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ON EXISTENCE AND STABILITY OF SOLUTIONS TO ELLIPTIC SYSTEMS WITH GENERALISED GROWTH

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We are concerned with existence and stability of solutions for system of equations with generalised $p(x)$ and $m(x)-$Laplace operators and where the nonlinearity satisfies some local growth conditions. We provide a variational approach that is based on investigation of the primal and the dual action functionals. As a consequence we consider the dependence of the the system on functional parameters.

1. INTRODUCTION

In this paper we consider existence and stability of solutions to the following family of systems of Dirichlet problems with generalised $p(x), m(x)-$Laplacian operators for $k = 0, 1, 2, \dots$

$$\begin{aligned} -\operatorname{div}\Big(a(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) &= F_u^k(x, u(x), v(x)), \\ -\operatorname{div}\Big(b(x)\big|\nabla v(x)\big|^{m(x)-2}\nabla v(x)\Big) &= F_v^k(x, u(x), v(x)), \\ u(x)\mid_{\partial\Omega} = 0,\ u \in W_0^{1,p(x)}(\Omega), &\qquad v(x)\mid_{\partial\Omega} = 0,\ v \in W_0^{1,m(x)}(\Omega) \end{aligned} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded region with Lipschitz boundary, $p, q, m, n \in C(\overline{\Omega})$, $1/p(x) + 1/q(x) = 1$, $1/m(x) + 1/n(x) = 1$ for $x \in \Omega$; $W_0^{1,p(x)}(\Omega)$, $W_0^{1,m(x)}(\Omega)$ denote the generalised Orlicz-Sobolev spaces, see [**3**, **5**]; $a, b \in C(\overline{\Omega})$ with $a(x) \geqslant a_0 > 0$, $b(x) \geqslant b_0 > 0$ on $\overline{\Omega}$ for $k = 0, 1, 2, \dots$. Let $p^- = \inf_{x\in\Omega} p(x) > N$, $m^- = \inf_{x\in\Omega} m(x) > N$.

We shall show – upon some conditions – that for all $k = 1, 2, \dots$ there exists a solution (u_k, v_k) to (1.2) and later that from the sequence (u_k, v_k) one can choose a subsequence (u_{k_i}, v_{k_i}) such that $u_{k_i} \rightharpoonup \overline{u}$ weakly in $W^{1,p(x)}(\Omega)$, $v_{k_i} \rightharpoonup \overline{v}$ weakly in $W^{1,m(x)}(\Omega)$ and

$$\begin{aligned} -\operatorname{div}\Big(a(x)\big|\nabla\overline{u}(x)\big|^{p(x)-2}\nabla\overline{u}(x)\Big) &= F_u^0(x, \overline{u}(x), \overline{v}(x)), \\ -\operatorname{div}\Big(b(x)\big|\nabla\overline{v}(x)\big|^{m(x)-2}\nabla\overline{v}(x)\Big) &= F_v^0(x, \overline{u}(x), \overline{v}(x)), \\ \overline{u}(x)\mid_{\partial\Omega} = 0, &\qquad \overline{v}(x)\mid_{\partial\Omega} = 0. \end{aligned}$$

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Such a property we shall call the stability of the system. Some general framework for studying stability of solutions to variational problems in sublinear case can be found in [**10, 12**] and [**13**] but our method provides suitable results for the family of systems of Dirichlet problems with generalised $p(x), m(x)$–Laplacian operators.

In order to obtain the solution to (1.2) we minimise J_k on a set $X_k \subset W^{1,p(x)}(\Omega) \times W^{1,m(x)}(\Omega)$ which has the following property: for all $(u,v) \in X_k$, the relation

$$\begin{aligned} &-\operatorname{div}\Big(a(x)\big|\nabla\widetilde{u}(x)\big|^{p(x)-2}\nabla\widetilde{u}(x)\Big) = F_u^k\big(x, u(x), v(x)\big),\\ &-\operatorname{div}\Big(b(x)\big|\nabla\widetilde{v}(x)\big|^{m(x)-2}\nabla\widetilde{v}(x)\Big) = F_v^k\big(x, u(x), v(x)\big)\\ &\quad \widetilde{u}(x)\mid_{\partial\Omega} = 0,\ \widetilde{u} \in W_0^{1,p(x)}(\Omega), \qquad \widetilde{v}(x)\mid_{\partial\Omega} = 0,\ \widetilde{v} \in W_0^{1,m(x)}(\Omega) \end{aligned} \tag{1.2}$$

implies $(\widetilde{u}, \widetilde{v}) \in X_k$.

First we show, with the aid growth conditions F1, F2, F3 (see Section 2), that the action functional

$$J_k(u,v) = \int_\Omega \frac{a(x)}{p(x)}\big|\nabla u(x)\big|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}\big|\nabla v(x)\big|^{m(x)}dx - \int_\Omega F^k\big(x, u(x), v(x)\big)dx$$

is bounded from below and achieves its minimum $(\overline{u}_k, \overline{v}_k)$ on X_k. Since X_k is not dense in $W^{1,p(x)}(\Omega) \times W^{1,m(x)}(\Omega)$ we may not apply the Euler-Lagrange equation. Our assumptions also do not allow us to use either the mountain pass geometry or the topological approach. In order to show that $(\overline{u}_k, \overline{v}_k)$ is indeed a solution we construct a dual functional $J_k^D : W^1 \times W^2 \to R$

$$\begin{aligned} J_k^D(w,z) &= \int_\Omega (F^k)^*\big(x, -\operatorname{div} w(x), -\operatorname{div} z(x)\big)dx\\ &\quad - \int_\Omega \frac{1}{(a(x))^{q(x)/p(x)}}\frac{1}{q(x)}\big|w(x)\big|^{q(x)}dx - \int_\Omega \frac{1}{(b(x))^{n(x)/m(x)}}\frac{1}{n(x)}\big|v(x)\big|^{n(x)}dx \end{aligned}$$

where

$$\begin{aligned} W^1 &= \big\{w \in L^{q(x)}(\Omega) \mid \operatorname{div} w \in L^{q(x)}(\Omega)\big\},\\ W^2 &= \big\{z \in L^{n(x)}(\Omega) \mid \operatorname{div} z \in L^{n(x)}(\Omega)\big\} \end{aligned}$$

and investigate relations between J and J_k^D. We relate critical values on X_k and X_k^d (on which J_k^D is considered) and later we relate the relevant critical points. These relations provide the existence of solutions. Construction of X_k and some convergence of F^k will further allow us to obtain stability. Here $(F^k)^*$ denotes the Fenchel–Young conjugate of F^k, see [**2**], that is,

$$(F^k)^*(x, w_1, w_2) = \sup_{z \in R^2}\big(\langle w, z\rangle - F^k(x, z_1, z_2)\big), \tag{1.3}$$

where $z = (z_1, z_2)$ and $w = (w_1, w_2)$. The only work – known to the authors – that concerns elliptic system with generalised growth is [**9**]. Following [**4**] the authors of [**9**] apply first a direct method of the calculus of variations and later a mountain pass geometry. Since problems with generalised growth conditions are applied in elastic mechanics and electrorheological fluid dynamics (see [**11, 14**] and references therein), we believe that our results may contribute to that research. Concerning some ideas on stability in variational problems we may mention [**6**].

2. ASSUMPTIONS AND AUXILIARY RESULTS

In what follows by C_S^1, C_S^2 we denote the best Sobolev constants

$$\|u\|_{p(x)} \leqslant C_S^1 \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$
$$\|u\|_{m(x)} \leqslant C_S^2 \|\nabla u\|_{m(x)} \text{ for all } u \in W_0^{1,m(x)}(\Omega).$$

Since $W_0^{1,p(x)}(\Omega)$ is continuously embedded into $W_0^{1,p^-}(\Omega)$, as well as $W_0^{1,m(x)}(\Omega)$ into $W_0^{1,m^-}(\Omega)$, [**3**], we denote by C_1^m, C_1^p the following constants

$$\|\nabla u\|_{p^-} \leqslant C_1^p \|\nabla u\|_{p(x)},$$
$$\|\nabla v\|_{m^-} \leqslant C_1^m \|\nabla v\|_{m(x)}. \tag{2.1}$$

Since $p^- > N$ and $m^- > N$ by Sobolev Imbedding Theorem [**1**] we get

$$\max_{x\in\Omega}|u(x)| \leqslant C_2^p \|\nabla u\|_{p^-} \text{ for all } u \in W_0^{1,p^-}(\Omega),$$
$$\max_{x\in\Omega}|v(x)| \leqslant C_2^m \|\nabla v\|_{m^-} \text{ for all } v \in W_0^{1,m^-}(\Omega). \tag{2.2}$$

Therefore by (2.2) and (2.3) for all $u \in W_0^{1,p(x)}(\Omega)$, $v \in W_0^{1,m(x)}(\Omega)$ we get

$$\max_{x\in\Omega}|u(x)| \leqslant C_2^p \|\nabla u\|_{p^-} \leqslant C_1^p C_2^p \|\nabla u\|_{p(x)},$$
$$\max_{x\in\Omega}|v(x)| \leqslant C_2^m \|\nabla v\|_{m^-} \leqslant C_1^m C_2^m \|\nabla v\|_{m(x)}. \tag{2.3}$$

Let us consider two nondecreasing sequences of positive numbers, bounded away from 0, $\{d_k\}_{k=1}^{\infty}$, $\{c_k\}_{k=1}^{\infty}$. We assume that

F1: $\|1\|_{q(x)} \leqslant (1/p^- + 1/q^-)^{-1}$, $\|1\|_{n(x)} \leqslant (1/m^- + 1/n^-)^{-1}$ and for $k = 0, 1, 2, \ldots$

$$C_S^1 C_1^p C_2^p \operatorname{ess\,sup}_{x\in\Omega} \max_{u\in[-d_k,d_k]} \max_{v\in[-c_k,c_k} \left|F_u^k(x,u,v)\right| \leqslant a_0 d_k,$$
$$C_S^2 C_1^m C_2^m \operatorname{ess\,sup}_{x\in\Omega} \max_{u\in[-d_k,d_k]} \max_{v\in[-c_k,c_k]} \left|F_v^k(x,u,v)\right| \leqslant b_0 c_k. \tag{2.4}$$

F2: $F^k, F_u^k, F_v^k : \Omega\times[-d_0,d_0]\times[-c_0,c_0] \to R$ are Caratheodory functions for all $k = 0, 1, 2, \ldots$, F^k is convex in the last two variables on $[-d_k,d_k]\times[-c_k,c_k]$ for all $k = 0, 1, 2, \ldots$ and almost all $x \in \Omega$.

We may define F^k on $\Omega\times(R\backslash[-d_0,d_0])\times(R\backslash[-c_0,c_0])$ by putting $F^k = +\infty$. Now F^k is convex and lower semicontinuous.

F3 $F_u^k(x,0,0) \neq 0$, $F_v^k(x,0,0) \neq 0$ for almost all $x \in \Omega$, functions $x \mapsto F^k(x,0,0)$ and $x \mapsto (F^k)^*(x,0,0)$ are integrable on Ω where $(F^k)^*$ is defined by (1.3).

We put

$$X_k = \Big\{(u,v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,m(x)}(\Omega),\ \|\nabla u\|_{p(x)} \leqslant \frac{d_k}{C_1^p C_2^p}, \|\nabla v\|_{m(x)} \leqslant \frac{c_k}{C_1^m C_2^m}, |u(x)| \leqslant d_k, |v(x)| \leqslant c_k\Big\}. \tag{2.5}$$

Reasoning exactly as in [8] we show that X_k has indeed property (1.3). The dual functional J_k^D will be considered on a set X_k^d which is a set of these $(w,z) \in W^1 \times W^2$ for which there exists a $(u,v) \in X_k$ such that

$$-\operatorname{div} w(x) = F_u^k(x, u(x), v(x)) \tag{2.6}$$

and

$$a(x)\big|\nabla\widetilde{u}(x)\big|^{p(x)-2}\nabla\widetilde{u}(x) = w(x), \tag{2.7}$$

$$-\operatorname{div} z(x) = F_v^k(x, u(x), v(x)) \tag{2.8}$$

and

$$b(x)\big|\nabla\widetilde{v}(x)\big|^{m(x)-2}\nabla\widetilde{v}(x) = z(x), \tag{2.9}$$

where $(\widetilde{u}, \widetilde{v})$ corresponds to (u,v) in (1.3).

J_k and J_k^D are well defined on X_k and X_k^d due to the following.

Lemma 2.1. *For any $k = 0, 1, 2, \ldots$, there exist constants $\gamma_k, \eta_k > 0$ such that*

$$\left|\int_\Omega F^k(x, u(x), v(x))\,dx\right| \leqslant \gamma_k \tag{2.10}$$

for all $(u,v) \in X_k$ and

$$\left|\int_\Omega (F^k)^*(x, -\operatorname{div} w(x), -\operatorname{div} z(x))\,dx\right| \leqslant \eta_k \tag{2.11}$$

for all $(w,z) \in X_k^d$.

Proof: Relation (2.10) follows by convexity of F^k, F1, F3 and the estimates

$$\big|F^k(x, u(x), v(x))\big| \leqslant \big|F^k(x,0,0)\big| + \sup_{x\in\Omega}\Big\{\big|F_u^k(x,0,0)\big|\big|u(x)\big|\Big\}.$$

By (2.6), (2.8), (2.10) and the definition of X_k^d we get that

$$\int_\Omega (F^k)^*(x, -\operatorname{div} w(x), -\operatorname{div} z(x))\,dx$$
$$= -\int_\Omega F^k(x, u(x), v(x))\,dx + \int_\Omega (u(x), v(x))(-\operatorname{div} w(x), -\operatorname{div} z(x))\,dx$$

is finite. Thus relation (2.11) follows. □

3. EXISTENCE OF SOLUTIONS

THEOREM 3.1. *Assume* F1, F2, F3. *For all* $k = 0,1,2,\dots$ *there exists* $(u_k, v_k, w_k, z_k) \in X_k \times X_k^d$ *such that*

$$-\operatorname{div} w_k(x) = F_u^k(x, u_k(x), v_k(x)), -\operatorname{div} z_k(x) = F_v^k(x, u_k(x), v_k(x)), \tag{3.1}$$

$$a(x)\left|\nabla u_k(x)\right|^{p(x)-2}\nabla u_k(x) = w_k(x),\ b(x)\left|\nabla v_k(x)\right|^{m(x)-2}\nabla v_k(x) = z_k(x). \tag{3.2}$$

Moreover

$$\inf_{(w,z)\in X_k^d} J_k^D(w,z) = J_k^D(w_k, z_k) = J_k(u_k, v_k) = \inf_{(u,v)\in X_k} J_k(u,v). \tag{3.3}$$

PROOF: We fix $k = 0,1,2\dots$. We observe that by Lemma 2.1

$$J_k(u,v) = \int_\Omega \frac{a(x)}{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}|\nabla v(x)|^{m(x)}dx - \int_\Omega F^k(x, u(x), v(x))dx \geqslant -\gamma_k.$$

Therefore $\inf_{(u,v)\in X_k} J_k(u,v)$ is finite. By the properties of X_k there exists a minimising sequence $\left\{(u_k^n, v_k^n)\right\}_{n=1}^\infty$ for functional J_k on X_k and this sequence may be assumed to be weakly convergent in $W_0^{1,p(x)}(\Omega)\times W_0^{1,m(x)}(\Omega)$ and therefore, up to a subsequence, strongly in $L^{p(x)}(\Omega) \times L^{m(x)}(\Omega)$. Thus it contains a subsequence convergent almost everywhere, still denoted by $\left\{(u_k^n, v_k^n)\right\}_{n=1}^\infty$ and its limit is denoted by (u_k, v_k). We see that

$$\|\nabla u_k^n\|_{L^{p(x)}(\Omega)} \leqslant \frac{d_k}{C_1^p C_2^p}$$

for all n and

$$\liminf_{n\to\infty}\|\nabla u_k^n\|_{L^{p(x)}(\Omega)} \geqslant \|\nabla u_k\|_{L^{p(x)}(\Omega)}.$$

Therefore $\|\nabla u_k\|_{L^{p(x)}(\Omega)} \leqslant (d_k)/(C_1^p C_2^p)$. By definition of sequence $\{u_k^n\}_{n=1}^\infty$ we also get $\left|u_k^n(x)\right| \leqslant d_k$. Since $\{u_k^n\}_{n=1}^\infty$ is convergent almost everywhere, we get $\left|u_k(x)\right| \leqslant d_k$. The same holds for $\{v_k^n\}_{n=1}^\infty$. So $(u_k, v_k) \in X_k$ and we get $\liminf_{n\to\infty} J_k(u_k^n, v_k^n) \geqslant J_k(u_k, v_k)$ since

$$\lim_{n\to\infty}\int_\Omega F_u^k(x, u_k^n(x), v_k^n(x))dx = \int_\Omega F_u^k(x, u_k(x), v_k(x))dx.$$

Thus J_k is weakly lower semicontinuous on X_k and since X_k is weakly compact we see that $J_k(u_k, v_k) = \inf_{(u,v)\in X_k} J_k(u,v)$.

We show that

$$\inf_{(w,z)\in X_k^d} J_k^D(w,z) = \inf_{(u,v)\in X_k} J_k(u,v). \tag{3.4}$$

We consider a functional $J_k^\# : X_k \times X_k^d \to R$ given by the formula

$$J_k^\#(u,v,w,z) = \int_\Omega (F^k)^*(x, -\operatorname{div} w(x), -\operatorname{div} z(x))dx + \int_\Omega \frac{a(x)}{p(x)}|\nabla u(x)|^{p(x)}dx - \int_\Omega \nabla u(x)w(x)dx + \int_\Omega \frac{b(x)}{m(x)}|\nabla v(x)|^{m(x)}dx - \int_\Omega \nabla v(x)z(x)dx.$$

We observe that for any $(u,v) \in X_k$

$$\inf_{(w,z)\in X_k^d} J_k^\#(u,v,w,z) = J_k(u,v) \tag{3.5}$$

and for any $(w,z) \in X_k^d$

$$\inf_{(u,v)\in X_k} J_k^\#(u,v,w,z) = J_k^D(w,z). \tag{3.6}$$

To show (3.5) we fix $(u,v) \in X_k$ and obtain by Fenchel–Young inequality

$$\sup_{(w,z)\in X_k^d} \int_\Omega \Big[(u(x),v(x))\big(-\operatorname{div} w(x), -\operatorname{div} z(x)\big) - (F^k)^*\big(x, -\operatorname{div} w(x), -\operatorname{div} z(x)\big)\Big]dx \leqslant \int_\Omega F^k\big(x,u(x),v(x)\big)dx. \tag{3.7}$$

By definition of X_k^d there exists $(\widehat{w},\widehat{z}) \in X_k^d$ satisfying

$$-\operatorname{div}\widehat{w}(x) = F_u^k\big(x,u(x),v(x)\big),$$
$$-\operatorname{div}\widehat{z}(x) = F_v^k\big(x,u(x),v(x)\big),$$

which provides

$$\int_\Omega \big(u(x),v(x)\big)\big(-\operatorname{div}\widehat{w}(x), -\operatorname{div}\widehat{z}(x)\big)dx - \int_\Omega (F^k)^*\big(x, -\operatorname{div}\widehat{w}(x), -\operatorname{div}\widehat{z}(x)\big)dx = \int_\Omega F^k\big(x,u(x),v(x)\big)dx. \tag{3.8}$$

Therefore equality holds in (3.7). This and integration by parts provides

$$\begin{aligned}&\inf_{(w,z)\in X_k^d} J_k^\#(u,v,w,z)\\ &= -\sup_{(w,z)\in X_k^d}\Big[-\int_\Omega (F^k)^*\big(x, -\operatorname{div} w(x), -\operatorname{div} z(x)\big)dx\\ &\quad + \int_\Omega \nabla u(x)w(x)dx + \int_\Omega \nabla v(x)z(x)dx\Big]\\ &\quad + \int_\Omega \frac{a(x)}{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}|\nabla v(x)|^{m(x)}dx\end{aligned}$$

$$\begin{aligned}
&= - \sup_{(w,z)\in X_k^d} \Big[- \int_\Omega (F^k)^*\big(x, -\operatorname{div} w(x), -\operatorname{div} z(x)\big)dx \\
&\qquad + \int_\Omega \big(u(x), v(x)\big)\big(-\operatorname{div} w(x), -\operatorname{div} z(x)\big)\Big] \\
&\qquad + \int_\Omega \frac{a(x)}{p(x)}\big|\nabla u(x)\big|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}\big|\nabla v(x)\big|^{m(x)}dx \\
&= - \int_\Omega F^k\big(x, u(x), v(x)\big)dx + \int_\Omega \frac{a(x)}{p(x)}\big|\nabla u(x)\big|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}\big|\nabla v(x)\big|^{m(x)}dx = J_k(u,v)
\end{aligned}$$

so (3.5) follows.

To show (3.6) we fix $(w,z)\in X_k^d$. We obtain by the Fenchel–Young inequalities

$$\begin{aligned}
&\sup_{(u,v)\in X_k} \Big\{ \int_\Omega w(x)\nabla u(x)dx - \int_\Omega \frac{a(x)}{p(x)}\big|\nabla u(x)\big|^{p(x)}dx \\
&\qquad + \int_\Omega \nabla v(x) z(x)dx - \int_\Omega \frac{b(x)}{m(x)}\big|\nabla v(x)\big|^{m(x)}dx \Big\} \\
&\leqslant \int_\Omega \frac{1}{(a(x))^{q(x)/p(x)}} \frac{1}{q(x)}\big|w(x)\big|^{q(x)}dx + \int_\Omega \frac{1}{(b(x))^{n(x)/m(x)}} \frac{1}{n(x)}\big|v(x)\big|^{n(x)}dx
\end{aligned} \tag{3.9}$$

For a given $(w,z)\in X_k^d$ there exists $(\widetilde{u}, \widetilde{v})\in X_k$ such that

$$a(x)\big|\nabla\widetilde{u}(x)\big|^{p(x)-2}\nabla\widetilde{u}(x) = w(x),\ \ b(x)\big|\nabla\widetilde{v}(x)\big|^{m(x)-2}\nabla\widetilde{v}(x) = z(x).$$

Thus we get

$$\begin{aligned}
&\int_\Omega w(x)\nabla\widetilde{u}(x)dx - \int_\Omega \frac{a(x)}{p(x)}\big|\nabla\widetilde{u}(x)\big|^{p(x)}dx + \int_\Omega z(x)\nabla\widetilde{v}(x)dx - \int_\Omega \frac{b(x)}{m(x)}\big|\nabla\widetilde{v}(x)\big|^{m(x)}dx \\
&\qquad = \int_\Omega \frac{1}{(a(x))^{q(x)/p(x)}} \frac{1}{q(x)}\big|w(x)\big|^{q(x)}dx + \int_\Omega \frac{1}{(b(x))^{n(x)/m(x)}} \frac{1}{n(x)}\big|v(x)\big|^{n(x)}dx.
\end{aligned}$$

Thus equality holds in (3.10) and relation (3.6) follows.

By (3.5) and (3.6) we obtain

$$\inf_{(u,v)\in X_k} J_k(u,v) = \inf_{(u,v)\in X_k}\ \inf_{(w,z)\in X_k^d} J_k^{\#}(u,v,w,z) = \inf_{(w,z)\in X_k^d}\ \inf_{(u,v)\in X_k} J_k^{\#}(u,v,w,z) = \inf_{(w,z)\in X_k^d} J_k^D(w,z)$$

and (3.4) follows.

Since $(u_k, v_k)\in X_k$ we may take $(w_k, z_k)\in X_k^d$ such that (3.1) hold. By the Fenchel–Young inequalities

$$\int_\Omega \frac{a(x)}{p(x)}\big|\nabla u_k(x)\big|^{p(x)}dx \geqslant \int_\Omega w_k(x)\nabla u_k(x)dx - \int_\Omega \frac{1}{(a(x))^{q(x)/p(x)}} \frac{1}{q(x)}\big|w_k(x)\big|^{q(x)}dx \tag{3.10}$$

and

$$\int_\Omega \frac{b(x)}{m(x)}|\nabla v_k(x)|^{m(x)}dx \geqslant \int_\Omega z_k(x)\nabla v_k(x)dx - \int_\Omega \frac{1}{(b(x))^{n(x)/m(x)}}\frac{1}{n(x)}|z_k(x)|^{n(x)}dx$$

and by a direct calculation we get $J_k(u_k, v_k) \geqslant J_k^D(w_k, z_k)$. By (3.4), it follows that $J_k(u_k, v_k) \leqslant \inf_{(w,z)\in X_k^d} J_k^D(w, z) \leqslant J_k^D(w_k, z_k)$. Hence $J_k(u_k, v_k) = J_k^D(w_k, z_k)$ and by a direct calculation we have actually equalities in (3.10). Therefore by the properties of the Fenchel-Young transformation (3.2) holds. Assertion (3.3) follows by (3.4) and since $J_k^D(w_k, z_k) = J_k(u_k, v_k)$. $\square$

4. STABILITY OF SOLUTIONS

Now we take up the stability problem. We assume F1 – F3 and

F4: F_u^k is differentiable in u on $[-d_0, d_0]$ and in v on $[-c_0, c_0]$ for almost all $x \in \Omega$, F_v^k is differentiable in u on $[-d_0, d_0]$ and in v on $[-c_0, c_0]$ for almost all $x \in \Omega$. There exist constants β_1, $\beta_2, \beta_3, \beta_4 > 0$ (independent of k) such that

$$\begin{aligned} &\max_{u\in[-d_0,d_0]}\max_{v\in[-c_0,c_0]}\left|F_{uu}^k(x,u,v)\right| \leqslant \beta_1, \\ &\max_{u\in[-d_0,d_0]}\max_{v\in[-c_0,c_0]}\left|F_{vv}^k(x,u,v)\right| \leqslant \beta_2, \\ &\max_{u\in[-d_0,d_0]}\max_{v\in[-c_0,c_0]}\left|F_{uv}^k(x,u,v)\right| \leqslant \beta_3, \\ &\max_{u\in[-d_0,d_0]}\max_{v\in[-c_0,c_0]}\left|F_{vu}^k(x,u,v)\right| \leqslant \beta_4. \end{aligned} \tag{4.1}$$

THEOREM 4.1. *Assume* F1, F2, F3, F4 *and that for all* $(u, v) \in X_0$ *there exists a subsequence* $\{k_i\}_{i=1}^\infty$ *such that*

$$\lim_{i\to\infty} F_u^{k_i}(x, u(x), v(x)) = F_u^0(x, u(x), v(x))$$

and

$$\lim_{i\to\infty} F_v^{k_i}(x, u(x), v(x)) = F_v^0(x, u(x), v(x))$$

almost everywhere in Ω. *For each* $k = 0, 1, 2, \ldots$ *there exists a solution* (u_k, v_k) *to the problem (1.2). There exists a subsequence* $\{(u_{k_n}, v_{k_n})\}_{n=1}^\infty$ *of the sequence* $\{(u_k, v_k)\}_{k=1}^\infty$ *and* $(\overline{u}, \overline{v}) \in X_0$ *such that*

$$(u_{k_n}, v_{k_n}) \rightharpoonup (\overline{u}, \overline{v}) \in X_0,\ \text{weakly in } W_0^{1,p(x)}(\Omega) \times W_0^{1,m(x)}(\Omega)$$

and

$$\begin{aligned} -\operatorname{div}\Big(a(x)\left|\nabla\overline{u}(x)\right|^{p(x)-2}\nabla\overline{u}(x)\Big) &= F_u^0(x, \overline{u}(x), \overline{v}(x)), \\ -\operatorname{div}\Big(b(x)\left|\nabla\overline{v}(x)\right|^{m(x)-2}\nabla\overline{v}(x)\Big) &= F_v^0(x, \overline{u}(x), \overline{v}(x)), \\ \overline{u}(x)\ |_{\partial\Omega} = 0, \quad & \overline{v}(x)\ |_{\partial\Omega} = 0. \end{aligned} \tag{4.2}$$

PROOF: By Theorem 3.1 it follows that for each $k = 0, 1, 2, 3, \dots$ there exists $(u_k, v_k) \in X_k$ satisfying (1.2). Since $X_k \subset X_0$ it follows that we may choose a weakly convergent subsequence in $W_0^{1,p(x)}(\Omega) \times W_0^{1,m(x)}(\Omega)$ which up to a subsequence may be assumed to be strongly convergent in $L^{p(x)}(\Omega) \times L^{m(x)}(\Omega)$ and convergent almost everywhere to $(\overline{u}, \overline{v})$. Due to F1

$$\begin{aligned} &\operatorname*{ess\,sup}_{x\in\Omega}\Big|F_u^k\big(x,\overline{u}(x),\overline{v}(x)\big) - F_u^0\big(x,\overline{u}(x),\overline{v}(x)\big)\Big| \leqslant \frac{2a_0d_0}{C_S^1C_1^pC_2^p},\\ (4.3)\qquad &\operatorname*{ess\,sup}_{x\in\Omega}\Big|F_v^k\big(x,\overline{u}(x),\overline{v}(x)\big) - F_v^0\big(x,\overline{u}(x),\overline{v}(x)\big)\Big| \leqslant \frac{2b_0c_0}{C_S^2C_1^mC_2^m}. \end{aligned}$$

By (4.3) and definition of X_k we obtain that $\Big\{-\operatorname{div}\Big(a(\cdot)\big|\nabla u_k(\cdot)\big|^{p(x)-2}\nabla u_k(\cdot)\Big)\Big\}_{k=1}^{\infty}$ and $\Big\{-\operatorname{div}\Big(b(\cdot)\big|\nabla v_k(\cdot)\big|^{m(x)-2}\nabla v_k(\cdot)\Big)\Big\}_{k=1}^{\infty}$ are weakly convergent in $L^{q(x)}(\Omega)$ and $L^{n(x)}(\Omega)$, respectively to functions $d_1 \in L^{q(x)}(\Omega)$, $d_2 \in L^{n(x)}(\Omega)$. We obtain that

$$\begin{aligned} 0 \leqslant \int_\Omega \Big\langle -\operatorname{div}\big(a(x)|\nabla u_k|^{p(x)-2}\nabla u_k\big) - \Big(-\operatorname{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big)\Big), u_k - u\Big\rangle dx \\ \to \int_\Omega \Big\langle d_1(x) - \Big(-\operatorname{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big)\Big), \overline{u} - u\Big\rangle dx. \end{aligned}$$

Thus by the monotonicity of the $p(x)$–Laplacian we see that $d_1(x) = -\operatorname{div}\big(a(x)\,|\nabla\overline{u}|^{p(x)-2}\,\nabla\overline{u}\big)$ and similarly $d_2(x) = -\operatorname{div}\big(b(x)|\nabla\overline{v}|^{m(x)-2}\nabla\overline{v}\big)$.

We now prove that

$$\begin{aligned} &\lim_{i\to\infty} F_u^{k_i}\big(x, u_{k_i}(x), v_{k_i}(x)\big) = F_u^0\big(x, \overline{u}(x), \overline{v}(x)\big) \text{ and}\\ (4.4)\qquad &\lim_{i\to\infty} F_v^{k_i}\big(x, u_{k_i}(x), v_{k_i}(x)\big) = F_v^0\big(x, \overline{u}(x), \overline{v}(x)\big) \text{ almost everywhere.} \end{aligned}$$

We show the first relation. We have

$$\begin{aligned} (4.5)\qquad F_u^{k_i}\big(x, u_{k_i}(x), v_{k_i}(x)\big) - F_u^0\big(x, \overline{u}(x), \overline{v}(x)\big) &= F_u^{k_i}\big(x, u_{k_i}(x), v_{k_i}(x)\big)\\ -F_u^{k_i}\big(x, \overline{u}(x), \overline{v}(x)\big) + F_u^{k_i}\big(x, \overline{u}(x), \overline{v}(x)\big) &- F_u^0\big(x, \overline{u}(x), \overline{v}(x)\big). \end{aligned}$$

By the mean value theorem we observe that

$$\begin{aligned} \big|F_u^{k_i}(x, u_{k_i}, v_{k_i}) - F_u^{k_i}(x, \overline{u}, \overline{v})\big| \leqslant \sup_{x\in\Omega}\ \sup_{(u,v)\in X_0} \sqrt{\big|F_{uu}^{k_i}(x, u_{k_i}, v_{k_i})\big|^2 + \big|F_{uv}^{k_i}(x, u_{k_i}, v_{k_i})\big|^2} \\ \cdot \sqrt{|u_{k_i} - \overline{u}|^2 + |v_{k_i} - \overline{v}|^2} \end{aligned}$$

Since $\{u_{k_n}\}_{n=1}^{\infty}$ and $\{v_{k_n}\}_{n=1}^{\infty}$ are convergent almost everywhere, by F4 it follows that

$$\lim_{i\to\infty}\Big(F_u^{k_i}\big(x, u_{k_i}(x), v_{k_i}(x)\big) - F_u^{k_i}\big(x, \overline{u}(x), \overline{v}(x)\big)\Big) = 0 \text{ almost everywhere.}$$

Thus from (4.6) using the above and by the assumption $\lim\limits_{i\to\infty} F_u^{k_i}(x,\overline{u}(x),\overline{v}(x)) = F_u^0(x,\overline{u}(x),\overline{v}(x))$ we obtain (4.4). Since the weak limit is equal to an almost everywhere limit we get (4.3). $\square$

By Theorem 3.1 there exists $(u_0,v_0)\in X_0$ such that $\inf\limits_{(u,v)\in X_0} J_0(u,v) = J_0(u_0,v_0)$. The following corollary shows that under some additional assumptions $(\overline{u},\overline{v})$ minimises J_0 on X_0.

Corollary 4.2. *Under the assumptions of Theorem 4.1 if*

$$\limsup_{k\to\infty}\left(\int_\Omega F^k(x,u_0,v_0)dx - \int_\Omega F^0(x,u_0,v_0)dx\right) \leqslant 0 \tag{4.6}$$

and $\lim\limits_{k\to\infty} F^k(x,\overline{u},\overline{v}) = F^0(x,\overline{u},\overline{v})$ *for almost everywhere* $x\in\Omega$, *then* $(\overline{u},\overline{v})$ *minimises* J_0 *on* X_0.

Proof: Let us suppose that $(\overline{u},\overline{v})$ does not minimise J_0 on X_0, that is,

$$J_0(\overline{u},\overline{v}) - J_0(u_0,v_0) > 0,$$

where (u_0,v_0) is a point minimising J_0 on X_0, provided by Theorem 3.1. Due to the weak lower semicontinuity of J_0 we have

$$\liminf_{k\to\infty}\big(J_0(u_k,v_k) - J_0(\overline{u},\overline{v})\big) \geqslant 0. \tag{4.7}$$

Hence, by

$$\begin{aligned}0 < J_0(\overline{u},\overline{v}) - J_0(u_0,v_0) &= \big(J_k(u_k,v_k) - J_0(u_0,v_0)\big)\\ &\quad - \big(J_k(u_k,v_k) - J_0(u_k,v_k)\big) - \big(J_0(u_k,v_k) - J_0(\overline{u},\overline{v})\big)\end{aligned}$$

and by (4.7) the proof will be finished by showing that

$$\lim_{k\to\infty}\big(J_k(u_k,v_k) - J_0(u_k,v_k)\big) = 0 \tag{4.8}$$

and

$$\liminf_{k\to\infty}\big(J_k(u_k,v_k) - J_0(u_0,v_0)\big) \leqslant 0. \tag{4.9}$$

We get

$$\lim_{k\to\infty}\big(J_k(u_k,v_k) - J_0(u_k,v_k)\big) = \lim_{k\to\infty}\left(\int_\Omega F^0(x,u_k,v_k)dx - \int_\Omega F^k(x,u_k,v_k)dx\right).$$

Since

$$\begin{aligned}\left|F^0(x,u_k,v_k) - F^k(x,u_k,v_k)\right| &\leqslant \left|F^0(x,u_k,v_k) - F^0(x,\overline{u},\overline{v})\right|\\ &\quad + \left|F^k(x,\overline{u},\overline{v}) - F^0(x,\overline{u},\overline{v})\right| + \left|F^k(x,u_k,v_k) - F^k(x,\overline{u},\overline{v})\right|\end{aligned}$$

we have by the mean value theorem and by F1

$$\begin{aligned}
&\left|F^0(x,u_k,v_k)-F^0(x,\overline{u},\overline{v})\right|\\
&\qquad\leqslant \sup_{x\in\Omega}\sup_{(u,v)\in X_0}\sqrt{\left|F_u^0(x,u,v)\right|^2+\left|F_v^0(x,u,v)\right|^2}\sqrt{|u_k-\overline{u}|^2+|v_k-\overline{v}|^2}\\
&\qquad\leqslant \sqrt{a_0^2d_k^2+b_0^2c_k^2}\sqrt{|u_k-\overline{u}|^2+|v_k-\overline{v}|^2}\to 0,\\
&\left|F^k(x,u_k,v_k)-F^k(x,\overline{u},\overline{v})\right|\to 0.
\end{aligned}$$

Since $F^k(x,\overline{u},\overline{v})\to F^0(x,\overline{u},\overline{v})$ we obtain

$$\lim_{k\to\infty}\left(\int_\Omega F^0(x,u_k,v_k)dx-\int_\Omega F^k(x,u_k,v_k)dx\right)=0,$$

so (4.8) is shown.

Now since (u_k,v_k) minimises J_k and by (4.6) we have

$$\begin{aligned}
\liminf_{k\to\infty}\big(J_k(u_k,v_k)-J_0(u_0,v_0)\big)&\leqslant \liminf_{k\to\infty}\big(J_k(u_0,v_0)-J_0(u_0,v_0)\big)\\
&=\liminf_{k\to\infty}\left(\int_\Omega F^0(x,u_0,v_0)dx-\int_\Omega F^k(x,u_0,v_0)dx\right)\leqslant 0,
\end{aligned}$$

so (4.9) is proved. □

Investigation of the proof of Theorem 4.1 shows that we may weaken a bit its assumptions. Precisely, instead of F4 we assume F_u^k and F_v^k have property as in (4.4). Thus we have the following corollary.

COROLLARY 4.3. *Assume* F1, F2, F3 *and that for all* $(u,v)\in X_0$ *there exists a subsequence* $\{k_i\}_{i=1}^\infty$ *such that* $\lim_{i\to\infty}F_u^{k_i}(x,u(x),v(x))=F_u^0(x,u(x),v(x))$ *and* $\lim_{i\to\infty}F_v^{k_i}(x,u(x),v(x))=F_v^0(x,u(x),v(x))$ *almost everywhere in* Ω. *For each* $k=0,1,2,\ldots$ *there exists a solution* (u_k,v_k) *to the problem* (1.2), *subsequence* $\left\{(u_{k_n},v_{k_n})\right\}_{n=1}^\infty$ *of the sequence* $\left\{(u_k,v_k)\right\}_{k=1}^\infty$ *and* $(\overline{u},\overline{v})\in X_0$ *satisfying* (4.3). *We assume that*

$$\begin{aligned}
&\lim_{i\to\infty}F_u^{k_i}\big(x,u_{k_i}(x),v_{k_i}(x)\big)=F_u^0\big(x,\overline{u}(x),\overline{v}(x)\big),\\
(4.10)\qquad&\lim_{i\to\infty}F_v^{k_i}\big(x,u_{k_i}(x),v_{k_i}(x)\big)=F_v^0\big(x,\overline{u}(x),\overline{v}(x)\big)\quad\textit{almost everywhere}
\end{aligned}$$

Then $(u_{k_n},v_{k_n})\rightharpoonup(\overline{u},\overline{v})\in X_0$ *weakly in* $W_0^{1,p(x)}(\Omega)\times W_0^{1,m(x)}(\Omega)$. *If additionally* $\lim_{k\to\infty}F^k(x,\overline{u},\overline{v})=F^0(x,\overline{u},\overline{v})$ *for almost all* $x\in\Omega$ *and* (4.6) *holds, then* $\inf_{(u,v)\in X_0}J_0(u,v)=J_0(\overline{u},\overline{v})$.

5. CONTINUOUS DEPENDENCE ON PARAMETERS

Now we prove that system (5.1) depends continuously on a functional parameter g_k. We are interested in giving conditions asserting that if only $g_k \to \overline{g}$ in $L^{q(x)}(\Omega)$, then solutions (u_k, v_k) to

$$\begin{aligned} &-\operatorname{div}\Big(a(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_u\big(x, u(x), v(x), g_k(x)\big),\\ &-\operatorname{div}\Big(b(x)\big|\nabla v(x)\big|^{m(x)-2}\nabla v(x)\Big) = F_v\big(x, u(x), v(x), g_k(x)\big),\\ &u(x)\mid_{\partial\Omega} = 0,\ u \in W_0^{1,p(x)}(\Omega), \qquad v(x)\mid_{\partial\Omega} = 0,\ v \in W_0^{1,m(x)}(\Omega) \end{aligned} \tag{5.1}$$

converge (up to a subsequence) to the solution $(\overline{u}, \overline{v})$ to

$$\begin{aligned} &-\operatorname{div}\Big(a(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_u\Big(x, u(x), v(x), \overline{g}(x)\Big),\\ &-\operatorname{div}\Big(b(x)\Big|\nabla v(x)\Big|^{m(x)-2}\nabla v(x)\Big) = F_v\big(x, u(x), v(x), \overline{g}(x)\big) \end{aligned}$$

Let g_k and $\overline{g}$ be functional parameters taken from the set

$$\big\{g : \Omega \to R^m :\ g \text{ is measurable, } g(x) \in M \text{ almost everywhere}\big\},$$

where M is a bounded and compact subset of R^m. Existence of solutions to (5.1) for each $k = 0, 1, \ldots$ is guaranteed by Theorem 3.1.

We assume that for some $d > 0$, $c > 0$ we have

F5: $\|1\|_{q(x)} \leqslant (1/p^-) + 1/q^-)^{-1}$, $\|1\|_{n(x)} \leqslant (1/m^- + 1n^-)^{-1}$ and for all $g \in M$

$$\begin{aligned} C_S^1 C_1^p C_2^p \operatorname*{ess\,sup}_{x\in\Omega} \max_{u\in[-d,d]} \max_{v\in[-c,c]} \big|F_u(x, u, v, g)\big| &\leqslant a_0 d,\\ C_S^2 C_1^m C_2^m \operatorname*{ess\,sup}_{x\in\Omega} \max_{u\in[-d,d]} \max_{v\in[-c,c]} \big|F_v(x, u, v, g)\big| &\leqslant b_0 c. \end{aligned}$$

F6: $F_u(x, 0, 0, 0) \neq 0$, $F_v(x, 0, 0, 0) \neq 0$ for almost all $x \in \Omega$, $x \mapsto F(x, 0, 0, 0)$ and $x \mapsto (F)^*(x, 0, 0, 0)$ are integrable on Ω for all $g \in M$.

F7: $F : \Omega \times [-d, d] \times [-c, c] \times M \to R$ is a Caratheodory function, that is, measurable in x and continuous in (u, v, g). F is convex in (u, v) on $[-d, d] \times [-c, c]$ for almost all $x \in \Omega$ and all $g \in M$.

Since the notation changes, we now rewrite the definitions of X_k and action functionals. We have for each $k = 0, 1, 2, \ldots$ that $X_k = X$, where

$$X = \Big\{(u, v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,m(x)}(\Omega),\ \|\nabla u\|_{p(x)} \leqslant \frac{d}{C_1^p C_2^p}, \|\nabla v\|_{m(x)} \leqslant \frac{c}{C_1^m C_2^m}, \big|u(x)\big| \leqslant d, \big|v(x)\big| \leqslant c\Big\}$$

and

$$J_k(u,v) = \int_\Omega \frac{a(x)}{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}|\nabla v(x)|^{m(x)}dx - \int_\Omega F(x,u(x),v(x),g_k(x))dx,$$
$$J_0(u,v) = \int_\Omega \frac{a(x)}{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega \frac{b(x)}{m(x)}|\nabla v(x)|^{m(x)}dx - \int_\Omega F(x,u(x),v(x),\overline{g}(x))dx.$$

THEOREM 5.1. *Assume* F5–F7, $g_k \to \overline{g}$ *in* $L^{q(x)}(\Omega)$ *and*

$$\limsup_{k\to\infty}\left(\int_\Omega F(x,u,v,g_k)dx - \int_\Omega F(x,u,v,\overline{g})dx\right) \leqslant 0.$$

Then for each $k = 0,1,2,\dots$ *there exists solutions* (u_k,v_k) *to* (5.1) *minimising* J_k *given by* (5.2) *on* X. *Moreover, up to a subsequence,* $\{(u_k,v_k)\}$ *converges in* $L^{p(x)}(\Omega)\times L^{m(x)}(\Omega)$ *to* $(\overline{u},\overline{v})$ *being solution to*

$$-\operatorname{div}\left(a(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = F_u\left(x,u(x),v(x),\overline{g}(x)\right),$$
$$-\operatorname{div}\left(b(x)|\nabla v(x)|^{m(x)-2}\nabla v(x)\right) = F_v\left(x,u(x),v(x),\overline{g}(x)\right)$$

where $J_0(\overline{u},\overline{v}) = \inf_{(u,v)\in X} J_0(u,v)$.

PROOF: We show that conditions of Corollary 4.3 are satisfied with $F^k(x,u,v) := F(x,u,v,g_k)$ and $F^0(x,u,v) := F(x,u,v,\overline{g})$. Let us fix $(u,v)\in X$. Clearly F5–F7 imply F1–F3. By F5 it follows that $|F_u(x,u,v,g_k)|$ and $|F_v(x,u,v,g_k)|$ are bounded on $\Omega\times[-d,d]\times[-c,c]\times M$. By the generalised Krasnosielski Theorem [7] the Nemytskij operators

$$L^{q(x)}(\Omega)\ni g\longmapsto F_u(\cdot,u(\cdot),v(\cdot),g(\cdot))$$
$$L^{q(x)}(\Omega)\ni g\longmapsto F_v(\cdot,u(\cdot),v(\cdot),g(\cdot))$$

are well defined and continuous, that is,

$$F_u\left(x,u(x),v(x),g_k(x)\right)\to F_u\left(x,u(x),v(x),\overline{g}(x)\right),$$
$$F_v\left(x,u(x),v(x),g_k(x)\right)\to F_v\left(x,u(x),v(x),\overline{g}(x)\right).$$

Clearly (4.10) holds. Moreover, by F7 it follows that $F(x,u,v,g_k)\to F(x,u,v,\overline{g})$ almost everywhere in Ω. Thus assertion follows by Corollary 4.3. ◻

Now we present a special form of system (5.1); that is, a special form of nonlinearities in which parameters are given linearly but the sequence of parameters is only weakly convergent. We provide also Theorem similar to Theorem 5.1.

5.1. LINEAR CASE. We assume that right hand side is in the form

$$F(x,u,v,g_k) = F^1(x,u,v)g_k + F^2(x,u,v).$$

THEOREM 5.2. *Assume* F5–F7, $g_k \rightharpoonup \overline{g}$ *weakly in* $L^{q(x)}(\Omega)$ *and*

$$x \mapsto F_u^1(\cdot, u(\cdot), v(\cdot)), \; x \mapsto F_v^1(\cdot, u(\cdot), v(\cdot))$$

are in $L^{p(x)}(\Omega)$. *Then for each* $k = 0, 1, 2, \ldots$ *there exists solution* (u_k, v_k) *to* (5.1) *minimising* J_k *on* X. *Moreover, up to a subsequence,* $\{(u_k, v_k)\}$ *converges in* $L^{p(x)}(\Omega) \times L^{m(x)}(\Omega)$ *to* $(\overline{u}, \overline{v})$ *being the solution to*

$$-\operatorname{div}\Big(a(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_u\big(x, u(x), v(x), \overline{g}(x)\big),$$
$$-\operatorname{div}\Big(b(x)\big|\nabla v(x)\big|^{m(x)-2}\nabla v(x)\Big) = F_v\big(x, u(x), v(x), \overline{g}(x)\big)$$

and $J_0(\overline{u}, \overline{v}) = \inf_{(u,v)\in X} J_0(u, v)$.

PROOF: We have since $\{g_k\}$ is weakly convergent and $F_u^1(\cdot, u(\cdot), v(\cdot))$, $F_v^1(\cdot, u(\cdot), v(\cdot))$ are in $L^{p(x)}(\Omega)$ that

$$\int_\Omega F_u^1\big(x, u(x), v(x)\big) g_k(x)\, dx \to \int_\Omega F_u^1\big(x, u(x), v(x)\big)\overline{g}(x)\, dx,$$
$$\int_\Omega F_v^1\big(x, u(x), v(x)\big) g_k(x)\, dx \to \int_\Omega F_v^1\big(x, u(x), v(x)\big)\overline{g}(x)\, dx,$$

so

$$\int_\Omega F_u\big(x, u(x), v(x)\big) g_k(x)\, dx \to \int_\Omega F_u\big(x, u(x), v(x)\big)\overline{g}(x)\, dx,$$
$$\int_\Omega F_v\big(x, u(x), v(x)\big) g_k(x)\, dx \to \int_\Omega F_v\big(x, u(x), v(x)\big)\overline{g}(x)\, dx.$$

The assertion follows by By Corollary 4.3. $\square$

6. EXAMPLES

We give now two examples of nonlinearities satisfying our growth assumptions.

EXAMPLE 1. Let us first take

$$F^k(x, u, v) = (C_S^1 C_1^p C_2^p)^{-1}\Big(e^u + f_k(x)u + \frac{1}{2}u^2\Big) + (C_S^2 C_1^m C_2^m)^{-1}\Big(\frac{1}{2}e^v + \frac{1}{12}v^4 - f_k(x)v\Big)$$

We assume that Ω is a bounded subset of R^N and

Z1 $a(x) \geqslant a_0 \geqslant 2\sqrt{e} + 3$, $b(x) \geqslant b_0 \geqslant \sqrt{3} + 25/12$ for all $x \in \Omega$.

Z2 $f_k \in L^1(\Omega)$, $\operatorname{ess\,sup}_{x\in\Omega}\big|f_k(x)\big| = 1$ and $\operatorname{meas}\big\{x \in \Omega \mid f_k(x) = -1 \vee f_k(x) = -1/2\big\} = 0$.

Clearly assumptions F2 and F3 are satisfied. F_u^k, F_v^k are integrable respectively in u, v, on every compact subset of R. To conclude that F4 holds we only need to show (4.2). By Z2 and since $0 < d_k \leqslant d_0$, $0 < c_k \leqslant c_0$ we get

$$\max_{u\in[-d_k,d_k]}\max_{v\in[-c_k,c_k]}\left|F_{uu}^k(x,u,v)\right| \leqslant (C_S^1C_1^pC_2^p)^{-1}(e^{d_0}+1),$$
$$\max_{u\in[-d_k,d_k]}\max_{v\in[-c_k,c_k]}\left|F_{vv}^k(x,u,v)\right| \leqslant (C_S^2C_1^mC_2^m)^{-1}\Bigl(\frac{1}{2}e^{c_0}+c_0^2\Bigr)$$

We demonstrate that relations (2.4) hold. We obtain

$$C_S^1C_1^pC_2^p\operatorname*{ess\,sup}_{x\in\Omega}\max_{u\in[-d_k,d_k]}\max_{v\in[-c_k,c_k]}\left|F_u^k(x,u,v)\right| \leqslant e^{d_k}+d_k+1,$$
$$C_S^2C_1^mC_2^m\operatorname*{ess\,sup}_{x\in\Omega}\max_{u\in[-d_k,d_k]}\max_{v\in[-c_k,c_k]}\left|F_v^k(x,u,v)\right| \leqslant \frac{1}{2}e^{c_k}+\frac{1}{3}c_k^3+1.$$

By Z1 the functions $x \mapsto e^x+(1-a_0)x+1$ and $x \mapsto (1/2)e^x+(1/3)x^3-b_0x+1$ are both nonpositive on the intervals $[1/2,5/2]$ and $[1/2,2]$, respectively. Thus

$$e^{d_k}+d_k+1 \leqslant a_0d_k,$$
$$\frac{1}{2}e^{c_k}+\frac{1}{3}c_k^3+1 \leqslant b_0c_k,$$

for $d_k \in [1/2,5/2]$ and $c_k \in \bigl[(1/2),2\bigr]$ and we conclude that (2.4) holds. Therefore we may take any nondecreasing sequences $\{d_k\}$, $\{c_k\}$ from $[1/2,5/2]$ and $[1/2,2]$, respectively and put X_k as in (2.5).

As for stability we consider for $k=1,2,\dots$ F^k as in the above but with f_k which now reads

$$f_k(x) = e^{-(kx^2/k+1)}.$$

Now F1, F2, F3, F4 and Z2 are obviously satisfied. Clearly (4.6) is also satisfied since $f_k(x) \to f_0(x) := e^{-x^2}$ uniformly on Ω. By Corollary 4.3 it follows that from the sequence $\bigl\{(u_k,v_k)\bigr\}$ of solutions to

$$\begin{aligned}
-\operatorname{div}\bigl(a(x)\left|\nabla u(x)\right|^{p(x)-2}\nabla u(x)\bigr) &= (C_S^1C_1^pC_2^p)^{-1}\bigl(e^u+u+e^{-(kx^2/k+1)}\bigr),\\
-\operatorname{div}\Bigl(b(x)\left|\nabla v(x)\right|^{m(x)-2}\nabla v(x)\Bigr) &= (C_S^2C_1^mC_2^m)^{-1}\Bigl(\frac{1}{2}e^v+\frac{1}{3}v^3-e^{-(kx^2/k+1)}\Bigr),\\
u(x)\mid_{\partial\Omega}=0,\ u\in W_0^{1,p(x)}(\Omega),&\qquad v(x)\mid_{\partial\Omega}=0,\ v\in W_0^{1,m(x)}(\Omega)
\end{aligned}$$

we may take subsequence converging to a certain (u_0,v_0) being a solution to

$$\begin{aligned}
-\operatorname{div}\Bigl(a(x)\left|\nabla u(x)\right|^{p(x)-2}\nabla u(x)\Bigr) &= (C_S^1C_1^pC_2^p)^{-1}(e^u+u+e^{-x^2}),\\
-\operatorname{div}\Bigl(b(x)\left|\nabla v(x)\right|^{m(x)-2}\nabla v(x)\Bigr) &= (C_S^2C_1^mC_2^m)^{-1}\Bigl(\frac{1}{2}e^v+\frac{1}{3}v^3-e^{-x^2}\Bigr),\\
u(x)\mid_{\partial\Omega}=0,\ u\in W_0^{1,p(x)}(\Omega),&\qquad v(x)\mid_{\partial\Omega}=0,\ v\in W_0^{1,m(x)}(\Omega).
\end{aligned}$$

Now we check that F^k with $g_k(x) := f_k(x) = e^{-(kx^2/k+1)}$ satisfies assumptions of Theorem 5.2. We may rewrite it as follows

$$\begin{aligned} F^k(x,u,v) = F(x,u,v,g_k) = ((C^1_S C^p_1 C^p_2)^{-1}u + (C^2_S C^m_1 C^m_2)^{-1}v)g_k \\ + (C^1_S C^p_1 C^p_2)^{-1}\Big(e^u + \frac{1}{2}u^2\Big) + (C^2_S C^m_1 C^m_2)^{-1}\Big(\frac{1}{2}e^v + \frac{1}{12}v^4\Big), \end{aligned}$$

so it is in the form required in Theorem 5.2. It is clear that $g_k \rightharpoonup \overline{g}$ in $L^{q(x)}(\Omega)$ and that F5–F7 hold. Obviously the functions $x \mapsto F^1_u(\cdot, u(\cdot), v(\cdot))$, $x \mapsto F^1_v(\cdot, u(\cdot), v(\cdot)) \in L^{p(x)}(\Omega)$. Thus assertion of Theorem 5.2 holds.

Example 2. We consider

$$\begin{aligned} -\operatorname{div}\Big(a(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\Big) &= |x|^2 \cdot |u(x)|^{\alpha_k(x)-1} \cdot u(x) \cdot v^2(x) \\ &\quad + \frac{2}{\beta_k(x)+1} \cdot |x| \cdot |v(x)|^{\beta_k(x)+1} \cdot u(x) + |x|, \\ -\operatorname{div}\Big(b(x)|\nabla v(x)|^{m(x)-2}\nabla v(x)\Big) &= \frac{2}{\alpha_k(x)+1} \cdot |x|^2 \cdot |u(x)|^{\alpha_k(x)+1} \cdot v(x) \\ &\quad + |x| \cdot |v(x)|^{\beta_k(x)-1} \cdot v(x) \cdot u^2(x) + |x|, \\ u(x)\,|_{\partial\Omega} = 0,\ u \in W^{1,p(x)}_0(\Omega), &\quad v(x)\,|_{\partial\Omega} = 0,\ v \in W^{1,m(x)}_0(\Omega). \end{aligned} \tag{6.1}$$

Here $\Omega = B(0,\delta)$ is a ball in R^3 and for all $k = 0,1,\dots$ $\alpha^+_k, \alpha^-_k > p^+ > 3$; $\beta^+_k, \beta^-_k > m^+ > 3$, where $\alpha^+_k = \sup_{x\in\Omega} \alpha_k(x)$, $\alpha^-_k = \inf_{x\in\Omega} \alpha_k(x)$, $\beta^+_k = \sup_{x\in\Omega} \beta_k(x)$, $\beta^-_k = \inf_{x\in\Omega} \beta_k(x)$. We also assume

W1 $a(x) \geqslant a_0 \geqslant C^1_S C^p_1 C^p_2$, $b(x) \geqslant b_0 \geqslant C^S_2 C^m_1 C^m_2$ for all $x \in \Omega$.

W2 $\alpha^+_k = \beta^+_k$ for all $k = 1,2,\dots$ and $\{\alpha^+_k\}$ is bounded by $\overline{\alpha}^+$.

W3 δ satisfies the following inequalities

$$\begin{aligned} 2^{\overline{\alpha}^+ + 2}\Big(\delta^2 + \frac{1}{2}\delta\Big) + \delta \leqslant 1, \\ 2^{\overline{\alpha}^+ + 2}\Big(\frac{1}{2}\delta^2 + \delta\Big) + \delta \leqslant 1. \end{aligned}$$

W4 $c_k = d_k = 2 - (1/k)$ for all $k = 1,2,\dots$

Here

$$F^k(x,u,v) = \frac{1}{\alpha_k(x)+1} \cdot |x|^2 \cdot |u|^{\alpha_k(x)+1} \cdot v^2 + \frac{1}{\beta_k(x)+1} \cdot |x| \cdot |v|^{\beta_k(x)+1} \cdot u^2 + |x|\big(|u| + |v|\big)$$

Obviously assumptions F2 and F3 are satisfied. We show now that (4.2) holds. Clearly F^k_u is differentiable in u on $[-d_k, d_k]$ and F^k_v is differentiable in v on $[-c_k, c_k]$ for almost all $x \in \Omega$. Moreover, for all k by W2 and W4 we have

$$\max_{u\in[-d_k,d_k]} \max_{v\in[-c_k,c_k]} \big|F^k_{uu}(x,u,v)\big| = \delta^2 \alpha^+_k d_k^{\alpha^+_k - 1} c_k^2 + \frac{2}{\beta^-_k + 1}\delta c_k^{\beta^+_k + 1} \leqslant 2^{\overline{\alpha}^+ + 1}\delta^2\overline{\alpha}^+ + 2^{\overline{\alpha}^+}\delta$$

and similarly

$$\max_{u\in[-d_k,d_k]}\max_{v\in[-c_k,c_k]}\left|F_{vv}^k(x,u,v)\right| \leqslant 2^{\overline{\alpha}^+}\delta^2 + 2^{\overline{\alpha}^+ +1}\overline{\alpha}^+\delta.$$

Now we show that (2.4) is also satisfied. By W1 and W2 it is equivalent to showing that for all $k = 1, 2, \ldots$

$$\begin{gathered}
\delta^2\cdot\Bigl(2-\frac{1}{k}\Bigr)^{\alpha_k^+ +2}+\frac{1}{2}\delta\cdot\Bigl(2-\frac{1}{k}\Bigr)^{\alpha_k^+ +2}+\delta\leqslant 2-\frac{1}{k},\\
\frac{1}{2}\delta^2\cdot\Bigl(2-\frac{1}{k}\Bigr)^{\alpha_k^+ +2}+\delta\cdot\Bigl(2-\frac{1}{k}\Bigr)^{\alpha_k^+ +2}+\delta\leqslant 2-\frac{1}{k}
\end{gathered}$$

or

$$\begin{gathered}
\Bigl(2-\frac{1}{k}\Bigr)^{\alpha_k^+ +2}\Bigl(\delta^2+\frac{1}{2}\delta\Bigr)\leqslant 2-\frac{1}{k}-\delta,\\
\Bigl(2-\frac{1}{k}\Bigr)^{\alpha_k^+ +2}\Bigl(\frac{1}{2}\delta^2+\delta\Bigr)\leqslant 2-\frac{1}{k}-\delta
\end{gathered}$$

which is true by W2 and W3. Thus we may put X_k as in (2.5) with c_k, d_k as in W4. Again we may conclude that from the sequence $\{(u_k, v_k)\}$ of solutions to (6.1) we may choose a subsequence converging to the solution $(u_0, v_0) \in X_0$ to

$$\begin{aligned}
-\operatorname{div}\Bigl(a(x)\bigl|\nabla u(x)\bigr|^{p(x)-2}\nabla u(x)\Bigr) &= |x|^2\cdot\bigl|u(x)\bigr|^{\alpha_0(x)-1}\cdot u(x)\cdot v^2(x)\\
&\quad+\frac{2}{\beta_0(x)+1}\cdot|x|\cdot\bigl|v(x)\bigr|^{\beta_0(x)+1}\cdot u(x)+|x|,\\
-\operatorname{div}\Bigl(b(x)\bigl|\nabla v(x)\bigr|^{m(x)-2}\nabla v(x)\Bigr) &= \frac{2}{\alpha_0(x)+1}\cdot|x|^2\cdot\bigl|u(x)\bigr|^{\alpha_0(x)+1}\cdot v(x)\\
&\quad+|x|\cdot\bigl|v(x)\bigr|^{\beta_0(x)-1}\cdot v(x)\cdot u^2(x)+|x|.
\end{aligned}$$

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