The Solution of Difference Equations by Continued Fractions.

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I. The theorems which furnish in C.F. form the roots of a quadratic equation, and the similar process which leads to particular integrals of an ordinary differential equation of the second order, may be applied to certain types of difference equation. The types which suggest themselves for examination are

$$u_r \,.\, u_{r-s} = A_r \,.\, u_r + B_r \,.\, u_{r-s} + C_r \,, \qquad\qquad (1)$$

the bilinear equation, and

$$u_r = A_r \,.\, u_{r-s} + B_r \,.\, u_{r-2s} \qquad\qquad (2)$$

a special form of the linear equation; the coefficients are functions of r, and s is constant.

II. (α). To solve (1) by a terminating C.F. we have immediately

$$u_r = \frac{B_r . u_{r-s} + C_r}{u_{r-s} - A_r} = B_r + \frac{C_r + A_r . B_r}{u_{r-s} - A_r} = B_r + \frac{D_r}{u_{r-s} - A_r} \text{ say,}$$

$$= B_r + \frac{D_r}{B_{r-s} - A_r} + \frac{D_{r-s}}{B_{r-2s} - A_{r-s}} + \ldots + \frac{D_{r-(k-1)s}}{u_{r-ks} - A_{r-(k-1)s}}$$

where k is any integer. We may conveniently assume it, in most cases, to be the integral part of r/s.

(β). The fact that the solution can be expressed as a C.F. suggests the equivalent form $u_r = \dfrac{a_r . u + b_r}{c_r . u + d_r}$, where u is arbitrary, and the coefficients are functions of r.

Substituting in the relation

$$u_r = \frac{B_r . u_{r-s} + C_r}{u_{r-s} - A_r},$$

we have $$\frac{a_r . u + b_r}{c_r . u + d_r} = \frac{B_r (a_{r-s} . u + b_{r-s}) + C_r (c_{r-s} . u + d_{r-s})}{a_{r-s} . u + b_{r-s} - A_r (c_{r-s} . u + d_{r-s})},$$

which is identical if $a_r = B_r . a_{r-s} + C_r . c_{r-s}$ .... (1)

$b_r = B_r . b_{r-s} + C_r . d_{r-s}$ .... (2)

$c_r = a_{r-s} - A_r . c_{r-s}$ .... (3)

and $d_r = b_{r-s} - A_r . d_{r-s}$ .... (4)

Eliminating a from (1) and (3) we get

$$c_{r+s} + (A_{r+s} - B_r)\, c_r - (A_r B_r + C_r) . c_{r-s} = 0, \quad \text{.........} \quad (5)$$

and d_r satisfies the same equation.

Similarly we find that a_r and b_r are solutions of the equation

$$C_r . a_{r+s} + (A_r . C_{r+s} - C_r . B_{r+s}) . a_r - C_{r+s}(A_r B_r + C_r)\, a_{r-s} = 0. \quad (6)$$

These equations are of the second type mentioned above, and it follows that *the solution of the bilinear equation can be made to depend on that of either of two linear equations.*

For example, we may solve (5) for c_r and d_r, and then a_r and b_r are completely determined by (3) and (4). This result, although of some interest in itself as showing the close connection between the two types of equation, is not of much practical value except in a few cases, for, as a rule, the solutions of the linear equations will be incapable of expression in any very simple form, and may be more complex than the C.F. form of the solution of the bilinear equation. In some few instances a simple result is obtainable. This occurs, for example, when the coefficients of one of the linear equations are constant, or if any one coefficient vanishes identically, or if either of the linear equations belongs to one of the general forms mentioned below in IV. (α) and V.

(γ). The question now arises as to the degree of generality of the solution we have obtained.

It is clear from the form of the equation (1) that the complete solution depends on s different quantities. Thus the equation will determine u_r for every value of r, provided we know the values of $u_0, u_1, \dots u_{s-1}$. Nevertheless, *the expression for u_r can contain only one arbitrary constant*; for in order to calculate u_r for any *given* value of r, only one of the s quantities is required, namely, u_λ, where λ is the remainder when r is divided by s. The C.F. form of the solution is therefore the most general one obtainable, provided that in any particular case the value of k be chosen so that u_{r-ks} is that difference upon which the value of u_r depends.

The number of partial quotients in the C.F. depends in fact on the particular u which is to be regarded as arbitrary, and different C.F. are required to express the value of u_r, according as $r = ks + 1$, $ks + 2$, ... or $ks + \overline{s-1}$.

Similarly, in the case of the equation $u_r = A_r \,.\, u_{r-s} + B_r \,.\, u_{r-2s}$ the complete solution depends on $2s$ arbitrary quantities. But the calculation of u_r for any given value of r depends on two only of these; therefore the expression for u_r can contain only two arbitrary constants, and there may be, as before, s different expressions for u_r, the particular one used depending on the form of r.

It only remains to note that the coefficients a, b, c, d do not contain any arbitrary constant, since they are conditioned by the relations

$$u_0 = \frac{a_0 . u_0 + b_0}{c_0 . u_0 + d_0} \text{ and } u_s = \frac{a_s . u_0 + b_s}{c_s . u_0 + d_s} = \frac{B_s . u_0 + C_s}{u_0 - A_s},$$

where, for the sake of simplicity, we have put $r - ks = 0$.

(δ). Three exceptional cases remain to be discussed.

(1). If $A_r B_r + C_r = 0$ identically, the bilinear equation reduces to two linear equations, $u_r - B_r = 0$ and $u_{r-s} - A_r = 0$, which need no further remark.

(2). The C.F. solution also fails if $A_r = B_{r-s}$ identically. We have then $$u_r \,.\, u_{r-s} = B_{r-s} \,.\, u_r + B_r \,.\, u_{r-s} + C_r ,$$
which may be written

$$(u_r - B_r)(u_{r-s} - B_{r-s}) = C_r + B_r \,.\, B_{r-s}$$

or $$z_r \,.\, z_{r-s} = D_r , \text{ say.}$$

Hence
$$z_r = \frac{D_r}{z_{r-s}} = \frac{D_r}{D_{r-s}} \,.\, z_{r-2s}$$
$$= \frac{D_r}{D_{r-s}} \,.\, \frac{D_{r-2s}}{D_{r-3s}} \cdots \,.\, \frac{D_{r-(2m-2)s}}{D_{r-(2m-1)s}} \,.\, z_{r-2ms}$$

where $r - 2ms$ is less than $2s$.

The solution now takes one or other of two forms, according as $r - 2ms$ is less than s or not.

If $r - 2ms < s$, we simply replace z_r and z_{r-2ms} by their values, and we have u_r expressed in terms of u_{r-2ms}, the latter being the arbitrary quantity in the solution.

If $r - 2ms \gtreqless s$, we have $z_{r-2ms} = \dfrac{D_{r-2ms}}{z_{r-(2m+1)s}}$, and the arbitrary quantity in the solution is $u_{r-(2m+1)s}$. It is clear then that we get two different expressions for u_r, according as the integral part of r/s is odd or even.

(3). In dealing with the third exception it is assumed that r may have only integral values.

The exception is due to the fact that there are in general values of r which satisfy the equation $C_r + A_r . B_r = 0$, and one or more of the roots of this equation may be integral. With other roots we are not concerned. For any such value of r we have

$$\begin{aligned}(u_r - B_r)(u_{r-s} - A_r) &= C_r + A_r B_r \\ &= 0,\end{aligned}$$

so that either $u_r = B_r$ or $u_{r-s} = A_r$.

The point is that one or other of the two quantities u_r, u_{r-s} has a *definite* value, depending on nothing arbitrary, and it is necessary therefore to examine more closely the form of the solution.

The net result is to introduce a kind of discontinuity into the solution. Instead of a single expression for u_r, valid for all values of the variable, we get a number of such expressions, each of which is applicable only for values of r between two adjacent integral roots of the equation $C_r + A_r B_r = 0$.

Take for example the equation $(u_r - 1)(u_{r-1} - 1) = r - 3$,

$$\text{or} \qquad z_r . z_{r-1} = r - 3,$$

$$\text{or} \qquad z_r = \frac{r-3}{r-4} . z_{r-2} .$$

If $r > 3$, and $= 2n$ say, we get

$$z_r = \frac{(2n-3)(2n-5)(2n-7)\ldots 5 . 3}{(2n-4)(2n-6)(2n-8)\ldots 4 . 2} . z_4 = \frac{(2n-3).(2n-5)\ldots 5 . 3}{(2n-4).(2n-6)\ldots 4 . 2} . \frac{1}{z_3}$$

and if $r = 2n + 1$,

$$z_r = \frac{(2n-2).(2n-4)\ldots 4 . 2}{(2n-3).(2n-5)\ldots 3 . 1} . z_3 .$$

Similarly, if $r < 2$, we get, writing $r = -2m$,

$$z_r = \frac{(2m+2) . 2m . (2m-2)\ldots 4 . 2}{(2m+1)(2m-1)(2m-3)\ldots 3 . 1} z_2 ,$$

and if $r = -(2m+1)$,

$$z_r = \frac{(2m+3)(2m+1)\ldots 3}{(2m+2) \cdot 2m \ldots\ldots 2}\left(-\frac{1}{z_2}\right).$$

These, together with the condition that either u_3 or u_2 must be $=1$, constitute the complete solution, and enable us to calculate u_r when any particular u is known. If, for instance, we are given $z_r = 8$, we have $z_r = \frac{4 \cdot 2}{3 \cdot 1} z_3 = 8$, $\therefore z_3 = 3$, $\therefore u_3 = 4$. Since $u_3 \neq 1$ we must have $u_2 = 1$; and therefore all the u's are fixed in value. Similar reasoning applies to cases in which the equation $C_r + A_r \cdot B_r = 0$ has more than one integral root.

Consider, for instance, the equation $(u_r - r)(u_{r-1} - r^2) = r(r-4)$. The critical values of r are 0 and 4; from the first we have $u_0 = 0$ or $u_{-1} = 0$; and corresponding to the second value, $u_4 = 4$ or $u_3 = 16$.

Let us construct the scheme

$$\ldots u_{-2}\, u_{-1} \mid u_0\, u_1\, u_2\, u_3 \mid u_4\, u_5 \ldots$$

Suppose now that we are given $u_{-5} = k$. Starting from u_{-5} we can calculate u_{-1}, and the value at which we arrive will not usually be 0. Since $u_{-1} \neq 0$ we must have $u_0 = 0$, and this determines u_1, u_2, and u_3 completely. The value obtained for u_3 will not in general be 16, so we must put $u_4 = 4$; and all the other u's are determinate.

Again, if u_1 is given, we can determine u_0 and u_3 from the equation, and as a rule their calculated values will not be the critical values 0 and 16 found above. Hence we must put $u_{-1} = 0$ and $u_4 = 4$, and all the remaining u's are again determinate. If the calculated value of u_0 were 0, then u_{-1} would remain arbitrary, and all the u's depending on it would be indeterminate to the extent of an arbitrary constant.

In general we should have a scheme such as

$$\ldots u_{\alpha-2}\, u_{\alpha-1} \mid u_\alpha\, u_{\alpha+1} \ldots u_{\beta-1} \mid u_\beta \ldots u_{\gamma-1} \mid u_\gamma \ldots$$

where $\ldots \alpha, \beta, \gamma \ldots$ are the integral roots of $C_r + A_r \cdot B_r = 0$. From each compartment we can obtain from the equation an expression involving (as arbitrary constant) any one of the u's in that compartment. When any particular u is given, we can determine from it the values of the end u's in that compartment, and proceed as in the particular examples given above.

It seems worthy of notice that if the equation $C_r + A_r B_r = 0$ possesses integral roots, the specification of the value of any one u may leave u_r (for certain values of r) still indeterminate to the extent of an arbitrary constant.

(ϵ). Finally, if the coefficients A, B, C are constant, the equation admits of the trivial solution $u_r = z$, where z is one of the roots of the quadratic equation

$$z^2 = (A + B) z + C.$$

These are obviously particular solutions only.

If the coefficients are not constant, we may obtain two particular solutions of the equation in the form of infinite C.F., provided the equation $C_r + A_r B_r = 0$ has no integral roots. One of these is

$$u_r = B_r + \frac{D_r}{B_{r-s} - A_r} + \frac{D_{r-s}}{B_{r-2s} - A_{r-s}} + \ldots \text{ to infinity.}$$

The other is obtainable similarly from the equation

$$u_{r+s} \, . \, u_r = A_{r+s} \, . \, u_{r+s} + B_{r+s} \, . \, u_r + C_{r+s}.$$

We find

$$u_r = A_{r+s} + \frac{D_{r+s}}{A_{r+2s} - B_{r+s}} + \frac{D_{r+2s}}{A_{r+3s} - B_{r+2s}} + \ldots \text{ to infinity.}$$

If the equation $C_r + A_r \, . \, B_r = 0$ possesses integral roots, one or both of these particular solutions may be no longer obtainable.

It is evident that these two particular solutions become the roots of the quadratic when the coefficients are constant.

III. *Examples.* (1). $u_r \, . \, u_{r-1} + u_r = 1$.

The C.F. solution is

$$u_r = \frac{1}{1 + u_{r-1}} = 1 + \frac{1}{1} + \frac{1}{1} + \ldots + \frac{1}{1 + u_0}$$

where there are r partial quotients.

Putting $u_r = \dfrac{a_r . u + b_r}{c_r . u + d_r}$ we find
$$\begin{aligned} a_r &= c_{r-1} \\ b_r &= d_{r-1} \\ c_r &= a_{r-1} + c_{r-1} \\ d_r &= b_{r-1} + d_{r-1}, \end{aligned}$$

so that $$u_r = \frac{x_r . u + x_{r+1}}{x_{r+1} . u + x_{r+2}}$$

where $$x_{r+2} = x_{r+1} + x_r\,, \text{ and } x_1 = 1,\ x_2 = 0.$$

That this furnishes a solution of the equation is easily verified.

By the method of IV. (α) we find $x_r = (-1)^{r+1}(A\beta^{r+1} + B\alpha^{r+1})$ where α and β are the roots of $x^2 + x - 1 = 0$, and

$$A = \frac{2\alpha - 1}{\alpha - \beta},\ \ B = \frac{2\beta - 1}{\beta - \alpha}.$$

We may easily verify that on putting $u = \alpha$, we get $u_r = \alpha$.

The infinite C.F. are

$$u = 1 + \frac{1}{1} + \frac{1}{1} + \dots \text{ to infinity } = \frac{\sqrt{5} - 1}{2},$$

$$u = -1 - \frac{1}{1} - \frac{1}{1} - \dots \text{ to infinity } = -\frac{\sqrt{5} - 1}{2}.$$

(2) $u_{x+1} . u_x + x . u_x = x^2$.

We obtain easily

$$u_x = -(x-1) + \frac{(x-1)^2}{-(x-2)} + \frac{(x-2)^2}{-(x-3)} + \dots + \frac{1^2}{u_1}.$$

The linear equations for a, b, c, d both reduce to

$$u_{r+1} = r^2 . u_{r-1} - r . u_r,$$

but the general solution of this linear equation is not so simple.

The C.F. may be transformed into a series without much difficulty. The result is

$$\frac{u_x}{1-x} = 1 - (x-1) + (x-1)(x-2) + \dots$$

$$+ (-1)^{x-1}\frac{(x-1)!}{2!} + \frac{(-1)^x u_1 (x-1)!}{1 + u_1}.$$

The same thing is possible in other cases, but in every case the trouble of transformation is considerable; verification, when possible at all, is no longer easy; the vexed questions of equivalence and convergence arise, particularly when C.F. and series are infinite; and no advantage is gained by the transformation.

IV. We now proceed to consider the equation

$$u_r = A_r \,.\, u_{r-s} + B_r \,.\, u_{r-2s}.$$

(α). When the coefficients are constant, the complete solution is easily effected by the aid of a recurring series.

Take, for example, the equation $x_r + a \,.\, x_{r-1} + b \,.\, x_{r-2} = 0$.

Writing $\qquad y = x_0 + x_1 z + x_2 z^2 + \ldots + x_r z^r + \ldots,$

we have $\qquad azy = ax_0 \,.\, z + ax_1 \,.\, x^2 + \ldots + ax_{r-1} \,.\, z^r + \ldots$

and $\qquad bz^2 y = bx_0 \,.\, z^2 + \ldots + bx_{r-2} \,.\, z^r + \ldots .$

Hence $\qquad y\,(1 + az + bz^2) = x_0 + (x_1 + ax_0)\, z$

if the coefficients satisfy the given equation.

Hence $$y = \frac{x_0 + (x_1 + ax_0)\, z}{1 + az + bz^2},$$

and to find x_r we need only expand in ascending powers of z, and equate x_r to the coefficient of z^r.

The equation

$$f(r) \,.\, x_r + p \,.\, f(r-1) \,.\, x_{r-1} + q \,.\, f(r-2) \,.\, x_{r-2} + \ldots = a \,.\, \lambda^r$$

is immediately soluble by this method, on putting $z_r = x_r \,.\, f(r)$ (a and λ being constants).

(β). The above method may be extended to a few cases in which the coefficients are not constant.

Take, for instance, $x_r + p\,(r+1)\, x_{r+1} + q\,(r+2)\,(r+1)\, x_{r+2} = 0$.

As before, we have $\quad y = x_0 + x_1 z + x_2 z^2 + \ldots + x_r \,.\, z^r + \ldots$

$$p \,.\, Dy = px_1 + 2px_2 z + \ldots + p\,(r+1)\, x_{r+1} \,.\, z^r + \ldots$$

$$q \,.\, D^2 y = 2q \,.\, x_2 + \ldots + q\,(r+2)\,(r+1)\, x_{r+2} \,.\, z^r + \ldots$$

so that on adding we obtain

$$(1 + p \,.\, D + q \,.\, D^2)\, y = 0.$$

If the roots of the quadratic $1 + p \,.\, \xi + q \,.\, \xi^2 = 0$ are unequal, (λ, μ) say, we have as the general solution

$$y = Ae^{\lambda z} + Be^{\mu z},$$

and therefore $$x_r = \frac{1}{r\,!}(A\lambda^r + B\mu^r),$$

where, of course, $\quad A + B = x_0,\;$ and $\; A\lambda + B\mu = x_1$.

If the roots are equal, the solution of the differential equation will be $$y = (A + Bz)\, e^{\lambda z},$$

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and therefore $$x_r = \frac{\lambda^{r-1}}{r!}(A\lambda + Br)$$

where $A = x_0$, and $A\lambda + B = x_1$.

In either case these expressions supply the general solution of the difference equation, the quantities x_0 and x_1 being arbitrary; and verification by substitution is easy.

This method is of very limited application. It evidently is only applicable when the differential equation which results is capable of solution in comparatively simple form, and when y can be expanded in a series of the assumed form. The failure of the method is easily illustrated.

Suppose that $x_n = nx_{n-1} + x_{n-2}$. We have then

$$\begin{aligned} y &= x_0 + x_1 z + \ldots + x_n . z^n \ldots \\ -zD(zy) &= \quad - x_0 z \ldots\ldots - n . x_{n-1} . z^n \ldots \\ -z^2 y &= \quad - x_0 z^2 \ldots\ldots - x_{n-2} . z^n \ldots, \end{aligned}$$

so that if the coefficients obey the assumed law

$$(1 - z^2) y - z D(zy) = x_0 + (x_1 - x_0) z,$$

which simplifies to

$$Dy + \left(1 + \frac{1}{z} - \frac{1}{z^2}\right) y = -\frac{x_0}{z^2} + \frac{x_0 - x_1}{z}$$

or $$yze^{z+\frac{1}{z}} = K + \int \left(x_0 - x_1 - \frac{x_0}{z}\right) e^{z+\frac{1}{z}} . dz.$$

It is evident that even if the quadrature could be performed, y is not expressible in a series of the assumed form; and the reason is obvious, for x_n is comparable with (and greater than) $n!$, so that the assumed series could not converge for any value of z whatever, except $z = 0$. Nor can we surmount the difficulty by assuming for y a series infinite both ways, and neglecting all but formal equivalence; the integral on the right hand side would then disappear, but the determination of K raises new difficulties, apart, of course, from the fact that we have no right at all to differentiate such a series.

(γ). It remains to show how the general solution may be obtained by the use of C.F.

We have $u_r = A_r \,.\, u_{r-s} + B_r \,.\, u_{r-2s}$.

$$\therefore \frac{u_r}{u_{r-s}} = A_r + \frac{B_r}{\dfrac{u_{r-s}}{u_{r-2s}}}$$

$$= A_r + \frac{B_r}{A_{r-s}\,+}\;\frac{B_{r-s}}{A_{r-2s}\,+}\;\cdots\;+\frac{B_{r-(k-2)s}}{\dfrac{u_{r-(k-1)s}}{u_{r-ks}}} = C_r, \text{ say.}$$

Similarly $\qquad \dfrac{u_{r-s}}{u_{r-2s}} = C_{r-s}, \quad \dfrac{u_{r-2s}}{u_{r-3s}} = C_{r-2s},$ and so on.

Hence
$$u_r = \frac{u_r}{u_{r-s}} \,.\, \frac{u_{r-s}}{u_{r-2s}} \cdots\cdots \frac{u_{r-(k-1)s}}{u_{r-ks}} \,.\, u_{r-ks}$$
$$= C_r \,.\, C_{r-s} \,.\, C_{r-2s} \cdots\cdots C_{r-(k-1)s} \,.\, u_{r-ks}\,.$$

This is the general solution required. It contains two arbitrary quantities, namely, $u_{r-(k-1)s}$ and u_{r-ks}. The ratio of these is the last partial quotient in each of the C's.

We can at once verify the solution by substitution in the original equation, for we find

$$C_r \,.\, C_{r-s} \ldots u_{r-2ks} = A_r \,.\, C_{r-s} \,.\, C_{r-2s} \ldots u_{r-ks} + B_r \,.\, C_{r-2s} \ldots u_{r-ks},$$

which is true if
$$C_r \,.\, C_{r-s} = A_r \,.\, C_{r-s} + B_r,$$
i.e. if $\qquad C_r = A_r + \dfrac{B_r}{C_{r-s}}$, which is true.

(δ). Those cases in which either or both of the coefficients vanish for some particular value or values of r may be best illustrated by examples.

(1). $\quad u_r = (r-6) \,.\, u_{r-1} + r^2 \,.\, u_{r-2}$.

When $r = 6$ we get $u_6 = 6^2 \,.\, u_4$, so that if $r > 6$,

$$\frac{u_r}{u_{r-1}} = r - 6 + \frac{r^2}{\dfrac{u_{r-1}}{u_{r-2}}} = r - 6 + \frac{r^2}{r-7\,+}\;\frac{(r-1)^2}{r-8\,+}\;\ldots\;+\frac{8^2}{1\,+}\;\frac{7^2}{\dfrac{u_6}{u_5}}$$

$$= r - 6 + \frac{r^2}{r-7\,+}\;\frac{(r-1)^2}{7-8\,+}\;\ldots\;+\frac{8^2}{1\,+}\;\frac{7^2}{6^2}\,.\,\frac{u_5}{u_4}.$$

But
$$\frac{u_5}{u_4} = -1 + \frac{5^2}{-2\,+}\;\frac{4^2}{-3\,+}\;\frac{3^2}{-4\,+}\;\frac{2^2}{\dfrac{u}{u_0}},$$

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so that evidently there is no failure of the solution. The only effect is a slight disturbance of the form of each C.F. The same happens in general if $A_r = 0$ for any particular value of r.

(2). As an instance of a value of r which makes $B_r = 0$, take the equation $$u_r = r^2 . u_{r-1} + (r-6)\, u_{r-2}.$$

Here we have $u_6 = 6^2 . u_5$, so that if $r > 6$,

$$\frac{u_r}{u_{r-1}} = r^2 + \frac{r-6}{(r-1)^2 +}\,\frac{r-7}{(r-2)^2 +} \dots + \frac{3}{8^2 +}\,\frac{2}{7^2 +}\,\frac{1}{\frac{u_6}{u_5}}$$

$$= r^2 + \frac{r-6}{(r-1)^2 +}\,\frac{r-7}{(r-2)^2 +} \dots + \frac{3}{8^2 +}\,\frac{2}{7^2 +}\,\frac{1}{6^2} = C_r,$$

and therefore

$$u_r = C_r . C_{r-1} . C_{r-2} \dots C_6 . u_5$$

where none of the C's contains an arbitrary quantity, so that u_r *depends only on* u_5, so long as $r > 5$.

If, however, we regard u_1 and u_0 as arbitrary, and express others in terms of these, we have

$$u_5 = \frac{u_5}{u_4} . \frac{u_4}{u_3} . \frac{u_3}{u_2} . \frac{u_2}{u_1} . \frac{u_1}{u_0} . u_0,$$

and since each of these ratios depends on u_1/u_0, it follows that u_5 alone introduces two arbitrary quantities into the expression for u_r.

The exception, so far as it exists, consists in the fact that if either u_5 or u_6 be arbitrary, u_r depends only on *one* quantity, and not on two.

(3). Suppose now that $A_r = 0$ and $B_r = 0$ for the same value of r, as in the equation $u_r = (r-6)(u_{r-1} + u_{r-2})$, so that $u_6 = 0$.

We find $$\frac{u_r}{u_{r-1}} = (r-6) + \frac{r-6}{r-5 +}\,\frac{r-5}{r-4 +} \dots + \frac{4}{3 +}\,\frac{3}{2 +}\,\frac{2u_6}{u_7}$$

$$= (r-6) + \frac{r-6}{r-5 +}\,\frac{r-5}{r-4 +} \dots + \frac{4}{3 +}\,\frac{3}{2}, \quad \text{since } u_6 = 0,$$

$$= C_r \text{ say.}$$

$$\therefore \quad u_r = C_r . C_{r-1} . C_{r-2} \dots C_8 . u_7$$
$$= C_r . C_{r-1} . C_{r-2} \dots C_8 . u_5, \quad \text{since } u_7 = (7-6)(u_6 + u_5) = u_5,$$

and as before, $$u_5 = \frac{u_5}{u_4} . \frac{u_4}{u_3} . \frac{u_3}{u_2} . \frac{u_2}{u_1} . \frac{u_1}{u_0} . u_0 .$$

Here again the exception is of the same nature.

(ϵ). Since the difference equation is linear, it follows that the sum of any number of particular solutions, each multiplied by an arbitrary constant, is itself a solution. It is interesting to note that the above expression may be reduced to this form.

For suppose that $C_r = \dfrac{a_r \lambda + b_r}{c_r \lambda + d_r}$, where $\lambda = \dfrac{u_s}{u}$, and we put $r - ks = 0$ for convenience.

Substituting in the last equation of IV. (γ), we find

$$\frac{a_r \lambda + b_r}{c_r \lambda + d_r} = A_r + \frac{B_r (c_{r-s} . \lambda + d_{r-s})}{a_{r-s} . \lambda + b_{r-s}},$$

which is an identical relation provided

$$\begin{aligned} a_r &= A_r . a_{r-s} + B_r . c_{r-s} \\ b_r &= A_r . b_{r-s} + B_r . d_{r-s} \\ c_r &= a_{r-s} \\ d_r &= b_{r-s}. \end{aligned}$$

From these equations it follows that a_r and b_r are particular solutions of the given difference equation, and from the last two, that

$$C_r = \frac{a_r . \lambda + b_r}{a_{r-s} . \lambda + b_{r-s}},$$

so that

$$\begin{aligned} u_r &= C_r . C_{r-s} \ldots C_{r-(k-1)s} u_{r-ks} \\ &= (a_r \lambda + b_r) u_0, \quad \text{when we take account of the initial values of } a \text{ and } b, \\ &= a_r . u_s + b_r u_0, \end{aligned}$$

the form required, since u_0 and u_s are arbitrary.

V. (1). The general solution of

$$x_{r-2} + p (r-1) x_{r-1} + q (r-1) r . x_r = 0$$

has been obtained in IV. (β).

It is
$$x_r = \frac{1}{r!} \left[\frac{(\mu x_0 - x_1) \lambda^r - (\lambda x_0 - x_1) \mu^r}{\mu - \lambda} \right]$$

where x_0 and x_1 are arbitrary, and λ, μ are the roots of $1 + p \xi + q \xi^2 = 0$.

The solution by C.F. is as follows:

The equation may be written

$$-r \frac{x_r}{x_{r-1}} = \frac{p}{q} + \frac{1}{q (r-1) \dfrac{x_{r-1}}{x_{r-2}}}$$

or $\quad z_r = \dfrac{p}{q} - \dfrac{1}{q . z_{r-1}} \quad$ where $\quad z = -r . \dfrac{x_r}{x_{r-1}}$, This is a bilinear equation with constant coefficients, whose solution is

$$z_r = \frac{a_r \,.\, z_1 + b_r}{a_{r-1} \,.\, z_1 + b_{r-1}}$$

where $a_r = \frac{1}{\beta - \alpha}\left[\frac{\beta}{\alpha^{r-1}} - \frac{\alpha}{\beta^{r-1}}\right]$ and $b_r = \frac{1}{\beta - \alpha}\left[\frac{1}{\beta^{r-1}} - \frac{1}{\alpha^{r-1}}\right]$,

α and β being the roots of $\eta^2 - p\eta + q = 0$.

$$\text{Hence}\quad z_r \,.\, z_{r-1} \,.\, z_{r-2} \ldots z_2 \,.\, z_1 = \frac{a_r \,.\, z_1 + b_r}{a_{r-1} \,.\, z_1 + b_{r-1}} \ldots \frac{a_2 z_1 + b_2}{a_1 z_1 + b_1} \,.\, z_1$$

$$= a_r z_1 + b_r,$$

$$i.e. \quad (-1)^r (r\,!) \frac{x_r}{x_0} = -a_r \,.\, \frac{x_1}{x_0} + b_r,$$

$$\text{or} \quad x_r = \frac{(-1)^{r-1}}{r\,!\,(\beta - \alpha)}\left[\frac{\beta x_1 + x_0}{\alpha^{r-1}} - \frac{\alpha x_1 + x_0}{\beta^{r-1}}\right]$$

where α, β are the roots of $\eta^2 - p\eta + q = 0$,

$$= \frac{1}{r\,!\,(\mu - \lambda)}\left[(\mu x_0 - x_1)\,\lambda^r - (\lambda x_0 - x_1)\,\mu^r\right]$$

where λ, μ are the roots of $1 + p\xi + q\xi^2 = 0$,

which is the previous form.

If, in solving the equation for z_r, we proceed to form an infinite C.F., we find

$$z_r = \frac{p}{q} - \frac{1}{p}\;\frac{1}{-p/q}\;\frac{1}{-p} \ldots \text{ to infinity,}$$

$= \alpha$, where α is a root of $1 - p\alpha + q\alpha^2 = 0$,

$= -\lambda$, where λ is a root of $1 + p\lambda + q\lambda^2 = 0$.

$$\therefore \quad \frac{x_r}{x_{r-1}} = \frac{\lambda}{r},$$

$$\text{and } \therefore \quad x_r = \frac{x_r}{x_{r-1}} \,.\, \frac{x_{r-1}}{x_{r-2}} \ldots \frac{x_1}{x_0} \,.\, x_0$$

$$= \frac{\lambda^r \,.\, x_0}{r\,!},$$

which is a particular solution derivable from the general expression by putting $x_1 = \lambda x_0$.

We may evidently obtain by an exactly similar process the solution of the equation

$$x_{r-2}+p\,.\,\phi(r-1)\,x_{r-1}+q\,.\,\phi(r-1)\,.\,x_r=0$$

where $\phi(r)$ is any function of r.

The result is then

$$x_r=\frac{P}{\mu-\lambda}\left[(\mu\,x_0-x_1)\,\lambda^r-(\lambda\,x_0-x_1)\,\mu^r\right]$$

where $$\frac{1}{P}=\phi(r)\,.\,\phi(r-1)\,.\,\phi(r-2)\ \dots\ \phi(2)\,.\,\phi(1).$$

The expression takes a slightly different but easily obtained form if the equation $\phi(r)=0$ has one or more integral roots.

(2). A difference equation of considerable importance in mathematical physics is that satisfied by the coefficients of Legendre, namely,

$$n\,.\,P_n=(2n-1)\,\mu\,.\,P_{n-1}-(n-1)\,.\,P_{n-2}\,.$$

From it we have

$$\frac{P_n}{P_{n-1}}=\frac{(2n-1)\,\mu}{n}-\frac{(n-1)/n}{P_{n-1}/P_{n-2}}$$

$$=\frac{(2n-1)\,\mu}{n}-\frac{(n-1)/n}{(2n-3)\,\mu/(n-1)-}\,\frac{(n-2)/(n-1)}{(2n-5)\,\mu/(n-2)-}\dots\frac{2/3}{P_2/P_1}$$

which may also be written

$$n\,.\,\frac{P_n}{P_{n-1}}=(2n-1)\,\mu-\frac{(n-1)^2}{(2n-3)\,\mu-}\,\frac{(n-2)^2}{(2n-5)\,\mu-}\dots\frac{3^2}{5\mu-}\,\frac{2}{P_2/P_1}\,.$$

But $P_2=\frac{1}{2}(3\mu^2-1)$
$P_1=\mu$

and $\therefore$ $$\frac{P_2}{P_1}=\frac{3\mu}{2}-\frac{1}{2\mu}\,.$$

Hence

$$n \cdot \frac{P_n}{P_{n-1}} = (2n-1)\mu - \frac{(n-1)^2}{(2n-3)\mu -} \frac{(n-2)^2}{(2n-5)\mu -} \cdots - \frac{3^2}{5\mu -} \frac{2^2}{3\mu -} \frac{1^2}{\mu}.$$

$$= C_n \text{, say,}$$

and $\quad \therefore \quad n! \cdot \frac{P_n}{P_1} = C_n \cdot C_{n-1} \cdot \ldots C_2$

and since

$$P_1 = \mu = C_1 \text{,}$$

$$P_n = \frac{1}{n!} \cdot \prod_1^n C$$

an expression for P_n which is, I believe, new.