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LOCAL PROPERTIES OF THE HOCHSCHILD COHOMOLOGY OF C^*-ALGEBRAS

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Abstract

Let A be a C^*-algebra, and let X be a Banach A-bimodule. Johnson [B. E. Johnson, 'Local derivations on C^*-algebras are derivations', *Trans. Amer. Math. Soc.* 353 (2000), 313–325] showed that local derivations from A into X are derivations. We extend this concept of locality to the higher cohomology of a C^*-algebra and show that, for every $n \in \mathbb{N}$, bounded local n-cocycles from $A^{(n)}$ into X are n-cocycles.

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1. Introduction

The study of the local properties of Hochschild cohomology of a Banach algebra was initiated by introducing the concept of 'local derivations'. Let A be a Banach algebra, and let X be a Banach A-bimodule. An operator $D\colon A \to X$ is a local derivation if, for each $a \in A$, there is a derivation $D_a\colon A \to X$ such that $D(a) = D_a(a)$. This concept was introduced independently by Kadison [4] and Larson [6] and it has been much studied since then. Kadison's motivation was based on his and Ringrose's earlier investigation of Hochschild cohomology of various operator algebras [5], whereas Larson's motivation was to investigate algebraic reflexivity (or reflexivity) of the linear space of derivations (respectively bounded derivations) from a Banach algebra. Local derivations have been investigated for various classes of Banach algebras such as operator algebras, Banach operator algebras, group algebras, and Fourier algebras (see [2, 8, 9] and the references therein).

In [4], Kadison showed that bounded local derivations from a von Neumann algebra into any of its dual bimodules are derivations. He then raised the question of whether the preceding result can be extended to the local higher cohomology. The purpose of this article is to answer this affirmatively in a more general setting. We show that if A is a C^*-algebra and $n \in \mathbb{N}$, then bounded *local n-cocycles* from $A^{(n)}$ into any Banach A-bimodule are n-cocycles. This has already been obtained by Johnson in [3] for the case $n = 1$. Our approach is as follows.

Let A be a Banach algebra, let X be a Banach A-bimodule, and let $n \in \mathbb{N}$. In Section 3, we introduce certain n-linear maps from $A^{(n)}$ into X which are more general than local n-cocycles. We call them *n-hyperlocal maps*. We show that in order to characterize bounded local n-cocycles from $A^{(n)}$ into X, it suffices to first extend them to $A^{\sharp(n)}$, where $A^{\sharp}$ is the unitization of A, and view them as n-hyperlocal maps from $A^{\sharp(n)}$ into X. Then, by imposing certain conditions, one can obtain the result. As is shown in Proposition 3.2, the advantage of this technique is that we can 'transfer the information' from the lower cohomology to the higher one if we consider n-hyperlocal maps rather than local n-cocycles.

In Section 4, we apply these ideas to hyper-Tauberian algebras. These algebras were introduced and studied in [9] because of their useful local properties. By using the results of the preceding section, together with the properties of hyper-Tauberian algebras, we show that bounded local n-cocycles from $A^{(n)}$ into X are n-cocycles when A is a hyper-Tauberian algebra.

In Section 5, we first show that every commutative C^*-algebra is a hyper-Tauberian algebra. We then apply the results of Section 4 to obtain our result for a general C^*-algebra. Finally, in the last section, we give a characterization of amenable C^*-algebras in terms of the 1-hyperlocal maps.

2. Preliminaries

Let X and Y be Banach spaces. For $n \in \mathbb{N}$, let $X^{(n)}$ be the Cartesian product of n copies of X, and let $L^n(X, Y)$ and $B^n(X, Y)$ be the spaces of n-linear maps and bounded n-linear maps from $X^{(n)}$ into Y, respectively.

Let A be a Banach algebra, and let X be a Banach A-bimodule. An operator $D \in L(A, X)$ is a derivation if, for all $a, b \in A$, $D(ab) = aD(b) + D(a)b$. For each $x \in X$, the operator $ad_x \in B(A, X)$ defined by $ad_x(a) = ax - xa$ is a bounded derivation, called an inner derivation. Let $Z^1(A, X)$ and $\mathcal{Z}^1(A, X)$ be the linear spaces of derivations and bounded derivations from A into X, respectively. For $n \in \mathbb{N}$ and $T \in L^n(A, X)$, define

$$\begin{aligned}\delta^n T : (a_1, \dots, a_{n+1}) \mapsto\ & a_1 T(a_2, \dots, a_n) \\ & + \sum_{j=1}^{n} (-1)^j T(a_1, \dots, a_{j-1}, a_j a_{j+1}, \dots, a_{n+1}) \\ & + (-1)^{n+1} T(a_1, \dots, a_n) a_{n+1}.\end{aligned}$$

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It is clear that δ^n is a linear map from $L^n(A, X)$ into $L^{n+1}(A, X)$; these maps are the *connecting maps*. The elements of $\ker \delta^n$ are the *n-cocycles*; we denote this linear space by $Z^n(A, X)$. If we replace $L^n(A, X)$ with $B^n(A, X)$ in the above, we will have the 'Banach' version of the connecting maps; we denote them with the same notation δ^n. In this case, δ^n is a bounded linear map from $B^n(A, X)$ into $B^{n+1}(A, X)$; these maps are the *bounded connecting maps*. The elements of $\ker \delta^n$ are the *bounded n-cocycles*; we denote this linear space by $\mathcal{Z}^n(A, X)$. It is easy to check that $Z^1(A, X)$ and $\mathcal{Z}^1(A, X)$ coincide with our previous definition of this notation.

Let A be a Banach algebra, and let X be a Banach A-bimodule. By [1, Section 2.8], for $n \in \mathbb{N}$, the Banach space $B^n(A, X)$ turns into a Banach A-bimodule by the actions defined by

$$(a \star T)\,(a_1, \ldots, a_n) = aT(a_1, \ldots, a_n);$$

$$\begin{aligned}(T \star a)\,(a_1, \ldots, a_n) = T(aa_1, \ldots, a_n)&\\ + \sum_{j=1}^{n}(-1)^j T(a, a_1, \ldots, a_j a_{j+1}, \ldots, a_n)&\\ + (-1)^{n+1} T(a, a_1, \ldots, a_{n-1})a_n.&\end{aligned}$$

In particular, when $n = 1$, $B(A, X)$ becomes a Banach A-bimodule with respect to the products

$$(a \star T)\,(b) = aT(b), \qquad (T \star a)\,(b) = T(ab) - T(a)b.$$

Let $\Lambda_n \colon B^{n+1}(A, X) \to B^n(A, B(A, X))$ be the identification given by

$$(\Lambda_n(T)\,(a_1, \ldots, a_n))\,(a_{n+1}) = T(a_1, \ldots, a_{n+1}).$$

Then Λ_n is an A-bimodule isometric isomorphism. If we denote the connecting maps for the complex $B^n(A, (B(A, X), \star))$ by Δ^n, then it is shown in [1] that

$$\Lambda_{n+1} \circ \delta^{n+1} = \Delta^n \circ \Lambda_n.$$

3. *n*-hyperlocal maps and local *n*-cocycles

DEFINITION 3.1. Let A be a Banach algebra, and let X be a Banach A-bimodule. For $n \in \mathbb{N}$, let T be an n-linear map from $A^{(n)}$ into X.

(i) T is *n-hyperlocal* if, for $a_0, \ldots, a_{n+1} \in A$,

$$a_0 a_1 = a_1 a_2 = \cdots = a_n a_{n+1} = 0 \quad \text{implies} \quad a_0 T(a_1, \ldots, a_n) a_{n+1} = 0.$$

For $n = 1$, 1-hyperlocal maps are simply called hyperlocal maps or hyperlocal operators.

(ii) T is a *local n-cocycle* if, for each $\tilde{a} = (a_1, \ldots, a_n) \in A^{(n)}$, there is an n-cocycle $T_{\tilde{a}}$ from $A^{(n)}$ into X such that $T(\tilde{a}) = T_{\tilde{a}}(\tilde{a})$. If, in addition, T is bounded, we say that T is a *bounded local n-cocycle*.

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It is easy to see that every (local) n-cocycle is a n-hyperlocal map. The following proposition states some sufficient conditions for a bounded n-hyperlocal map to be an n-cocycle. This is critical for us to obtain our result.

PROPOSITION 3.2. *Let A be a unital Banach algebra with unit 1 which satisfies the following two conditions:*

(i) *for every unital Banach A-bimodule X, a bounded operator $D\colon A \to X$ is a left multiplier if and only if $ba = 0$ implies $D(b)a = 0$;*

(ii) *for every unital Banach A-bimodule X, a bounded operator $D\colon A \to X$ is hyperlocal if and only if*

$$D(acb) - aD(cb) - D(ac)b + aD(c)b = 0,$$

for all $a, b, c \in A$.

Let X be a unital Banach A-bimodule, let $n \in \mathbb{N}$, and let $T \in B^n(A, X)$ be an n-hyperlocal map such that $T(a_1, \ldots, a_n) = 0$ if any one of $a_1, \ldots, a_n$ is 1. Then T is an n-cocycle.

PROOF. We prove the statement by induction on n. For $n = 1$, by hypothesis,

$$T(acb) - aT(cb) - T(ac)b + aT(c)b = 0,$$

for all $a, b, c \in A$. Since $T(1) = 0$, by putting $c = 1$ we get the result.

Now suppose that the result is true for $n = k$ ($k \geq 1$). We show that it is also true for $n = k + 1$. Let $T \in B^{k+1}(A, X)$ be $k + 1$-hyperlocal such that $T(a_1, \ldots, a_{k+1}) = 0$ if any one of $a_1, \ldots, a_{k+1}$ is 1. We first show that $\Lambda_k(T) \in B^k(A, B(A, X))$ is k-hyperlocal. Let $a_0, \ldots, a_{k+1} \in A$ be such that $a_0a_1 = \cdots = a_ka_{k+1} = 0$, and put

$$S = a_0 \star \Lambda_k(T)\,(a_1, \ldots, a_k) \star a_{k+1}.$$

Then $S\colon A \to X$ is a bounded operator. We claim that S satisfies the following condition:

$$bc = 0 \quad \text{implies} \quad S(b)c = 0. \tag{1}$$

Let $b, c \in A$ be such that $bc = 0$. Then

$$\begin{aligned} S(b)c &= [a_0 \star \Lambda_k(T)\,(a_1, \ldots, a_k) \star a_{k+1}]\,(b)c \\ &= a_0(\Lambda_k(T)\,(a_1, \ldots, a_k))\,(a_{k+1}b)c - a_0(\Lambda_k(T)\,(a_1, \ldots, a_k))\,(a_{k+1})bc \\ &= a_0T(a_1, \ldots, a_k, a_{k+1}b)c - a_0T(a_1, \ldots, a_k, a_{k+1})bc \\ &= a_0T(a_1, \ldots, a_k, a_{k+1}b)c. \end{aligned}$$

However, $a_0a_1 = \cdots = a_k(a_{k+1}b) = (a_{k+1}b)c = 0$, and T is $(k + 1)$-hyperlocal. Hence

$$a_0T(a_1, \ldots, a_k, a_{k+1}b)c = 0.$$

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Thus (1) holds, and so, by hypothesis, S is a left multiplier. Therefore, $S(a) = S(1)a$ for all $a \in A$. However,

$$\begin{aligned} S(1) &= [a_0 \star \Lambda_k(T)\,(a_1, \ldots, a_k) \star a_{k+1}]\,(1) \\ &= a_0(\Lambda_k(T)\,(a_1, \ldots, a_k))\,(a_{k+1}1) - a_0(\Lambda_k(T)\,(a_1, \ldots, a_k))\,(a_{k+1})1 \\ &= a_0 T(a_1, \ldots, a_k, a_{k+1}) - a_0 T(a_1, \ldots, a_k, a_{k+1}) \\ &= 0. \end{aligned}$$

Thus $S = 0$. Hence $\Lambda_k(T)$ is k-hyperlocal.

Let q be the natural quotient mapping from $B(A, X)$ into $B(A, X)/B_A(A, X)$, where $B_A(A, X)$ is the space of left multipliers. Since $\Lambda_k(T)$ is k-hyperlocal and q is an A-bimodule morphism with the $\star$ actions, $q \circ \Lambda_k(T)$ is k-hyperlocal. Moreover, because of the assumption on T, $q \circ \Lambda_k(T)\,(a_1, \ldots, a_k) = 0$ if any one of $a_1, \ldots, a_k$ is 1. On the other hand, for every $T \in B(A, X)$,

$$1 \star T = T \quad \text{and} \quad T \star 1 - T \in B_A(A, X).$$

Thus $B(A, X)/B_A(A, X)$ is a unital Banach A-bimodule. Therefore, by the inductive hypothesis, $q \circ \Lambda_k(T)$ is a k-cocycle. This means that, for $a_1, \ldots, a_{k+1} \in A$,

$$\Delta^k(q \circ \Lambda_k(T))\,(a_1, \ldots, a_{k+1}) = 0.$$

Hence, from the equation $\Lambda_{k+1} \circ \delta^{k+1} = \Delta^k \circ \Lambda_k$,

$$\Lambda_{k+1}(\delta^{k+1}(T))\,(a_1, \ldots, a_{k+1}) = \Delta^k(\Lambda_k(T))\,(a_1, \ldots, a_{k+1}) \in B_A(A, X).$$

Thus, for every $a_{k+2} \in A$,

$$\begin{aligned} \delta^{k+1}(T)\,(a_1, \ldots, a_{k+1}, a_{k+2}) &= [\Lambda_{k+1}(\delta^{k+1}(T))\,(a_1, \ldots, a_{k+1})]\,(a_{k+2}) \\ &= [\Lambda_{k+1}(\delta^{k+1}(T))\,(a_1, \ldots, a_{k+1})]\,(1)a_{k+2} \\ &= \delta^{k+1}(T)\,(a_1, \ldots, a_{k+1}, 1)a_{k+2}. \end{aligned}$$

On the other hand, by the assumption on T,

$$a_1 T(a_2, \ldots, a_{k+1}, 1) + \sum_{j=1}^{k} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots, a_{k+1}, 1) = 0.$$

Also,

$$\delta^{k+1}(T)\,(a_1, \ldots, a_k, a_{k+1}1) - \delta^{k+1}(T)\,(a_1, \ldots, a_k, a_{k+1})1 = 0.$$

Hence $\delta^{k+1}(T)\,(a_1, \ldots, a_{k+1}, 1) = 0$. Therefore, $\delta^{k+1}(T) = 0$, and so $T \in B^{k+1}(A, X)$. This completes the proof. □

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We are now ready to state the main result of this section. We recall that the unitization of A is $A^\sharp := A \oplus \mathbb{C}$ with multiplication

$$(a, \lambda)\,(b, \mu) = (ab + a\mu + b\lambda, \lambda\mu) \quad (a, b \in A, \lambda, \mu \in \mathbb{C}),$$

and norm

$$\|(a, \lambda)\| = \|a\| + |\lambda| \quad (a \in A, \lambda \in \mathbb{C}).$$

Thus $A^\sharp$ is a unital Banach algebra with unit (0, 1) which is denoted by 1 if there is no case of ambiguity.

THEOREM 3.3. *Let A be a Banach algebra such that $A^\sharp$ satisfies conditions* (i) *and* (ii) *of Proposition 3.2. Then, for any Banach A-bimodule X and $n \in \mathbb{N}$, every bounded local n-cocycle T from $A^{(n)}$ into X is an n-cocycle.*

PROOF. We can extend X to a Banach $A^\sharp$-bimodule by defining $1x = x1 = x$.

Let $\sigma : L^n(A, X) \to L^n(A^\sharp, X)$ be a linear map defined by

$$\sigma(T)\,(a_1 + \lambda_1, \ldots, a_n + \lambda_n) = T(a_1, \ldots, a_n),$$

for $a_1, \ldots, a_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. It is straightforward to check that $T \in L^n(A, X)$ is an n-cocycle if and only if $\sigma(T)$ is an n-cocycle. Now let $T \in B^n(A, X)$ be a bounded local n-cocycle, and let $(a_1 + \lambda_1, \ldots, a_n + \lambda_n) \in A^{\sharp(n)}$. By the assumption on T, for $\tilde{a} = (a_1, \ldots, a_n) \in A^{(n)}$, there is an n-cocycle $T_{\tilde{a}}$ from $A^{(n)}$ into X such that $T(a_1, \ldots, a_n) = T_{\tilde{a}}(a_1, \ldots, a_n)$. Thus

$$\begin{aligned}\sigma(T)\,(a_1 + \lambda_1, \ldots, a_n + \lambda_n) &= T(a_1, \ldots, a_n)\\ &= T_{\tilde{a}}(a_1, \ldots, a_n)\\ &= \sigma(T_{\tilde{a}})\,(a_1 + \lambda_1, \ldots, a_n + \lambda_n).\end{aligned}$$

Hence $\sigma(T)$ is a bounded local n-cocycle, and so it is a bounded n-hyperlocal map. Moreover,

$$\sigma(T)\,(a_1, \ldots, a_n) = 0,$$

if any one of $a_1, \ldots, a_n$ is 1. Thus, by Proposition 3.2, $\sigma(T)$ is an n-cocycle. Therefore, T is an n-cocycle. □

4. Hyper-Tauberian algebras

Throughout this section, A and B are commutative regular semisimple Banach algebras with the carrier spaces Φ_A and Φ_B, respectively. Let I be a closed ideal in A. The *hull* of I is

$$\{t \in \Phi_A \mid a(t) = 0 \text{ for all } a \in I\},$$

and it is denoted by $h(I)$.

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Let X and Y be Banach left (right) A-modules. For $x \in X$, the annihilator $\mathrm{Ann}_A(x)$ of x is

$$\mathrm{Ann}_A(x) = \{a \in A \mid ax = 0 \ (xa = 0)\}.$$

$\mathrm{Ann}_A(x)$ is clearly a closed ideal in A. The hull of $\mathrm{Ann}_A(x)$ is called the *support* of x (in Φ_A), denoted by $\mathrm{supp}_A\, x$. We will write 'supp x' instead of 'supp$_A$ x' whenever there is no risk of ambiguity. By [7, Lemma 2.1], $t \notin \mathrm{supp}\, x$ if and only if there is a compact neighborhood V of t in Φ_A such that, for every element $a \in A$, if $\mathrm{supp}\, a \subseteq V$, then $ax = 0$ $(xa = 0)$. In the case $X = A$ where we regard A as a Banach (left or right) A-module on itself, the support of an element $a \in A$ coincides with the usual definition of supp a, namely $\mathrm{cl}\{t \in \Phi_A \mid a(t) \neq 0\}$.

An operator $T : X \to Y$ is *local* with respect to the left (right) A-module action if $\mathrm{supp}\, T(x) \subseteq \mathrm{supp}\, x$ for all $x \in X$. We recall from [9, Definition 4] that A is a *hyper-Tauberian algebra* if every bounded local operator from A into A^* is a multiplier. If A is unital, then the definition of hyper-Tauberian algebras coincides with the definition of (SD) algebras introduced in [11].

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let X be both a Banach left $\mathfrak{A}$-module and a Banach right $\mathfrak{B}$-module such that, for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and $x \in X$, $a(xb) = (ax)b$. Then we write $X \in \mathfrak{A}$-mod-$\mathfrak{B}$. If, in addition, X is essential both as a Banach left $\mathfrak{A}$-module and Banach right $\mathfrak{B}$-module, then we write $X \in$ ess. $\mathfrak{A}$-mod-$\mathfrak{B}$.

DEFINITION 4.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $X, Y \in \mathfrak{A}$-mod-$\mathfrak{B}$. An operator $D : X \to Y$ is *hyperlocal* with respect to $\mathfrak{A}$-mod-$\mathfrak{B}$ actions if, for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and $x \in X$,

$$ax = xb = 0 \quad \text{implies} \quad aD(x)b = 0.$$

The preceding definition was introduced in [10] in order to extend the concept of locality for operators in the non-commutative setting (see also [2]). It is easily seen that, for commutative C^*-algebras, this locality condition coincides with the usual one. However, as is shown in Remark 4.4, in general the concept of being hyperlocal is weaker than the concept of being local.

In the following proposition, we use the properties of hyper-Tauberian algebras to characterize bounded hyperlocal operators that are defined from essential modules over these algebras.

PROPOSITION 4.2. *Let A and B be hyper-Tauberian algebras. Then, for all $X, Z \in$ ess. $A-mod-B$ and $Y \in$ ess. $B-mod-A$:*

(i) *a bounded operator $D : X \to Y^*$ is hyperlocal if and only if*

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0,$$

for all $a \in A$, $b \in B$ and $x \in X$;

(ii) *if A and B have bounded approximate identities, then the result in* (i) *is also true for all bounded hyperlocal operators from X into Z.*

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PROOF. (i) First assume that $Y = B\widehat{\otimes}A$, where the B-mod-A actions on $B\widehat{\otimes}A$ are specified by

$$d(b \otimes a) = db \otimes a, \quad (b \otimes a)c = b \otimes ac \quad (a, c \in A, b, d \in B).$$

Let $D: X \to (B\widehat{\otimes}A)^*$ be a bounded hyperlocal operator, and let $x \in X$ and $a \in A$. Define the bounded operator $\widetilde{D}: B \to (B\widehat{\otimes}A)^*$ by

$$\widetilde{D}(b) = D(axb) - aD(xb) \quad (b \in B).$$

We claim that D is local with respect to the right B-module action. Let $b \in B$ and $t \notin \operatorname{supp}_B b$. There is a compact neighborhood V of t (in Φ_B) such that $V \cap \operatorname{supp}_B b = \emptyset$. Let $c \in B$ with $\operatorname{supp}_B c \subseteq V$. By the regularity of B, there is an $e \in B$ such that $e = 1$ on V and $e = 0$ on $\operatorname{supp}_B b$. So

$$ec = c \quad \text{and} \quad eb = 0. \tag{1}$$

Put

$$K_0(V) = \overline{\operatorname{span}}\{n \otimes m \mid m \in A,\ n \in B \text{ and } n = 0 \text{ on } \Phi_B \setminus V\}.$$

Since $e = 1$ on V, for all $\theta \in (B\widehat{\otimes}A)^*$,

$$\theta e - \theta = 0 \quad \text{on } K_0(V). \tag{2}$$

Let $z \in X$, and define the bounded operator $T: A \to (B\widehat{\otimes}A)^*/K_0(V)^\perp$ by

$$T(u) = D(uzb) + K_0(V)^\perp \quad (u \in A).$$

Let $h \in A$ such that $hu = 0$. Then, from (1), $huzb = 0 = uzbe$. Since D is hyperlocal, $hD(uzb)e = 0$. Hence, from (2),

$$\begin{aligned} hT(u) &= hD(uzb) + K_0(V)^\perp \\ &= hD(uzb)e + K_0(V)^\perp \\ &= 0. \end{aligned}$$

In particular, T is local with respect to the left A-module action. Since

$$(B\widehat{\otimes}A)^*/K_0(V)^\perp \cong K_0(V)^*,$$

and $K_0(V)$ is an essential Banach right A-module, from [9, Proposition 3], it follows that T is a right multiplier. Therefore, $T(uv) = uT(v)$ for all $u, v \in A$. Hence, if we put $u = a$, then $D(avzb) - aD(vzb) \in K_0(V)^\perp$. Thus, from essentiality of X, we have

$$\widetilde{D}(b) = D(axb) - aD(xb) \in K_0(V)^\perp.$$

Therefore, $\widetilde{D}(b)c = 0$, since $\operatorname{supp}_B c \in V$. This means that $t \notin \operatorname{supp}_B \widetilde{D}(b)$, and so $\widetilde{D}$ is a bounded local operator. Hence, from [9, Proposition 3], $\widetilde{D}$ is a left multiplier. Thus $\widetilde{D}(bd) = \widetilde{D}(b)d$ for all $b, d \in B$. Therefore,

$$D(axbd) - aD(xbd) = D(axb)d - aD(xb)d.$$

The final result follows from the essentiality of X.

Now consider the general case. Let $y \in Y$ and define $S_y \colon Y^* \to (B \widehat{\otimes} A)^*$ by

$$\langle S_y(y^*), b \otimes a\rangle = \langle y^*, bya\rangle \quad (a \in A, b \in B, y^* \in Y^*).$$

It is easy to see that S_y is both a bounded left A-module morphism and a bounded right B-module morphism, and so $S_y \circ D$ is a bounded hyperlocal operator from X into $(B \widehat{\otimes} A)^*$. Thus, for all $a \in A$, $b \in B$, $x \in X$ and $y \in Y$,

$$S_y[D(axb) - aD(xb) - D(ax)b + aD(x)b] = 0.$$

Hence, for all $c \in A$ and $d \in B$,

$$\langle D(axb) - aD(xb) - D(ax)b + aD(x)b, dyc\rangle = 0.$$

The final result follows from the essentiality of Y.

(ii) Let $\{e_\alpha\}_{\alpha \in \Lambda}$ and $\{f_\beta\}_{\beta \in \Omega}$ be bounded approximate identities for A and B, respectively. With a similar argument to that in (i) (by replacing Z with Z^{**}), we can show that

$$c[D(axb) - aD(xb) - D(ax)b + aD(x)b]\, d = 0, \tag{3}$$

for all $a, c \in A$, $b, d \in B$ and $x \in X$. On the other hand, since A and B have bounded approximate identities, by Cohen's factorization theorem [1, Corollary 2.9.26], there exist $e \in A$, $f \in B$ and $z \in Z$ such that

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = ezf.$$

So we have the final result if we put $c = e_\alpha$ and $d = f_\beta$ in (3), and let $\alpha, \beta \to \infty$. □

THEOREM 4.3. *Let A be a hyper-Tauberian algebra, and let X be a Banach A-bimodule. Then, for $n \in \mathbb{N}$, every bounded local n-cocycle T from $A^{(n)}$ into X is an n-cocycle.*

PROOF. Let $A^\sharp$ be the unitalization of A. By [9, Corollary 10], $A^\sharp$ is hyper-Tauberian. Therefore, by [9, Proposition 3] and Proposition 4.2, $A^\sharp$ satisfies the conditions (i) and (ii) of Proposition 3.2. Hence the result follows from Theorem 3.3. □

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REMARK 4.4. Let $\mathbb{T}$ be the unit circle, and let $A := A(\mathbb{T})$ be the Fourier algebra on $\mathbb{T}$. It is shown in [9, Remark 24(ii)] that there is a closed ideal I in A such that I is weakly amenable but I is not hyper-Tauberian. Hence there are bounded local operators from I into I^* which are not multipliers. However, this is not the case if we consider bounded hyperlocal operators. To see this, let $D\colon I \to I^*$ be a bounded hyperlocal operator. First, we show that D is hyperlocal with respect to A-bimodule actions. Let $a, b \in A$ and $c \in I$ such that $ac = cb = 0$. Take $e, f \in I$. Then $ea, bf \in I$ and $(ea)c = c(bf) = 0$. Thus $eaD(c)bf = 0$. Hence $aD(c)b = 0$ on I^3. However, I is weakly amenable, and so, by [1, Theorem 2.8.69(ii)], I^2 is dense in I. Hence $aD(c)b = 0$. Therefore, D is hyperlocal with respect to A-bimodule actions. On the other hand, A is a hyper-Tauberian algebra [9, Proposition 18]. Hence, from Proposition 4.2, for all $a, b \in A$ and $c \in I$,

$$D(abc) - aD(bc) - D(ab)c + aD(b)c = 0. \tag{4}$$

Define the bounded operator $\mathcal{D}\colon I \to \mathcal{B}_I(I, I^*)$ by

$$\mathcal{D}(a)\,(b) = D(ab) - aD(b) \quad (a, b \in I).$$

From (4), it is easy to verify that $\mathcal{D}$ is well defined. Moreover, upon setting

$$\langle a \cdot S, b\rangle = \langle S \cdot a, b\rangle = \langle S, ab\rangle,$$

the space $\mathcal{B}_I(I, I^*)$ becomes a symmetric Banach I-module and $\mathcal{D}$ becomes a bounded derivation from I into $\mathcal{B}_I(I, I^*)$. Hence $\mathcal{D} = 0$ since I is weakly amenable. Thus D is a multiplier.

5. C^*-algebras

It follows from the works of Johnson that $C_0(\mathbb{R})$ is a hyper-Tauberian algebra [3, Proposition 3.1]. One the other hand, Shulman showed that every unital commutative C^*-algebra is hyper-Tauberian [11]. We extend these results by showing that $C_0(\Omega)$ is hyper-Tauberian for every locally compact topological space Ω. For the sake of completeness, we first prove it for the case when Ω is compact.

THEOREM 5.1. *Let Ω be a locally compact topological space. Then $C_0(\Omega)$ is a hyper-Tauberian algebra.*

PROOF. First consider the case when Ω is compact. Let $T\colon C(\Omega) \to C(\Omega)^*$ be a bounded local operator. First we show that T satisfies the following condition:

$$ab = 0 \quad \text{implies} \quad aT(b) = 0. \tag{$\star$}$$

Let $a, b \in C(\Omega)$ with $ab = 0$. So if we put $E = \operatorname{supp} b$, then $a = 0$ on E. Since E is a closed subset of Ω, E is a set of synthesis (see [1, Definition 4.1.12 and Theorem 4.2.1]). Thus there is a sequence $\{a_n\}$ in $C(\Omega)$ such that, for each

n, $\operatorname{supp} a_n$ is compact and disjoint from E, and $a_n \to a$ as $n \to \infty$. On the other hand, since T is local and $\operatorname{supp} a_n$ is disjoint from E,

$$\begin{aligned}\operatorname{supp} a_n \cap \operatorname{supp} T(b) &\subseteq \operatorname{supp} a_n \cap \operatorname{supp} b \\ &= \operatorname{supp} a_n \cap E \\ &= \emptyset.\end{aligned}$$

Therefore, since $\operatorname{supp} a_n$ is compact, $a_n T(b) = 0$. Hence, by letting $n \to \infty$, we have $aT(b) = 0$. This proves ($\star$).

Now let $a \in C(\Omega)$ be a self-adjoint element, and let $A(a)$ be the C^*-subalgebra of $C(\Omega)$ generated by $\{a, 1\}$. It is well known that there is a compact subset K of $\mathbb{R}$ such that $A(a)$ is isometrically isomorphic to $C(K)$. In particular, $C(\Omega)$ is an essential and symmetric Banach $C(K)$-module. Let $d \in C(\Omega)$ and $c \in C(K)$ with $cd = 0$. Then, since $c \in A(a)$ and T satisfies condition ($\star$), $cT(d) = 0$. Hence $\operatorname{Ann}_{C(K)} d \subseteq \operatorname{Ann}_{C(K)} T(d)$, and so $\operatorname{supp}_{C(K)} T(d) \subseteq \operatorname{supp}_{C(K)} d$. Therefore, T is local with respect to $C(K)$-module actions. On the other hand, the restriction map $f \mapsto f|_K$ is a bounded algebra homomorphism from $C_0(\mathbb{R})$ onto $C(K)$. Hence, from [9, Theorem 12], $C(K)$ is hyper-Tauberian, and so, from [9, Proposition 3], T is a $C(K)$-module morphism. Hence, for each self-adjoint $b \in C(\Omega)$, $T(ab) = aT(b)$. The final result follows since $C(\Omega)$ is the linear span of its self-adjoint elements.

We now consider the general case. Let Ω be a locally compact topological space, and let $\Omega \cup \{\infty\}$ be its one-point compactification. Then, from the first case, $C(\Omega \cup \{\infty\})$ is hyper-Tauberian. On the other hand,

$$C_0(\Omega) = \{a \in C(\Omega \cup \{\infty\}) \mid a(\infty) = 0\},$$

and $\{\infty\}$ is a set of synthesis for $C(\Omega \cup \{\infty\})$. Thus, from [9, Theorem 7(ii)], $C_0(\Omega)$ is hyper-Tauberian. □

We are now ready to obtain our results for C^*-algebras. We start with the following critical theorem which characterizes bounded hyperlocal operators defined over essential modules of a C^*-algebra. This was partially obtained in [10, Theorem 2.2] and [2, Theorem 2.17].

Theorem 5.2. *Let A be a C^*-algebra, let X be an essential Banach A-bimodule, and let Y be an essential or the dual of an essential Banach A-bimodule. Then a bounded operator $D\colon X \to Y$ is hyperlocal if and only if*

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0,$$

for all $a, b \in A$ and $x \in X$.

Proof. First suppose that $Y = (A\widehat{\otimes}A)^*$. Let $D\colon X \to (A\widehat{\otimes}A)^*$ be a bounded hyperlocal operator, and let $A^\sharp$ be the unitalization of A [1, Definition 3.2.1]. We show that D is hyperlocal with respect to $A^\sharp$-module actions. Let $u, v \in A^\sharp$ and $x \in X$ such that $ux = xv = 0$. So, for all $a, b \in A$, $(au)x = x(vb) = 0$.

Thus $auD(x)vb = 0$. Hence $uD(x)v = 0$ on $A^2 \otimes A^2$ which is dense in $A\widehat{\otimes}A$. So $uD(x)v = 0$. Now let c and d be self-adjoint elements in A, and let $A(c)$ and $A(d)$ be the commutative C^*-subalgebras of $A^\sharp$ generated by $\{c, 1\}$ and $\{d, 1\}$, respectively. Clearly $D\colon X \to (A\widehat{\otimes}A)^*$ is hyperlocal with respect to $A(c)$-mod-$A(d)$ actions. Thus, from Theorem 5.1, for every $x \in X$,

$$D(cxd) - cD(xd) - D(cx)d + cD(x)d = 0.$$

The final result follows since A is the linear span of its self-adjoint elements. The general case follows from a similar argument made in the proof of Proposition 4.2. □

REMARK 5.3. In the preceding theorem, if we replace the locality condition that we used in the definition of a hyperlocal operator with the condition

$$ax = 0 \quad \text{implies} \quad aD(x) = 0,$$

then, by a similar argument and using [9, Proposition 3] instead of Proposition 4.2, we can show that D is a left A-module morphism. We can also have a similar result regarding bounded right A-module morphisms.

Let A be a C^*-algebra which is not unital. We can see that, in general, our unitization, $A^\sharp = A \oplus^1 \mathbb{C}$, is not a C^*-algebra (as the norm dose not satisfy the correct condition). However, there is an equivalent norm on $A^\sharp$ that turns it into a C^*-algebra (see [1, Definition 3.2.1]). Thus we can state the our main result.

THEOREM 5.4. *Let A be a C^*-algebra, and let X be a Banach A-bimodule. Then, for $n \in \mathbb{N}$, every bounded local n-cocycle T from $A^{(n)}$ into X is an n-cocycle.*

PROOF. The result follows from Theorem 5.2, Remark 5.3, and Theorem 3.3. □

6. Hyperlocal operators and amenable C^*-algebras

In this final section, we present a characterization of cohomological properties of C^*-algebras, i.e. amenability and weak amenability, with respect to hyperlocal operators.

THEOREM 6.1. *Let A be a C^*-algebra, and let X be an essential Banach A-bimodule. Then a bounded operator $D\colon A \to X^*$ is hyperlocal if and only if there is a derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. In particular, D is a derivation if and only if weak* $\lim_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ in A.*

PROOF. It is easy to see that all derivations and multipliers are hyperlocal. On the other hand, let $D\colon A \to X^*$ be a bounded hyperlocal operator. By Theorem 5.2, for all $a, b, c \in A$,

$$D(acb) - D(ac)b - aD(cb) + aD(c)b = 0.$$

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By putting $c = e_\alpha$ and letting $\alpha \to \infty$ we obtain

$$D(ab) - D(a)b - aD(b) + \lim_{\alpha\to\infty} aD(e_\alpha)b = 0.$$

Since $\{D(e_\alpha)\}$ is bounded, there is an $x^* \in X^*$ and a subnet $\{D(e_{\alpha_i})\}$ such that $D(e_{\alpha_i}) \to x^*$ in the weak* topology. So $D(ab) - D(a)b - aD(b) + ax^*b = 0$. Define $T\colon A \to X^*$ by $T(a) = ax^*$ and put $\mathcal{D} = D - T$. It is straightforward to check that T is a right multiplier and $\mathcal{D}$ is a derivation. Finally, D is a derivation if and only if T is zero. However, it is easy to verify that T is zero if and only if weak* $\lim_{\alpha\to\infty} D(e_\alpha) = 0$. □

We recall that a Banach algebra A is amenable if, for any Banach A-bimodule X, every bounded derivation $D\colon A \to X^*$ is inner.

COROLLARY 6.2. *Let A be a C^*-algebra. Then A is amenable if and only if for any essential Banach A-bimodule X and every bounded hyperlocal operator $D\colon A \to X^*$, there exist x^*, $y^* \in X^*$ such that $D(a) = ax^* - y^*a$ $(a \in A)$.*

PROOF. Let A be amenable, let X be an essential Banach A-bimodule, and let $D\colon A \to X^*$ be a bounded hyperlocal operator. By Theorem 6.1, there is a derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. Since A is amenable, there exist y^* and z^* in X^* such that $\mathcal{D}(a) = ay^* - y^*a$ and $T(a) = az^*$ for all $a \in A$. Thus $D(a) = a(y^* + z^*) - y^*a$. The converse follows immediately from Theorem 6.1 and [1, Corollary 2.9.27]. □

COROLLARY 6.3. *Let A be a C^*-algebra. Then, for every bounded hyperlocal operator $D\colon A \to A^*$, there exist x^*, $y^* \in A^*$ such that $D(a) = ax^* - y^*a$ $(a \in A)$.*

PROOF. The result follows from a similar argument to that made in the proof of the preceding corollary together with the fact that every C^*-algebra is weakly amenable [1, Theorem 5.6.77]. □

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References

[1] H. G. Dales, *Banach algebras and automatic continuity* (Oxford University Press, New York, 2000).

[2] D. Hadwin and J. Li, 'Local derivations and local automorphism', *J. Math. Anal. Appl.* **290** (2004), 702–714.

https://doi.org/10.1017/S1446788708000049 Published online by Cambridge University Press

[3] B. E. Johnson, 'Local derivations on C^*-algebras are derivations', *Trans. Amer. Math. Soc.* **353** (2000), 313–325.
[4] R. V. Kadison, 'Local derivations', *J. Algebra* **130** (1990), 494–509.
[5] R. V. Kadison and J. R. Ringrose, 'Cohomology of operator algebras. I, Type 1 Von Neumann algebras', *Acta Math.* **126** (1971), 227–243.
[6] D. R. Larson, 'Reflexivity, algebraic reflexivity and linear interpolation', *Amer. J. Math.* **110** (1988), 283–299.
[7] E. Samei, 'Bounded and completely bounded local derivations from certain commutative semisimple Banach algebras', *Proc. Amer. Math. Soc.* **133** (2005), 229–238.
[8] E. Samei, 'Approximately local derivations', *J. London Math. Soc.* (2) **71**(3) (2005), 759–778.
[9] E. Samei, 'Hyper-Tauberian algebras and weak amenability of Figà-Talamanca-Herz algebras', *J. Funct. Anal.* **231**(1) (2006), 195–220.
[10] J. Schweizer, 'An analogue of Peetr's theorem in non-commutative topology', *Quart. J. Math.* **52** (2001), 499–506.
[11] V. S. Shulman, 'Spectral synthesis and the Fuglede–Putnam–Rosenblum theorem', *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* **54** (1990), 25–36 (in Russian) (Engl. Transl. *J. Soviet Math.* **58**(4) (1992), 312–318).

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