal{A}$. Moreover if the group Γ also acts transitively on the vertices of $\mathcal{B}$ (which is the case in the examples of Section 7) then $\mathcal{A}(\Gamma)$ is isomorphic to $\mathcal{A}$.

Lemma 5.1 *If M_1, M_2 are associated with an $\tilde{A}_2$ building as in Section 2, then there is a permutation matrix $S\colon \mathbb{Z}^A \to \mathbb{Z}^A$ such that $S^2 = I$ and $SM_1S = M_2^t$, $SM_2S = M_1^t$. In particular* $\operatorname{coker}\left(\, I-M_1,\ I-M_2\,\right) = \operatorname{coker}\left(\, I-M_1^t,\ I-M_2^t\,\right)$

Proof Define $s\colon \mathfrak{t} \to \mathfrak{t}$ by $s(i)(j,k) = i(1-k, 1-j)$ for $i \in \mathfrak{T}$ and $0 \le j,k \le 1$. Then s is the type preserving isometry of $\mathfrak{t}$ given by reflection in the edge $[(0,1),(1,0)]$. (See Figure 9.) Now define a permutation $s\colon A \to A$ by $s(\Gamma i) = \Gamma s(i)$. If $a = \Gamma i_a$, $b = \Gamma i_b \in A$ then it is clear that $M_1(b,a) = 1 \Leftrightarrow M_2\big(s(a), s(b)\big) = 1$. The situation is illustrated, not too cryptically we hope, in Figure 10, where the tiles are located in the building $\mathcal{B}$ and, for example, the tile labeled a is the range of a suitable isometry $i_a\colon \mathfrak{t} \to \mathcal{B}$ with $a = \Gamma i_a$. Let S be the permutation matrix corresponding to s. Then

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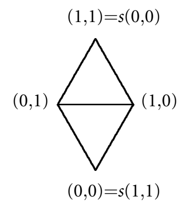

Figure 10: The reflection s of a tile t

$M_1(b, a) = 1$ $\qquad$ $M_2\big(s(a), s(b)\big) = 1$

Figure 11: Reversing transitions between tiles

$M_1(b, a) = 1 \Leftrightarrow SM_2S^{-1}(a, b) = 1$. Clearly $S^2 = I$. Therefore $M_1^t = SM_2S$. A similar argument proves the other equality. ∎

The proof of Theorem 2.1 now follows immediately from Proposition 4.13 and Lemma 5.1. The next result identifies which of the algebras $\mathcal{A}(\Gamma)$ are rank one algebras.

Corollary 5.2 *Continue with the hypotheses of Theorem 2.1. The following are equivalent.*

1. *The algebra $\mathcal{A}(\Gamma)$ is isomorphic to a rank one Cuntz-Krieger algebra;*
2. *the algebra $\mathcal{A}(\Gamma)$ is stably isomorphic to a rank one Cuntz-Krieger algebra;*
3. *the group $K_0\big(\mathcal{A}(\Gamma)\big)$ is torsion free.*

Proof The K-theory of a rank one Cuntz Krieger algebra $\mathcal{O}_A$ can be characterized as follows (see [C1]):

$$K_0(\mathcal{O}_A) = (\text{finite group}) \oplus \mathbb{Z}^k \,;\, K_1(\mathcal{O}_A) = \mathbb{Z}^k.$$

By Theorem 2.1, we have $K_0 = K_1$ for the algebra $\mathcal{A}(\Gamma)$. Since stably isomorphic algebras have the same K-theory, it follows that if $\mathcal{A}(\Gamma)$ is stably isomorphic to a rank one Cuntz-Krieger algebra then $K_0\big(\mathcal{A}(\Gamma)\big)$ is torsion free.

On the other hand, suppose that $G_0 = K_0\big(\mathcal{A}(\Gamma)\big)$ is torsion free. Let $g_0 = [\mathbf{1}] \in G_0$ be the class in K_0 of the identity element of $\mathcal{A}(\Gamma)$. By a result of M. Rordam [Ror, Proposition 6.6], there exists a simple rank one Cuntz-Krieger algebra $\mathcal{O}_A$ such that

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$K_0(\mathcal{O}_A) = G_0$ with the class of the identity in $\mathcal{O}_A$ being g_0. Since G_0 is torsion free we necessarily have $K_1(\mathcal{O}_A) = G_0$ and by Theorem 2.1 we also have $K_1\big(\mathcal{A}(\Gamma)\big) = G_0$. Thus $K_*\big(\mathcal{A}(\Gamma)\big) = K_*(\mathcal{O}_A)$ and the identity elements of the two algebras have the same image in K_0. Since the algebras $\mathcal{A}(\Gamma)$ and $\mathcal{O}_A$ are purely infinite, simple, nuclear and satisfy the Universal Coefficient Theorem, it now follows from the Classification Theorem of [Kir, Ph] that they are isomorphic. ■

Remark 5.3 Corollary 5.2 can be used (see Remark 8.4) to verify that almost all the examples of rank 2 Cuntz-Krieger algebras described later are not stably isomorphic to ordinary (rank 1) Cuntz-Krieger algebras.

6 Reduction of Order

Continue with the assumptions of Section 5. The following lemma will simplify the calculation of the K-groups, by reducing the order of the matrices involved.

Lemma 6.1 *Suppose that M_1, M_2 are $\{0,1\}$-matrices acting on $\mathbb{Z}^A$.*

(i) *Let $\hat{A}$ be a set and let $\hat{\pi}\colon A \to \hat{A}$ be a surjection. Suppose that $M_j(b,a) = M_j(b,a')$ if $\hat{\pi}(a) = \hat{\pi}(a')$. Let the matrix $\hat{M}_j$ acting on $\mathbb{Z}^{\hat{A}}$ be given by $\hat{M}_j\big(\hat{b}, \hat{\pi}(a)\big) = \sum_{\hat{\pi}(b)=\hat{b}} M_j(b,a)$. Then the canonical map from $\mathbb{Z}^A$ onto $\mathbb{Z}^{\hat{A}}$ which sends generators to generators induces an isomorphism from* coker $(\, I-M_1,\ I-M_2\,)$ *onto* coker $(\, I-\hat{M}_1,\ I-\hat{M}_2\,)$.

(ii) *Let $\check{A}$ be a set and let $\check{\pi}\colon A \to \check{A}$ be a surjection. Suppose that $M_j(b,a) = M_j(b',a)$ if $\check{\pi}(b) = \check{\pi}(b')$. Let the matrix $\check{M}_j$ acting on $\mathbb{Z}^{\check{A}}$ be given by $\check{M}_j\big(\check{\pi}(b), \check{a}\big) = \sum_{\check{\pi}(a)=\check{a}} M_j(b,a)$. Then the canonical map from $\mathbb{Z}^A$ onto $\mathbb{Z}^{\check{A}}$ which sends generators to generators induces an isomorphism from* coker $(\, I-M_1,\ I-M_2\,)$ *onto* coker $(\, I-\check{M}_1,\ I-\check{M}_2\,)$.

Proof (i) Let $(e_a)_{a\in A}$ be the standard set of generators for the free abelian group $\mathbb{Z}^A$ and $(e_{\hat{a}})_{\hat{a}\in\hat{A}}$ that of $\mathbb{Z}^{\hat{A}}$. Define the map $\pi\colon \mathbb{Z}^A \to \mathbb{Z}^{\hat{A}}$ by $\pi(e_a) = e_{\hat{\pi}(a)}$. Observe that the diagram

$$\begin{array}{ccc} \mathbb{Z}^A & \xrightarrow{M_j} & \mathbb{Z}^A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{Z}^{\hat{A}} & \xrightarrow{\hat{M}_j} & \mathbb{Z}^{\hat{A}} \end{array}$$

commutes. Therefore so does the diagram

$$\begin{array}{ccc} \mathbb{Z}^A \oplus \mathbb{Z}^A & \xrightarrow{(\, I-M_1,\ I-M_2\,)} & \mathbb{Z}^A \\ {\scriptstyle\pi\oplus\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{Z}^{\hat{A}} \oplus \mathbb{Z}^{\hat{A}} & \xrightarrow{(\, I-\hat{M}_1,\ I-\hat{M}_2\,)} & \mathbb{Z}^{\hat{A}} \end{array}$$

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Hence there is a well defined map of cokernels, which is surjective because π is. The kernel of π is generated by $\{e_a - e_{a'} \,;\, \hat{\pi}(a) = \hat{\pi}(a')\}$. Now if $\hat{\pi}(a) = \hat{\pi}(a')$ then according to the hypothesis of the lemma, $M_j e_a = M_j e_{a'}$ and so $(I - M_j)(e_a - e_{a'}) = e_a - e_{a'}$. Hence the kernel of π is contained in the image of the map $(\, I-M_1,\ I-M_2 \,)$. It follows by diagram chasing that the map on cokernels is injective.

(ii) The argument in this case is a little harder but similar. Note that the vertical maps in the diagrams go up rather than down. ■

We now explain how Lemma 6.1 is used in our calculations to reduce calculations based on rhomboid tiles to calculations based on triangles. Let $\hat{\mathsf{t}}$ be the model triangle with vertices $\{(1,1),(0,1),(1,0)\}$, which is the upper half of the model tile t. Let $\hat{\mathfrak{T}}$ denote the set of type rotating isometries $i\colon \hat{\mathsf{t}} \to \mathcal{B}$, and let $\hat{A} = \Gamma \setminus \hat{\mathfrak{T}}$. We think of $\hat{A}$ as labels for triangles in $\mathcal{B}$, just as A is thought of as labels for parallelograms. Each type rotating isometry $i\colon \mathsf{t} \to \mathcal{B}$ restricts to a type rotating isometry $\hat{\imath} = i|_{\hat{\mathsf{t}}}\colon \hat{\mathsf{t}} \to \mathcal{B}$. Define $\hat{\pi}\colon A \to \hat{A}$ by $\hat{\pi}(a) = \Gamma \hat{\imath}_a$ where $a = \Gamma i_a$. It is clear that $M_j(b,a) = M_j(b,a')$

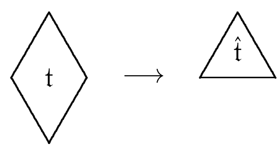

Figure 12: The restriction $\mathsf{t} \to \hat{\mathsf{t}}$

if $\hat{\pi}(a) = \hat{\pi}(a')$. This is illustrated in Figure 12 for the case $M_1(b,a) = 1$. Thus the hypotheses of Lemma 6.1(i) are satisfied.

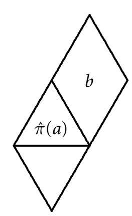

Figure 13: $M_1(b,a) = 1$

Each matrix $\hat{M}_j$ has entries in $\{0,1\}$. For example Figure 13 illustrates the configuration for $\hat{M}_1(\hat{b},\hat{a}) = 1$.

Note that although the matrices $\hat{M}_1$ and $\hat{M}_2$ are used to simplify the final computation, they could not be used to define the algebra $\mathcal{A}$ because their product $\hat{M}_1\hat{M}_2$ need not have entries in $\{0,1\}$. In fact in the gallery of Figure 14 the triangle labels $\hat{a}$ and $\hat{c}$ do not uniquely determine the triangle label $\hat{b}$. In other words there is more than one such two step transition from $\hat{a}$ to $\hat{c}$.

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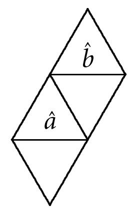

Figure 14: $\hat{M}_1(\hat{b}, \hat{a}) = 1$

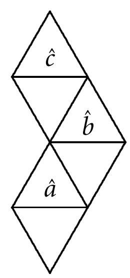

Figure 15: Non uniqueness of two-step transition

Similar arguments apply to the set $\check{A}$ obtained by considering the model triangle $\check{\mathrm{t}}$ with vertices $\{(0,0),(0,1),(1,0)\}$, which is the lower half of the model tile t (Figure 15).

The map $\check{\pi}\colon A \to \check{A}$ is induced by the restriction of Figure 15 and one applies Lemma 6.1(ii) to the resulting matrices $\check{M}_1$, $\check{M}_2$. Figure 16 illustrates the configuration for $\check{M}_1(\check{b}, \check{a}) = 1$.

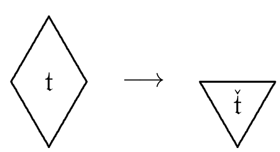

Figure 16: The restriction $\mathrm{t} \to \check{\mathrm{t}}$

The same argument as in Lemma 5.1 shows that there is an isomorphism $V : \mathbb{Z}^{\hat{A}} \to \mathbb{Z}^{\check{A}}$ such that $\hat{M}_1 = V^{-1}\check{M}_2 V$ and vice versa.

We may summarize the preceding discussion as follows.

Corollary 6.2 *Assume the notation and hypotheses of Theorem 2.1. Let* $\hat{M}_j$, $\check{M}_j$ *(*$j =$

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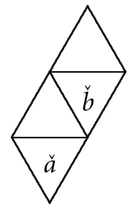

Figure 17: $\check{M}_1(\check{b}, \check{a}) = 1$

$1, 2$*) be the matrices defined as above. Then*

$$K_0(\mathcal{A}_D) = K_1(\mathcal{A}_D) = \mathbb{Z}^{2r} \oplus \operatorname{tor}\big(\operatorname{coker}\,(\, I-\hat{M}_1,\, I-\hat{M}_2 \,)\big)$$
$$= \mathbb{Z}^{2r} \oplus \operatorname{tor}\big(\operatorname{coker}\,(\, I-\check{M}_1,\, I-\check{M}_2 \,)\big)$$

where $r = \operatorname{rank}\big(\operatorname{coker}\,(\, I-\hat{M}_1,\, I-\hat{M}_2 \,)\big) = \operatorname{rank}\big(\operatorname{coker}\,(\, I-\check{M}_1,\, I-\check{M}_2 \,)\big)$.

7 K-Theory for the Boundary Algebra of an $\tilde{A}_2$ Group

Now suppose that Γ is an $\tilde{A}_2$ group. This means that Γ is a group of automorphisms of the $\tilde{A}_2$ building $\mathcal{B}$ which acts freely and transitively in a type rotating manner on the vertex set of $\mathcal{B}$. If Ω is the boundary of $\mathcal{B}$ then the algebra $\mathcal{A} = \mathcal{A}(\Gamma)$ was studied in [RS1], [RS2, Section 7]. Suppose that the building $\mathcal{B}$ has order q. If $q = 2$ there are eight $\tilde{A}_2$ groups Γ, all of which embed as lattices in a linear group $\mathrm{PGL}(3, \mathbb{F})$ over a local field $\mathbb{F}$. If $q = 3$ there are 89 possible $\tilde{A}_2$ groups, of which 65 do not embed naturally in linear groups.

The 1-skeleton of $\mathcal{B}$ is the Cayley graph of the group Γ with respect to its canonical set P of $(q^2 + q + 1)$ generators. The set P is identified with the set of points of a finite projective plane (P, L) and the set of lines L is identified with $\{x^{-1}\ ;\ x \in P\}$. The relations satisfied by the elements of P are of the form $xyz = 1$. There is such a relation if and only if $y \in x^{-1}$, that is the point y is incident with the line x^{-1} in the projective plane (P, L). See [CMSZ] for details.

Since Γ acts freely and transitively on the vertices of $\mathcal{B}$, each element $a \in A$ has a unique representative isometry $i_a \colon \mathfrak{t} \to \mathcal{B}$ such that $i_a(0, 0) = O$, the fixed base vertex of $\mathcal{B}$. We assume for definiteness that the vertex O has type 0. It then follows that the vertex $i_a(1, 0)$ has type 1, $i_a(0, 1)$ has type 2 and $i_a(1, 1)$ has type 0. The combinatorics of the finite projective plane (P, L) shows that there are precisely $q(q + 1)(q^2 + q + 1)$ possible choices for i_a. That is $\#(A) = q(q + 1)(q^2 + q + 1)$. Thinking of the 1-skeleton of $\mathcal{B}$ as the Cayley graph of the group Γ with $O = e$, we shall identify elements of Γ with vertices of $\mathcal{B}$ via $\gamma \mapsto \gamma(O)$.

We now examine the transition matrices M_1, M_2 in this situation. If $a, b \in A$, we have $M_1(b, a) = 1$ if and only if there are representative isometries in a and b respectively whose ranges are tiles which lie as shown in Figure 17. More precisely this means that the ranges $i_a(\mathfrak{t})$ and $i_a(1, 0)b(\mathfrak{t})$ lie in the building as shown in Figure 17, where they are labeled a and b respectively.

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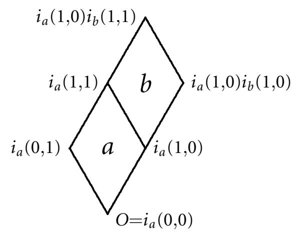

$$M_1(b,a) = 1$$

Figure 18: Transitions between tiles

Lemma 7.1 *The $\{0,1\}$-matrices M_j ($j = 1,2$) have order $\#(A) = q(q+1)(q^2+q+1)$ and each row or column has precisely q^2 nonzero entries.*

Proof Suppose that $a \in A$ has been chosen. Refer to Figure 17. In the link of the vertex $i_a(1,1)$, let the vertices of type 1 correspond to points in P and the vertices of type 2 correspond to lines in L. There are then $q+1$ choices for a line incident with the point $i_a(0,1)$; therefore there are q choices for $i_a(1,0)i_b(1,0)$. After choosing $i_a(1,0)i_b(1,0)$ there are q choices for the point $i_a(1,0)i_b(1,1)$. That choice determines b. There are therefore q^2 choices for b. This proves that for each $a \in A$, there are q^2 choices for $b \in A$ such that $M_1(b,a) = 1$. That is, each column of the matrix M_1 has precisely q^2 nonzero entries. A similar argument applies to rows. ■

In order to compute the K-theory of $\mathcal{A} = \mathcal{A}(\Gamma)$, it follows from Section 6 that we need only compute coker $\left(I-\hat{M}_1, I-\hat{M}_2 \right)$ or equivalently coker $\left(I-\check{M}_1, I-\check{M}_2 \right)$. For definiteness we deal in detail with the former. We shall see that this reduces the order of the matrices by a factor of q. Since Γ acts freely and transitively on the vertices of $\mathcal{B}$, each class $\hat{a} \in \hat{A}$ contains a unique representative isometry $\hat{\imath}_a \colon \mathfrak{t} \to \mathcal{B}$ such that $\hat{\imath}_a(1,1) = O$, the fixed base vertex of $\mathcal{B}$. The isometry $\hat{\imath}_a$ is completely determined by its range which is a triangle in $\mathcal{B}$ whose edges are labeled by generators in P, according to the structure of the 1-skeleton of $\mathcal{B}$ as a Cayley graph. In this way the element $\hat{a} \in \hat{A}$ may be identified with an ordered triple (a_0, a_1, a_2), where $a_0, a_1, a_2 \in P$ and $a_0a_1a_2 = 1$. See Figure 18.

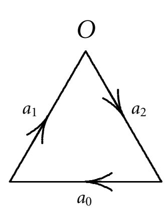

Figure 19: Representation of $\hat{a}$

Note that in this representation of $\hat{a}$ there are (q^2+q+1) choices for a_0. Having chosen a_0, there are $q+1$ choices for a_1, since a_1 is incident with a_0^{-1}. The element a_2 is then uniquely determined. This shows that $\#(\hat{A}) = (q+1)(q^2+q+1)$.

Given $\hat{a} \in \hat{A}$, an element $\hat{b} \in \hat{A}$ satisfies $\hat{M}_1(\hat{b}, \hat{a}) = 1$ if and only if the 1-skeleton of $\mathcal{B}$ contains a diagram of the form shown in Figure 19. In terms of the projective plane (P, L), this diagram is possible if and only if $b_1 \notin a_1^{-1}$ (q^2 choices for b_1). Then b_0 is uniquely specified by $b_0^{-1} = b_1 \vee a_2$, the line containing the points b_1 and a_2. This determines b_2 and hence $\hat{b}$. Thus $\hat{M}_1$ is a $\{0,1\}$-matrix of order $(q+1)(q^2+q+1)$, whose entries are specified by

$$\hat{M}_1(\hat{b}, \hat{a}) = 1 \Leftrightarrow b_1 \notin a_1^{-1}, \quad b_0^{-1} = b_1 \vee a_2. \tag{7.1}$$

In particular for a fixed $\hat{a} \in \hat{A}$ we have $\hat{M}_1(\hat{b}, \hat{a}) = 1$ for precisely q^2 choices of $\hat{b} \in \hat{A}$. That is, each column of the matrix $\hat{M}_1$ has precisely q^2 nonzero entries. Analogously, for a fixed $\hat{b} \in \hat{A}$ we have $\hat{M}_1(\hat{b}, \hat{a}) = 1$ for precisely q^2 choices of $\hat{a} \in \hat{A}$. Again refer to Figure 19. There are precisely q^2 choices of the line a_1^{-1} such that $b_1 \notin a_1^{-1}$. Then a_2 is specified by $a_2 = a_1^{-1} \wedge b_0^{-1}$ and this determines $\hat{a}$ completely.

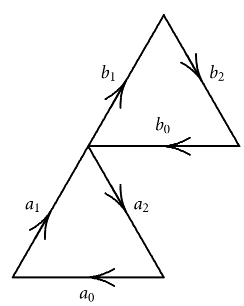

Figure 20: $\hat{M}_1(\hat{b}, \hat{a}) = 1$

A similar argument shows that the $\{0,1\}$-matrix $\hat{M}_2$ is specified by

$$\hat{M}_2(\hat{b}, \hat{a}) = 1 \Leftrightarrow a_2 \notin b_2^{-1}, \quad b_0 = a_1^{-1} \wedge b_2^{-1}. \tag{7.2}$$

Using the preceding discussion and the explicit triangle presentations for $\tilde{A}_2$ groups given in [CMSZ], we may now proceed to compute the K-theory of the algebra $\mathcal{A}$ by means of Corollary 6.2, with "upward pointing" triangles. The authors have done extensive computations for more than 100 different groups with $2 \le q \le 11$, including all possible $\tilde{A}_2$ groups for $q = 2, 3$. The complete results are available at http://maths.newcastle.edu.au/~guyan/Kcomp.ps.gz or from either of the authors.

Everything above applies mutatis mutandis for "downward pointing" triangles. In an obvious notation, illustrated in Figure 20, we have

$$\check{M}_1(\check{b}, \check{a}) = 1 \Leftrightarrow b_2 \notin a_2^{-1}, \quad a_0 = b_1^{-1} \wedge a_2^{-1}. \tag{7.3}$$

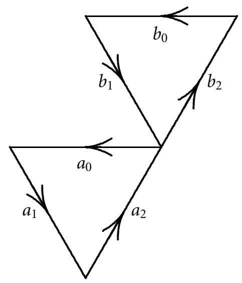

Figure 21: $\check{M}_1(\check{b},\check{a}) = 1$

The accuracy of our computations was confirmed by repeating them with "downward pointing" triangles.

It is convenient to summarize the general structure of the matrices we are considering.

Lemma 7.2 *The $\{0,1\}$-matrices $\hat{M}_j$, $\check{M}_j$ $(j = 1,2)$ have order $\#(\hat{A}) = (q+1)(q^2+q+1)$ and each row or column has precisely q^2 nonzero entries.*

Example 7.3 Consider the following two $\tilde{A}_2$ groups, which are both torsion free lattices in $\mathrm{PGL}(3,\mathbb{Q}_2)$, where $\mathbb{Q}_2$ is the field of 2-adic numbers [CMSZ].

The group B.2 of [CMSZ], which we shall denote $\Gamma_{\mathrm{B.2}}$ has presentation

$$\langle x_i, 0 \le i \le 6 \mid x_0x_1x_4, x_0x_2x_1, x_0x_4x_2, x_1x_5x_5, x_2x_3x_3, x_3x_5x_6, x_4x_6x_6\rangle.$$

The group C.1 of [CMSZ], which we shall denote $\Gamma_{\mathrm{C.1}}$ has presentation

$$\langle x_i, 0 \le i \le 6 \mid x_0x_0x_6, x_0x_2x_3, x_1x_2x_6, x_1x_3x_5, x_1x_5x_4, x_2x_4x_5, x_3x_4x_6\rangle.$$

These groups are not isomorphic. Indeed the MAGMA computer algebra package shows that $\Gamma_{\mathrm{B.2}}$ has a subgroup of index 5, whereas $\Gamma_{\mathrm{C.1}}$ does not. This non isomorphism is revealed by the K-theory of the boundary algebras. Performing the computations above shows that

$$K_0\big(\mathcal{A}(\Gamma_{\mathrm{B.2}})\big) = K_1\big(\mathcal{A}(\Gamma_{\mathrm{B.2}})\big) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z},$$
$$K_0\big(\mathcal{A}(\Gamma_{\mathrm{C.1}})\big) = K_1\big(\mathcal{A}(\Gamma_{\mathrm{C.1}})\big) = (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}.$$

These examples are not typical in that K_* of a boundary algebra usually has a free abelian component. Note also that in both these cases $[\mathbf{1}] = 0$ in $K_0\big(\mathcal{A}(\Gamma)\big)$. See Remark 8.4.

On the other hand, using the results of V. Lafforgue [La], the K-theory of the reduced group C^*-algebras of these groups can easily be computed. The result is the same for these two groups:

$$K_0\big(C_r^*(\Gamma_{\mathrm{B.2}})\big) = K_0\big(C_r^*(\Gamma_{\mathrm{C.1}})\big) = \mathbb{Z},$$
$$K_1\big(C_r^*(\Gamma_{\mathrm{B.2}})\big) = K_1\big(C_r^*(\Gamma_{\mathrm{C.1}})\big) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}.$$

8 The Class of the Identity in K-Theory

Continue with the assumptions of Section 7; that is Γ is an $\tilde{A}_2$ group. Since the algebras $\mathcal{A}(\Gamma)$ are purely infinite, simple, nuclear and satisfy the Universal Coefficient Theorem [RS2, Remark 6.5], it follows from the Classification Theorem of [Kir, Ph] that they are classified by their K-groups together with the class $[\mathbf{1}]$ in K_0 of the identity element $\mathbf{1}$ of $\mathcal{A}(\Gamma)$. It is therefore important to identify this class. We prove that $[\mathbf{1}]$ is a torsion element of K_0.

Let $i \in \mathfrak{T}$, that is, suppose that $i\colon \mathfrak{t} \to \mathcal{B}$ is a type rotating isometry. Let $\Omega(i)$ be the subset of Ω consisting of those boundary points represented by sectors which originate at $i(0,0)$ and contain $i(\mathfrak{t})$. Clearly $\Omega(\gamma i) = \gamma\Omega(i)$ for $\gamma \in \Gamma$. For each $i \in \mathfrak{t}$ let $\mathbf{1}_i$ be the characteristic function of the set $\Omega(i)$.

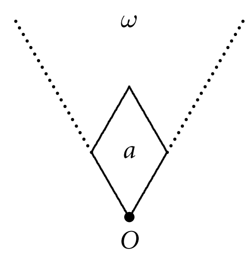

Figure 22: $\mathbf{1}_a(\omega) = 1$

Lemma 8.1 *If $i_1, i_2 \in \mathfrak{T}$ with $\Gamma i_1 = \Gamma i_2$ then $[\mathbf{1}_{i_1}] = [\mathbf{1}_{i_2}]$.*

Proof If $i_1 = \gamma i_2$ with $\gamma \in \Gamma$ then the covariance condition for the action of Γ on $C(\Omega)$ implies that $\mathbf{1}_{i_1} = \gamma \mathbf{1}_{i_2} \gamma^{-1}$. The result now follows because equivalent idempotents belong to the same class in K_0. ■

For each $a \in A$ let $\mathbf{1}_a = \mathbf{1}_{i_a}$. See Figure 21. It follows from the discussion in [CMS, Section 2] that the identity function in $C(\Omega)$ may be expressed as $\mathbf{1} = \sum_{a \in A} \mathbf{1}_a$.

Proposition 8.2 *In the group $K_0\big(\mathcal{A}(\Gamma)\big)$ we have $(q^2 - 1)[\mathbf{1}] = 0$.*

Proof Referring to Figure 17, we have for each $a \in A$, $\mathbf{1}_a = \sum \mathbf{1}_{i_a(1,0)i_b}$, where the sum is over all $b \in A$ such that $i_b(\mathfrak{t})$ lies as shown in Figure 17; that is the sum is over all $b \in A$ such that $M_1(b, a) = 1$. Now by Lemma 8.1, $[\mathbf{1}_{i_a(1,0)i_b}] = [\mathbf{1}_{i_b}] = [\mathbf{1}_b]$ and so

$$[\mathbf{1}_a] = \sum_{b \in A} M_1(b, a)[\mathbf{1}_b].$$

It follows that

$$[\mathbf{1}] = \sum_{a \in A} [\mathbf{1}_a] = \sum_{a \in A} \sum_{b \in A} M_1(b, a)[\mathbf{1}_b].$$

By Lemma 7.1, there are $q^3(q+1)(q^2+q+1)$ nonzero terms in this double sum and each term $[\mathbf{1}_b]$ occurs exactly q^2 times. Thus $[\mathbf{1}] = q^2\sum_{a\in A}[\mathbf{1}_a] = q^2[\mathbf{1}]$, which proves the result. ■

Proposition 8.3 *For $q \not\equiv 1 \pmod 3$, $q-1$ divides the order of $[\mathbf{1}]$. For $q \equiv 1 \pmod 3$, $(q-1)/3$ divides the order of $[\mathbf{1}]$.*

Proof By [RS2, Theorem 7.7], the algebra $\mathcal{A}(\Gamma)$ is isomorphic to the algebra $\mathcal{A}$, which is in turn stably isomorphic to the algebra $\mathcal{F} \rtimes \mathbb{Z}^2$ [RS2, Theorem 6.2]. We refer to Section 1 for notation and terminology. Recall that $\mathcal{F} = \varinjlim \mathcal{C}^{(m)}$ where $\mathcal{C}^{(m)} \cong \bigoplus_{a\in A}\mathcal{K}(\mathcal{H}_a)$. The isomorphism $\mathcal{A}(\Gamma) \to \mathcal{A}$ has the effect $\mathbf{1}_a \mapsto s_{a,a}$ and the isomorphism $\mathcal{A}\otimes\mathcal{K} \to \mathcal{F}\rtimes\mathbb{Z}^2$ sends $s_{a,a}\otimes E_{1,1}$ to a minimal projection $P_a \in \mathcal{K}(\mathcal{H}_a) \subset \mathcal{C}^{(0)} \subset \mathcal{F}$.

As an abelian group in terms of generators and relations, we have

$$\operatorname{coker}\left(\, I-M_1,\ I-M_2 \,\right) = \Big\langle e_a\ ;\ e_a = \sum_b M_j(b,a)e_b,\ j=1,2 \Big\rangle. \tag{8.1}$$

By Lemma 4.11 $\operatorname{coker}(\, I-M_1,\ I-M_2\,)$ is isomorphic to $\mathfrak{H}_0 = \operatorname{coker}(1-T_{2*}, T_{1*}-1)$. Under this identification, $\sum_{a\in A} e_a$ maps to the coset of $[\mathbf{1}] \in K_0(\mathcal{F})$. By Remark 4.3 that coset maps to $[\mathbf{1}] \in K_0(\mathcal{F}\rtimes\mathbb{Z}^2)$ under the injection of (4.3). Thus the order of $[\mathbf{1}] \in K_0(\mathcal{F}\rtimes\mathbb{Z}^2)$ is equal to the order of $\sum_{a\in A} e_a$ in $\operatorname{coker}(\, I-M_1,\ I-M_2\,)$.

Each of the relations in the equation (8.1) expresses a generator e_a as the sum of exactly q^2 generators. It follows that there exists a homomorphism ψ from $\operatorname{coker}(\, I-M_1,\ I-M_2\,)$ to $\mathbb{Z}/(q^2-1)$ which sends each generator to $1+(q^2-1)\mathbb{Z}$. As $\sum_{a\in A} e_a$ has $q(q+1)(q^2+q+1)$ terms,

$$\psi\Big(\sum_{a\in A} e_a\Big) \equiv q(q+1)(q^2+q+1) \equiv 3(q+1) \pmod{q^2-1}$$

Consequently, the order of $\psi(\sum_{a\in A} e_a)$ is

$$\begin{aligned}\frac{q^2-1}{\big(q^2-1, 3(q+1)\big)} = \frac{q^2-1}{(q+1)(q-1,3)} &= \frac{q-1}{(q-1,3)}\\ &= \begin{cases} q-1 & \text{if } q\not\equiv 1 \pmod 3,\\ (q-1)/3 & \text{if } q \equiv 1 \pmod 3.\end{cases}\end{aligned}$$

The result follows since the order of $\sum_{a\in A} e_a$ is necessarily a multiple of the order of $\psi(\sum_{a\in A} e_a)$. ■

Remark 8.4 Propositions 8.2 and 8.3 give upper and lower bounds for the order of $[\mathbf{1}]$ in K_0. The authors have computed the K-groups for the boundary algebras associated with more than one hundred different $\tilde{A}_2$ groups, for $2 \le q \le 11$. These numerical results strongly suggest that if $q \not\equiv 1 \pmod 3$ [respectively $q \equiv 1 \pmod 3$] then the order of $[\mathbf{1}]$ is precisely $q-1$ [respectively $(q-1)/3$]. Our computational results are complete in two cases: if $q = 2$ the $[\mathbf{1}] = 0$ and if $q = 3$ then $[\mathbf{1}]$ has order 2.

Propositions 8.2 and 8.3 show that if $q \neq 2, 4$ then $[\mathbf{1}]$ is a nonzero torsion element in $K_0\big(\mathcal{A}(\Gamma)\big)$. It follows from Corollary 5.2 that for $q \neq 2, 4$ the corresponding algebras are not isomorphic to any rank one Cuntz-Krieger algebra. The only group Γ among the eight groups for $q = 2$, for which $K_0\big(\mathcal{A}(\Gamma)\big)$ is torsion free is the group B.3. We do not know if such a group exists for $q = 4$.

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