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# A note on regular modules

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Kaplansky's observation, namely, a commutative ring $R$ is (von Neumann) regular if and only if each simple $R$-module is injective, is generalized to projective modules over a commutative ring.

It is a well-known observation due to Kaplansky that a commutative ring $R$ is (von Neumann) regular if and only if every simple epi-image of $R$ is injective. One may ask whether this can be extended to projective modules over arbitrary commutative rings. To be precise, call a module $P$ over a ring $R$

(i) a *regular module* if, for each $p \in P$, there is $f \in \hom_R(P, R)$ such that $p = p(fp)$ (Zelmanowitz [10]) and

(ii) a *V-module* if each simple epi-image of $P$, when it exists, is injective.

Then the question is: is a projective module $P$ over a commutative ring $R$ a regular module if and only if it is a $V$-module? In [9] Ware proved the implication 'only if' for all $P$ and the implication 'if' for finitely generated $P$, leaving unsettled the validity of 'if' for arbitrary $P$. This problem was solved by S. Alamelu (private communication). In this note we formulate this problem in a more general setting and prove the validity of 'if' for all $P$. We call a module $P$ over a ring $R$

(i) a *weakly regular module*, if for each $p \in P$, there are $f_1, \ldots, f_n \in \hom_R(P, R)$ and $r_1, \ldots, r_n \in R$ such that

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$$p = p \sum_1^n f_i(pr_i) \text{ , and}$$

(ii) an *HV-module* if every simple image of every cyclic submodule of $P$ is injective.

We show that any projective $HV$-module is weakly regular. For commutative $R$ , this yields the desired result; namely, any projective $V$-module is regular.

Throughout this paper $R$ will denote an associative ring with identity and all modules considered will be unitary right $R$-modules. Homomorphisms will be written opposite to scalars. For any module $M$ , $M^*$ will denote $\hom_R(M, R)$ , and for any $m \in M$ ,

$$T_m = \left\{\sum_1^n f_i(mr_i) ; \; f_i \in M^*, \; r_i \in R\right\} .$$

Note that $T_m$ is a two-sided ideal of $R$ .

**DEFINITION 1.** An $R$-module $M$ is said to be

(i) *regular*, if, for every $m \in M$ , there is $f \in M^*$ with $m = m(fm)$ (*cf.* [*10*]);

(ii) *weakly regular* if, for every $m \in M$ , $m \in mT_m$ ;

(iii) a *V-module* if each simple epi-image of $M$ (when it exists) is injective;

(iv) an *HV-module* if each cyclic submodule of $M$ is a $V$-module.

**EXAMPLES.** (i) If $R$ is a simple ring with identity and $I$ is a right ideal of $R$ , then $I$ is a weakly regular module. For, if $a \in I$ , then $a = a \cdot 1 = a \sum r_i a s_i$ $(r_i, s_i \in R)$ and clearly the $r_i$ can be considered as elements of $I^*$ . More generally, any right ideal of a weakly regular ring is a weakly regular module. ($R$ is called a *weakly regular ring* if $a \in aRaR$ for every $a \in R$ . For information on these rings, see [*6*].)

(ii) Any regular module is weakly regular. The converse is not true.

For instance, if $R$ is a simple integral domain which is not a division ring (for an example of such a ring see [5], p. 211), then any right ideal of $R$ is weakly regular but not regular.

(iii) A ring $R$ has been called a $V$-ring, if each simple right $R$-module is injective (see [3], [4] for properties and examples of such rings). It is clear that any module over a $V$-ring is an $HV$-module.

(iv) Any $HV$-module is a $V$-module. The converse is not true (see Remark 3 (ii) below).

PROPOSITION 2. *Any projective $HV$-module is weakly regular.*

Proof. Let $P$ be a projective $HV$-module and $p \in P$. If $p \notin pT_p$, choose a submodule $X$ of $P$ maximal with respect to $p \notin X$, $pT_p \subset X$. Then $Y = (pR+X)/X \simeq pR/pR \cap X$ is a simple $R$-module and as it is an epi-image of $pR \subset P$, $Y$ is injective, by hypothesis. But $Y$ is an essential submodule of $P/X$ so that $Y = P/X$. Now consider the diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow g \\ R & \xrightarrow{f} & P/X = (pR+X)/X , \end{array}$$

where $g$ is the projection map and $f$ is the canonical map obtained by using the fact that $P/X$ is simple. As $P$ is projective, this diagram yields a map $h : P \to R$ with $fh = g$. In particular,

$$p + X = gp = (fh)p = (f1)(hp) ,$$

and if $f1 = pr + X$, then this gives $p - (pr)(hp) \in X$. But $(pr)(hp) \in pT_p$. Thus $p \in X$, a contradiction.

REMARKS 3. (i) A projective weakly regular module need not be an $HV$-module. This follows from the fact that there are (von Neumann) regular rings which are not $V$-rings (see [4]).

(ii) A projective $V$-module need not be weakly regular. To see this, let $R$ be a non-semi-simple artinian, right hereditary ring such that the injective hull $E$ of the right $R$-module $R$ is projective. Then $R$ is a semi-primary QF-3 ring and $E$ is projective (*cf.* Colby and Rutter [2]). As $R$ is hereditary, $E$ is a projective $V$-module, and if it is a weakly regular module, then $R$ (being a submodule of $E$) will be a weakly regular module over itself. In other words, $R$ will be a weakly regular ring and consequently its Jacobson radical will be equal to zero [6]. Since $R$ is already semi-primary, this will imply that $R$ is semi-simple artinian, contradicting our assumption that $R$ is not so. Thus $E$ is not a weakly regular module. By Proposition 2, it also follows that $E$ is not an $HV$-module.

(iii) The above proposition generalizes to projective modules the fact that any $V$-ring is a weakly regular ring; proved in [7].

(iv) A more detailed study of weakly regular modules will be presented in a subsequent paper.

**THEOREM 4.** *If $R$ is a commutative ring, then the following are equivalent on any projective $R$-module $P$ :*

*(i) $P$ is weakly regular;*

*(ii) $P$ is regular;*

*(iii) $P$ is a $V$-module;*

*(iv) $P$ is an $HV$-module.*

Proof. We give a cyclic proof.

*(i) ⇒ (ii).* Let $p \in P$. Then $p = p \sum f_i(pr_i)$ where $f_i \in P^*$ and $r_i \in R$. But

$$\sum f_i(pr_i) = \sum (f_ip)r_i = \sum r_i(f_ip) = \sum (r_if_i)p = \left(\sum r_if_i\right)p .$$

Thus $p = p(fp)$ where $f = \sum r_if_i \in P^*$ and hence $P$ is regular.

*(ii) ⇒ (iii)* has been proved by Ware ([9], Proposition 2.5).

*(iii) ⇒ (iv).* Let $p \in P$. As $P$ is projective, $p = \sum p_i(f_ip)$ for $p_i \in P$, $f_i \in P^*$. Let $m_i = f_ip$ and

$$T = \left\{\sum f(\bar{p}),\ f \in P^* \text{ and } \bar{p} \in P\right\} = \text{trace ideal of } P \text{ in } R .$$

It is well known (and easily proved by considering localizations of $P$ and $T$ at maximal ideals) that $R/T$ is a flat module (see, for instance, [8]). Hence, by Proposition 2.2 of [1], there is a morphism $f$ from $R$ to $T$

such that $f(m_i) = m_i$ for each $i$. Writing $f(1) = t$, we have $m_i = m_i t$ for each $i$ and hence $p = pt$. Hence, if $t = \sum_1^n g_i x_i$ for some $g_i \in P^*$, $x_i \in P$, then $(q_i) \mapsto p\left(\sum g_i q_i\right)$ defines an epimorphism from $P \oplus \ldots \oplus P$ ($n$ copies) to $pR$. It follows from this, that any simple epi-image of $pR$ is also an epi-image of $P$ and hence is injective. Thus $P$ is an $HV$-module.

*(iv)* $\Rightarrow$ *(i)* follows from Proposition 2 above.

This completes the proof.

Note added in proof (11 November 1974). It seems worthwhile noting explicitly the following result, which is a consequence of the proofs given in this paper.

*Let $R$ be a (not necessarily commutative) ring with identity, and let $P$ be a projective $V$-module with trace ideal $T$. Then the following are equivalent:*

*(i)' $R/T$ is left $R$-flat;*

*(ii)' $P$ is an $HV$-module;*

*(iii)' $P$ is a weakly regular module.*

The proof of *(iii)* $\Rightarrow$ *(iv)* of Theorem 4 actually establishes *(i)'* $\Rightarrow$ *(ii)'* here. *(ii)'* $\Rightarrow$ *(iii)'* is Proposition 2. *(iii)'* $\Rightarrow$ *(i)'* follows by using the dual basis property of projective modules. Note that if $R$ is commutative, then *(i)'* holds for all projectives $P$.

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