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TRANSITIVE ORIENTATION OF GRAPHS AND IDENTIFICATION OF PERMUTATION GRAPHS

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1. Introduction. The graphs considered in this paper are assumed to be finite, with no edge joining a vertex to itself and with no two distinct edges joining the same pair of vertices. An undirected graph will be denoted by G or (V, E), where V is the set of vertices and E is the set of edges. An edge joining the vertices $i, j \in V$ will be denoted by the unordered pair (i, j).

An *orientation* of $G = (V, E)$ is an assignment of a unique direction $i \rightarrow j$ or $j \rightarrow i$ to every edge $(i, j) \in E$. The resulting directed image of G will be denoted by $G^{\rightarrow}$ or $(V, E^{\rightarrow})$, where $E^{\rightarrow}$ is now a set of ordered pairs $E^{\rightarrow} = \{[i, j] | (i, j) \in E \text{ and } i \rightarrow j\}$. Notice the difference in notation (brackets versus parentheses) for ordered and unordered pairs. Also, if $[i, j] \in E^{\rightarrow}$, then $[j, i] \notin E^{\rightarrow}$.

A graph G with n vertices is called a *permutation graph* if there exist:

(a) A labeling $v_1, v_2, \ldots, v_n$ of the vertices of G, and

(b) A permutation $P = [P(1), P(2), \ldots, P(n)]$ of the integers $1, 2, \ldots, n$ under which v_i and v_j are joined by an edge in G if and only if

$$(1) \qquad (i - j)[P^{-1}(i) - P^{-1}(j)] < 0,$$

where $P^{-1}(i)$ is the integer which P maps onto i.

In a recent paper [**1**] it was shown that permutation graphs, due to their special structural properties, are very helpful in modeling and solving various problems such as determining intersection-free layouts for connection boards [**4**] or optimal schedules for reallocation of memory space in a computer [**3**].

It was also shown that the family of permutation graphs is embedded in a broader family of so-called *transitively orientable graphs* whose basic property is that they admit an orientation under which the relation $\rightarrow$ on the set of vertices is transitive. That is, a graph (V, E) is transitively orientable (in short, TRO) if and only if there exists a directed image $(V, E^{\rightarrow})$ of (V, E) such that for $i, j, k \in V$,

$$(2) \qquad [i, j] \in E^{\rightarrow} \quad \text{and} \quad [j, k] \in E^{\rightarrow} \qquad \text{imply } [i, k] \in E^{\rightarrow}.$$

An equivalent definition of TRO graphs is that they admit a labeling $v_1, v_2, \ldots, v_n$ of their vertices under which for $i < j < k$ the existence of an edge joining v_i to v_j and one joining v_j to v_k implies the existence of an edge joining v_i to v_k.

This property of transitively orientable graphs, which is also shared by permutation graphs, led to the derivation of highly efficient procedures for

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finding maximal cliques and minimal chromatic decompositions of TRO graphs. These procedures require, however, an *a priori* knowledge of a valid labeling of the vertices. In some of the mentioned applications, such a labeling is evident from the nature of the problem. To determine maximal cliques or minimal chromatic decompositions of an arbitrary given graph is still a formidable problem in graph theory. The efficiency by which this can be done for TRO and permutation graphs motivated us to look for algorithms that will decide whether a given graph belongs to one of these families.

In § 2 of this paper we describe a simple iterative procedure for testing transitive orientability. In each iteration a certain subset of edges of the given graph is processed and then removed, leaving a graph with fewer edges for the next iteration. Whenever the graph G under test is TRO, the algorithm will produce a transitive orientation of G. A simple method of obtaining a valid labeling of the vertices is also presented.

In § 3 we show that a graph G is a permutation graph if and only if both G and its complement G_c are TRO. (G_c is a graph with the same vertices as G and two vertices are joined by an edge in G_c if and only if there is no such edge in G.) Thus, by applying the TRO test to both G and G_c, we can decide whether either of them is a permutation graph.

In § 4 we prove two theorems that establish the validity of the TRO algorithm described in § 2.

Having completed the preparation of this report, we became aware of an earlier paper by Gilmore and Hoffman [**2**] in which a characterization of TRO graphs (under the name of *comparability graphs*) and an algorithm for transitively orienting them are presented. The algorithm described in this paper is believed to be more efficient as it employs only one orientation rule and progressively reduces the size of the graph under test with each iteration. Naturally, the difference between the two algorithms is also reflected in their completely different proofs as well as in a variety of other secondary results pertaining to the structure of TRO graphs.

2. An algorithm for transitive orientation. The algorithm described in this section decides whether a given graph is TRO and in all affirmative cases produces a transitively directed (in short, TRD) image of the graph under test. The key device of the algorithm is an orientation rule which applies to a pair of adjacent edges (i, j) and (j, k) when there is no edge joining vertices i and k. To facilitate future reference to such pairs of edges, we associate with every graph (V, E) a symmetric and irreflexive relation Γ on E which is defined as follows.

If $i \neq k$ and (i, j), $(j, k) \in E$, then

$$(i, j)\ \Gamma\ (j, k) \text{ if and only if } (i, k) \notin E. \tag{3}$$

Note that only pairs of edges sharing a common vertex should be considered when looking for Γ-related pairs.

To describe the orientation rule, we need some further definitions involving partially oriented graphs; i.e., graphs such that some (possibly all or none) of their edges are directed.

Let G^* or (V, E^*) denote a partially oriented graph. An edge in G^* is called an *implicant* if it is directed and Γ-related to at least one undirected edge in G^*. A graph G^* is said to be *stable* if it contains no implicants; otherwise G^* is called *unstable*.

All statements made in the following description of the algorithm are proved in § 4.

To test a given undirected graph G for transitive orientability, start by arbitrarily choosing and directing one edge of G. If the resulting partially directed graph is unstable, proceed according to the following orientation rule as long as it is applicable.

The Γ-implied orientation rule. Choose an implicant edge $[i, j]$. Then, to every undirected edge (i, i') such that $[i, j]\ \Gamma\ (i, i')$ assign the direction $i \to i'$, and to every undirected edge (j, j') such that $[i, j]\ \Gamma\ (j, j')$, assign the direction $j' \to j$.

It is clear that repeated application of the above rule will result, eventually, in a stable graph G which is either fully oriented or has none of its undirected edges being Γ-related to a directed one.

Next, consider separately the subgraphs $G_d{}^*$ and $G_u{}^*$ of G^* which are formed by the directed and undirected edges of G^*, respectively. First apply the following test to determine whether $G_d{}^*$ is TRD.

TRD test for directed graphs. Let $(V, E^{\rightarrow})$ be a directed graph. For every $i \in V$ form the subset

$$V(i) = \{j \in V \mid [i, j] \in E^{\rightarrow}\}. \tag{4}$$

Then $(V, E^{\rightarrow})$ is TRD if and only if for all $i \in V$,

$$V(i) \supset W(i) = \bigcup_{j \in V(i)} V(j). \tag{5}$$

The validity of this test is evident directly from the definition of TRD graphs.

If $G_d{}^*$ fails to be TRD, then the given graph G is not TRO. If $G_d{}^*$ turns out to be TRD, repeat the whole procedure for the remaining undirected graph $G_u{}^*$.

The cycle described above will be referred to as a *phase* of the algorithm. Thus, a phase starts by choosing and directing arbitrarily one edge of the undirected graph at hand and terminates with the TRD test of the directed subgraph of a stable graph. The result of the TRD test at any phase is independent of the order in which the implicants are chosen during the iterative orientation process at that phase. Successive phases start with smaller subgraphs of the given graph G and the algorithm terminates either with the first failure of the TRD test and the conclusion that G is not TRO, or with a fully oriented TRD image $G^{\rightarrow}$ of G.

The following is a formal summary of the TRO algorithm. Initially, let G' be the given graph G and proceed as follows.

(A) Choose and direct arbitrarily an edge in G'. If the resulting graph is unstable, go to (B); if it is stable, go to (C).
(B) Apply repeatedly the Γ-implied orientation rule until no implicants are left. Then go to (C).
(C) Apply the TRD test to the subgraph formed by the directed edges. If the test fails, go to (F); otherwise, go to (D).
(D) If there are no more undirected edges, go to (S); otherwise, let G' be the remaining undirected subgraph and go back to (A).
(F) Stop. The given graph G is not TRO.
(S) Stop. The given graph G is TRO and the assigned orientation is transitive.

In § 4 we establish the validity of this algorithm by proving the following two theorems.

THEOREM 1. *If the TRO algorithm terminates in state* (*S*), *the resulting directed graph is TRD.*

THEOREM 2. *If the graph under test is TRO, the algorithm will terminate in state* (S).

The two possible terminations of the algorithm are illustrated by the following examples.

Example 1. Consider the graph G shown in Figure 1(a). Let us start (A) by choosing the direction $1 \to 2$ for $(1, 2)$. Since $[1, 2]$ is Γ-related to both $(2, 3)$ and $(1, 5)$, we proceed to (B) and assign the directions $3 \to 2$ and $1 \to 5$. We have $[3, 2]\ \Gamma\ (3, 4)$ which implies $3 \to 4$ and then $[3, 4]\ \Gamma\ (4, 5)$ which implies $5 \to 4$. At this stage there are no more implicants and we proceed to (C). The edges directed so far form the subgraph $G_d{}^*$ shown in Figure 1(b). Inspection of $G_d{}^*$ readily reveals that it is not TRD because of $1 \to 5$, $5 \to 4$, and no edge from 1 to 4. Also, according to the formal TRD test, condition (5) is violated for $i = 1$. Thus, we have reached state (F) and we conclude that G is not TRO.

Example 2. Consider the graph G of Figure 2(a). Starting (A) with $1 \to 2$, the algorithm proceeds as follows.

(B) $[1, 2] \Rightarrow [3, 2] \Rightarrow [3, 4]$, $[5, 2]$,
$[3, 4] \Rightarrow [5, 4] \Rightarrow [5, 1] \Rightarrow$ (C).

(C) The directed subgraph $G_d{}^*$ is shown in Figure 2(b). Applying the TRD test we obtain:

$V(1) = \{2\}$, $V(2) = V(4) = \emptyset$, $V(3) = \{2, 4\}$, and $V(5) = \{1, 2, 4\}$. Hence, $W(i) = \emptyset$ for all $i = 1, 2, 3, 4$ and $W(5) = \{2\} \subset V(5)$. Thus, $G_d{}^*$ is TRD and we proceed to (D) and from there back to (A) with the subgraph $G_u{}^*$ shown in Figure 2(c).

(A) $1 \to 6 \Rightarrow$ (B).
(B) $[1, 6] \Rightarrow [2, 6]$, $[3, 6]$, $[4, 6]$, $[5, 6] \Rightarrow$ (C).
(C) $\Rightarrow$ (D).
(D) $\Rightarrow$ (S).

Having reached state (S) we decide that G is TRO. The obtained TRD image of G is shown in Figure 2(d).

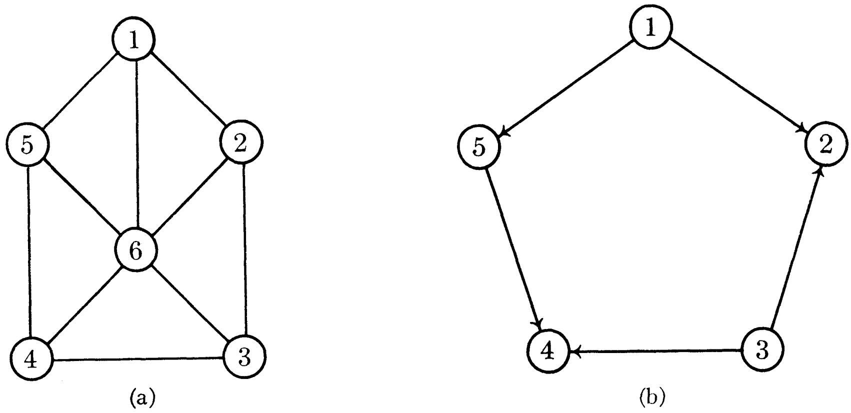

FIGURE 1. *The graphs of Example* 1

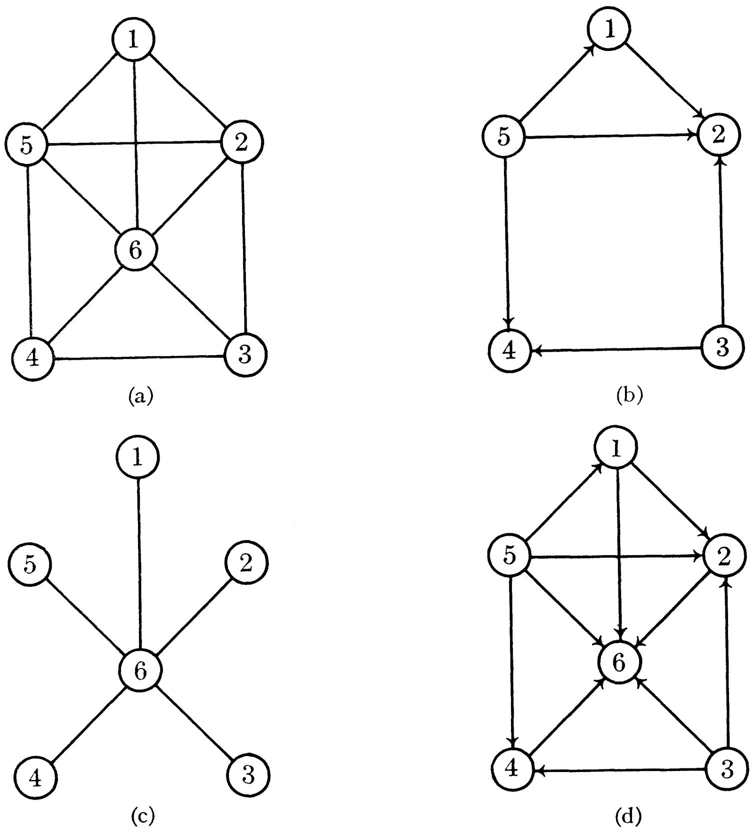

FIGURE 2. *The graphs of Example* 2

We conclude this section by describing how to achieve a labeling $v_1, v_2, \ldots, v_n$ for the vertices of a TRD graph $(V, E^{\rightarrow})$ under which $[v_i, v_j] \in E^{\rightarrow}$ implies $i < j$. One readily observes that, by the transitivity of $(V, E^{\rightarrow})$, such a labeling for V is equivalent to the one mentioned in the introduction, following the definition of TRO graphs. The procedure described below is, essentially, an embedding of a partially ordered set in a linearly ordered one and it applies to every directed graph which contains no directed circuits. (A circuit is said to be directed if the directions of all the edges forming the circuit are confluent.)

First, we observe that every TRD graph is free of directed circuits, for the existence of a directed circuit with k edges in a TRD graph would imply, by transitivity, the existence of one with $k - 1$ edges. Since there can be no directed circuit with only two edges, there can be no one at all.

It is also clear that if $(V, E^{\rightarrow})$ contains no directed circuit then it must contain at least one sink; that is, a vertex with no outgoing edges. Removal of a sink and all its incoming edges from $(V, E^{\rightarrow})$ leaves a graph which is also free of directed circuits.

Now, let $G^{\rightarrow}_n = (V, E^{\rightarrow})$ be a graph with n vertices and with no directed circuits. A labeling $v_1, v_2, \ldots, v_n$ of the vertices of $G^{\rightarrow}_n$ is called *valid* if and only if $i < j$ for all $[v_i, v_j] \in E^{\rightarrow}$. One easily verifies that the following procedure results in a valid labeling for $G^{\rightarrow}_n$.

Valid labeling procedure. Choose a sink in $G^{\rightarrow}_n$ and label it v_n. Remove v_n and all its incoming edges from $G^{\rightarrow}_n$ and call the resulting graph $G^{\rightarrow}_{n-1}$. Repeat the same with $G^{\rightarrow}_{n-1}$ using the label v_{n-1} and so forth until a single vertex is left and label it v_1.

For example, a valid labeling for the graph of Figure 2(d) would be: $v_6 = 6$, $v_5 = 4$, $v_4 = 2$, $v_3 = 3$, $v_2 = 1$, and $v_1 = 5$.

3. Identification of permutation graphs. Let G be a permutation graph with n vertices. A labeling $v_1, v_2, \ldots, v_n$ for the vertices of G is called *admissible* if there exists a permutation P on $N = \{1, 2, \ldots, n\}$ under which the inequality (1) is satisfied if and only if (v_i, v_j) is an edge in G. Consider a permutation graph (V, E) with an admissible labeling and a corresponding permutation P. If (V, E_c) is the complement of (V, E), then it is clear that for the same labeling and permutation, $(v_i, v_j) \in E_c$ if and only if the reversed inequality, that is

$$(i - j)[P^{-1}(i) - P^{-1}(j)] > 0, \tag{6}$$

is satisfied.

A graph G is called *complete* if every pair of distinct vertices is joined by an edge in G. Thus, if (V, E_c) is the complement of (V, E), then $(V, E \cup E_c)$ is complete.

THEOREM 3. *A graph $G = (V, E)$ is a permutation graph if and only if both G and its complement $G_c = (V, E_c)$ are TRO graphs.*

Proof.† Assume that G is a permutation graph. Then there exist a labeling $v_1, v_2, \ldots, v_n$ and a permutation P on N under which inequalities (1) and (6) are satisfied for G and G_c, respectively. Orient G and G_c so that all of their edges are directed from low to high. That is, let

(7) $$E^{\rightarrow} = \{[v_i, v_j] \mid (v_i, v_j) \in E \text{ and } i < j\}$$

and

(8) $$E^{\rightarrow}{}_c = \{[v_i, v_j] \mid (v_i, v_j) \in E_c \text{ and } i < j\}.$$

We claim that $G^{\rightarrow} = (V, E^{\rightarrow})$ and $G^{\rightarrow}{}_c = (V, E^{\rightarrow}{}_c)$ are both TRD. For, suppose that $[v_i, v_j]$ and $[v_j, v_k]$ belong to $G^{\rightarrow}$. Then $i < j < k$ and, by inequality (1), $P^{-1}(i) > P^{-1}(j) > P^{-1}(k)$. Hence inequality (1) is also satisfied for i and k, which means that $[v_i, v_k]$ belongs to $G^{\rightarrow}$. A similar argument applies to $G^{\rightarrow}{}_c$. This proves the "only if" part of the theorem.

Assume now that G and G_c are both TRO and let $G^{\rightarrow} = (V, E^{\rightarrow})$ and $G^{\rightarrow}{}_c = (V, E^{\rightarrow}{}_c)$ be a pair of corresponding TRD images. Consider the directed complete graph $H^{\rightarrow} = (V, E^{\rightarrow} \cup E^{\rightarrow}{}_c)$. First, we claim that $H^{\rightarrow}$ contains no directed circuits. For, if it does, let $i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow \ldots \rightarrow i_k \rightarrow i_1$ be a directed circuit in $H^{\rightarrow}$ which contains the minimum possible number of edges. It is clear that $k \geqq 3$. If $k > 3$, then since $H^{\rightarrow}$ is complete, there must be an edge joining i_1 and i_3. This edge, regardless of its direction, violates the minimality assumption with regard to the length of the chosen circuit. Therefore, we must have $k = 3$. However, two of the three edges $i_1 \rightarrow i_2$, $i_2 \rightarrow i_3$, and $i_3 \rightarrow i_1$ must belong either to $E^{\rightarrow}$ or to $E^{\rightarrow}{}_c$. Since (V, E) and (V, E_c) are both TRD, the graph that contains two of those edges must contain the third one, but with an opposite direction. This contradiction rules out the possibility of having a directed circuit in $H^{\rightarrow}$.

Now, let $G^{\leftarrow} = (V, E^{\leftarrow})$ be the graph obtained by reversing the direction of every edge in $G^{\rightarrow} = (V, E^{\rightarrow})$. That is,

(9) $$E^{\leftarrow} = \{[i, j] \mid [j, i] \in E^{\rightarrow}\}.$$

One easily verifies that $G^{\leftarrow}$ is also a TRD image of G and, hence, also the complete graph $R^{\rightarrow} = (V, E^{\leftarrow} \cup E^{\rightarrow}{}_c)$ is free of directed circuits. Thus, we may apply the valid labeling procedure, defined in § 2, to both $H^{\rightarrow}$ and $R^{\rightarrow}$. Let $h_1, h_2, \ldots, h_n$ be the labeling of V obtained by applying the procedure to $H^{\rightarrow}$ and let $r_1, r_2, \ldots, r_n$ be the labeling obtained with respect to $R^{\rightarrow}$. Every element of V now has two labels, an h-label h_i and an r-label r_j. Let L denote the set of the n label pairs (h_i, r_j) associated with the elements of V, and let for all $i, j \in N = \{1, 2, \ldots, n\}$,

(10) $$P(j) = i \text{ if and only if } (h_i, r_j) \in L.$$

It is clear that $P = [P(1), P(2), \ldots, P(n)]$ is a permutation on N and we will show now that under this permutation, inequality (1) holds for $i, j \in N$ if and only if the vertices with h-labels h_i and h_j are joined by an edge in

†A weaker theorem, that involves a prescribed labeling of the vertices, has been proved in [1].

$G = (V, E)$. To this end, consider the two label pairs which contain h_i and h_j. Let these pairs be (h_i, r_s) and (h_j, r_t). Then $P(s) = i$, $P(t) = j$, and

$$(11) \qquad P^{-1}(i) - P^{-1}(j) = s - t.$$

The left-hand side of inequality (1) is therefore given by

$$(12) \qquad m = (i - j)(s - t).$$

Since only the edges belonging to E have opposing directions in $H^{\rightarrow}$ and $R^{\rightarrow}$, it is evident that m is negative if and only if the edge that joins the vertex whose label pair is (h_i, r_s) to the vertex whose label pair is (h_j, r_t) belongs to E.

The proof of Theorem 3 suggests the following procedure for identifying permutation graphs.

(P.1) Apply the TRO algorithm to the given graph (V, E) and to its complement (V, E_c). If any of these graphs fails to be TRO, then neither is a permutation graph. If both are TRO, let $(V, E^{\rightarrow})$ and $(V, E^{\rightarrow}_c)$ be the corresponding TRD images and go to (P.2).

(P.2) Construct the graphs $H^{\rightarrow} = (V, E^{\rightarrow} \cup E^{\rightarrow}_c)$ and $R^{\rightarrow} = (V, E^{\leftarrow} \cup E^{\rightarrow}_c)$, where $E^{\leftarrow}$ is the set defined by (9), and apply the valid labeling procedure to obtain a labeling $h_1, h_2, \ldots, h_n$ for $H^{\rightarrow}$ and a labeling $r_1, r_2, \ldots, r_n$ for $R^{\rightarrow}$. The h-labels provide an admissible labeling for both (V, E) and (V, E_c). The corresponding permutation for (V, E) is the one defined by (10). If P is the permutation for (V, E), then the permutation P_c for (V, E_c) is given by

$$(13) \qquad P_c(i) = P(n + 1 - i), \qquad i \in N = \{1, 2, \ldots, n\}.$$

Verification of the validity of (13) is left to the interested reader.

Example 3. In Example 2 we found that the graph G of Figure 2(a) is TRO. Its complement G_c consists of only four edges (1, 3), (1, 4), (2, 4), and (3, 5), which may be transitively oriented as $1 \rightarrow 3$, $1 \rightarrow 4$, $2 \rightarrow 4$, and $5 \rightarrow 3$. Adding these directed edges to $G^{\rightarrow}$ of Figure 2(d) and to its reversal $G^{\leftarrow}$, we obtain the complete graphs $H^{\rightarrow}$ and $R^{\rightarrow}$, shown in Figures 3(a) and 3(b),

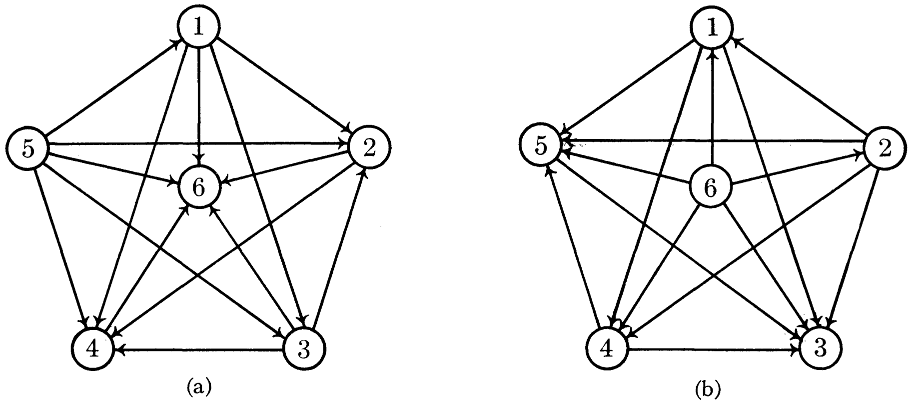

FIGURE 3. *The graphs $H^{\rightarrow}(a)$ and $R^{\rightarrow}(a)$ for Example* 3

respectively. The label pairs (h_i, r_j) obtained by applying the valid labeling procedure are: $1 \Leftrightarrow (h_2, r_3)$, $2 \Leftrightarrow (h_4, r_2)$, $3 \Leftrightarrow (h_3, r_6)$, $4 \Leftrightarrow (h_5, r_4)$, $5 \Leftrightarrow (h_1, r_5)$, and $6 \Leftrightarrow (h_6, r_1)$. Hence, an admissible labeling for G and G_c is given by $1 = h_2$, $2 = h_4$, $3 = h_3$, $4 = h_5$, $5 = h_1$, and $6 = h_6$; the corresponding permutation for G is $P(1) = 6$, $P(2) = 4$, $P(3) = 2$, $P(4) = 5$, $P(5) = 1$, and $P(6) = 3$, and the permutation for G_c is $P_c(1) = 3$, $P_c(2) = 1$, $P_c(3) = 5$, $P_c(4) = 2$, $P_c(5) = 4$, and $P_c(6) = 6$.

4. Proof of the TRO algorithm. The following notation and definitions will be useful in proving the TRO algorithm. Let (V, E) be a graph. For $A \subset E$ let $V(A)$ denote the subset of V which consists of all the vertices which belong to at least one edge in A. The graph $(V(A), A)$ will be referred to as the *edge-section* of (V, E) spanned by $A \subset E$. For $X \subset V$ let $E(X)$ denote the subset of E which consists of all the edges that have both of their vertices in X. The graph $(X, E(X))$ will be referred to as the *vertex-section* of (V, E) spanned by $X \subset V$.

Notice that $V(E(X)) \subset X$ for all $X \subset V$, while $E(V(A)) \supset A$ for all $A \subset E$.

For $X, Y \subset V$ let (X, Y) denote the set of all unordered vertex pairs formed by one vertex from X and one from Y. That is,

$$(X, Y) = \{(x, y) | x \in X, y \in Y\}.$$

The notation $[X, Y]$ will be used for ordered vertex pairs.

We define now an equivalence relation $\tilde{\Gamma}$ on E which is the closure of the Γ-relation defined in (3). The relation $\tilde{\Gamma}$ is defined as follows:

$$(14) \qquad \begin{cases} \text{(a)} & e \, \tilde{\Gamma} \, e \text{ for all } e \in E; \\ \text{(b)} & \text{If } e_1 \, \Gamma \, e_2, \text{ then } e_1 \, \tilde{\Gamma} \, e_2; \\ \text{(c)} & \text{If } e_1 \, \tilde{\Gamma} \, e_2 \text{ and } e_2 \, \tilde{\Gamma} \, e_3, \text{ then } e_1 \, \tilde{\Gamma} \, e_3. \end{cases}$$

The relation $\tilde{\Gamma}$ induces a partition of E into equivalence classes which will be referred to as the $\tilde{\Gamma}$-classes of (V, E). The set whose elements are the $\tilde{\Gamma}$-classes of a graph (V, E) will be denoted by $\tilde{E}$. Thus, the elements of $\tilde{E}$ are disjoint subsets of E, every edge in E belongs to some $\tilde{\Gamma}$-class $A \in \tilde{E}$, and all edges belonging to the same $A \in \tilde{E}$ are equivalent under $\tilde{\Gamma}$. For example, if (V, E) is a complete graph, then no two edges of (V, E) are Γ-related and, therefore, every edge forms a $\tilde{\Gamma}$-class of its own.

The proof of Theorem 1 is quite simple and is presented below in one continuous argument. The proof of Theorem 2 is more complicated and is preceded by a series of lemmas which, besides leading toward the final argument, also reveal the specific structure of TRO graphs and some far-reaching aspects of the relation Γ with regard to arbitrary graphs.

Proof of Theorem 1. The proof is by induction on the number of phases it takes to reach state (S). If state (S) is reached after one phase, then the resulting directed graph is obviously TRD. Assume that the theorem is true when the number of phases is k or less, and consider a graph (V, E) whose test runs through $k + 1$ phases and terminates in state (S) with the directed graph $(V, E^{\rightarrow})$. If $A^{\rightarrow} \subset E^{\rightarrow}$ is the set of edges oriented in the first phase and $B^{\rightarrow} = E^{\rightarrow} - A^{\rightarrow}$, then, by the inductive hypothesis, both $(V, A^{\rightarrow})$ and $(V, B^{\rightarrow})$ are TRD because the first graph results after one phase and the second graph after k phases of the TRO algorithm. It is also clear that the set of edges oriented in the first phase is always a $\tilde{\Gamma}$-class of the graph under test. Now, assume $i \rightarrow j$ and $j \rightarrow k$ in $(V, E^{\rightarrow})$. If both edges belong to $A^{\rightarrow}$ or both belong to $B^{\rightarrow}$, then also $i \rightarrow k$ in $(V, E^{\rightarrow})$ because each of $(V, A^{\rightarrow})$ and $(V, B^{\rightarrow})$ is TRD. If $[i, j] \in A^{\rightarrow}$ and $[j, k] \in B^{\rightarrow}$, then these edges belong to different $\tilde{\Gamma}$-classes of (V, E) and we must have $(i, k) \in E$. If $(i, k) \in A$, then $i \rightarrow k$ in $(V, E^{\rightarrow})$, or else $(V, A^{\rightarrow})$ is not TRD. If $(i, k) \in B$ we must also have $i \rightarrow k$ in $(V, E^{\rightarrow})$, or else $(V, B^{\rightarrow})$ is not TRD. A similar conclusion is reached if we assume $[i, j] \in B^{\rightarrow}$ and $[j, k] \in A^{\rightarrow}$.

We proceed now with a series of lemmas which lead to the proof of Theorem 2. The first six lemmas deal with the interconnections among edge and vertex sections of an arbitrary graph which are determined by the Γ-relation of the graph. The next six lemmas deal with subgraphs of a TRO graph which preserve the TRO property.

LEMMA 1. *For every $\tilde{\Gamma}$-class $A \in \tilde{E}$, the edge section $(V(A), A)$ of (V, E) is connected.*

Proof. The connectedness of $(V(A), A)$ is an immediate consequence of the fact that any two Γ-related edges have a common vertex.

LEMMA 2. *For every $A \in \tilde{E}$, if $x \in V(A)$, $y \in V - V(A)$, and $(x, y) \in E$, then $(V(A), y) \subset E$.*

Proof. Assume that $(x, y) \in E$ with $x \in V(A)$ and $y \in V - V(A)$. Let $X = \{x \in V(A) | (x, y) \in E\}$. If $X \neq V(A)$, then by Lemma 1 there exists an edge $(x', z) \in A$ such that $x' \in X$ and $z \in V(A) - X$. From the construction of X it follows that $(z, y) \notin E$ and $(x', y) \in E$. Therefore, $(x', z) \ \Gamma \ (x', y)$, and since $(x', z) \in A$ we must have $(x', y) \in A$. This, however, contradicts the assumption that $y \in V - V(A)$ and, hence, we must have $X = V(A)$.

The property established in Lemma 2 is now defined in general as follows. A non-empty subset X of V in a graph (V, E) is said to be *uniformly hinged* on another non-empty subset Y of V if

(i) X and Y are disjoint;
(ii) $(X, Y) \subset E$;
(iii) $x \in X$, $x' \in V - X$, and $(x, x') \in E$ imply $x' \in Y$.

From Lemma 2 it follows that if (V, E) is connected, then for every $A \in \tilde{E}$ either $V(A) = V$ or there exists a set $Y \subset V - V(A)$ such that $V(A)$ is uniformly hinged on Y. For an appropriate choice of $A \in \tilde{E}$ this conclusion may further be strengthened as follows.

LEMMA 3. *If* (V, E) *is connected, then there exists a* $\tilde{\Gamma}$*-class* $A \in \tilde{E}$ *such that either* $V(A) = V$ *or* $V(A)$ *is uniformly hinged on* $V - V(A)$.

Proof. If there exists no $A \in \tilde{E}$ such that $V(A) = V$, let $B \in \tilde{E}$ be a $\tilde{\Gamma}$-class for which $V(B)$ contains the maximum possible number of vertices over all $A \in \tilde{E}$. Consider the set $Y_B \subset V$ on which $V(B)$ is uniformly hinged. If $Y_B = V - V(B)$ there is nothing to prove. Assume that $Y_B \neq V - V(B)$ and let $Z_B = V - (V(B) \cup Y_B)$. Note that there is no edge in (V, E) which joins a vertex of Z_B to one of $V(B)$. Since (V, E) is assumed to be connected, there must exist an edge $(y, z) \in E$ such that $y \in Y_B$ and $z \in Z_B$. Observing that for all $x \in V(B)$ we have $(x, y) \in E$ while $(x, z) \notin E$, it follows that $(x, y)\ \Gamma\ (y, z)$ for all $x \in V(B)$. Consider now the $\tilde{\Gamma}$-class $A \in \tilde{E}$ which contains the edge (y, z). We have just shown that in addition to (y, z), A also contains the set $(V(B), y)$. This implies that $V(B) \subset V(A)$, and since $z \in V(A)$ but $z \notin V(B)$, $V(B) \neq V(A)$. Thus, $V(A)$ contains more vertices than $V(B)$ does, which contradicts the maximality of $V(B)$ and invalidates the assumption with regard to Y_B. Hence, $Y_B = V - V(B)$ which completes the proof.

A graph (V, E) is said to be Γ-*connected* if there exists a $\tilde{\Gamma}$-class $A \in \tilde{E}$ such that $V(A) = V$.

The next three lemmas deal with Γ-connected graphs. The $\tilde{\Gamma}$-class which meets all the vertices in such a graph (V, E) will be denoted by R and the edges belonging to R will be called *red*. If $E \neq R$, let $B = E - R$ and call the edges belonging to B *blue*.††

LEMMA 4. *A* Γ*-connected graph* (V, E) *contains no circuit which is formed by only one red and the rest blue edges.*

Proof. Suppose that there exists a circuit in (V, E) which contains only one red edge. Among all such circuits consider one with the smallest possible number of edges. Let the edges of this circuit be (i_1, i_2), (i_2, i_3), . . . , (i_{k-1}, i_k), (i_k, i_1) and let (i_1, i_2) be the only red edge. Clearly, $k \geqq 3$. If $k \neq 3$, then since (i_1, i_2) is red and (i_2, i_3) is blue, we must have $(i_1, i_3) \in E$. This edge (i_1, i_3) cannot be red because then we would have a circuit formed by (i_1, i_3), (i_3, i_4), . . . , (i_{k-1}, i_k), (i_k, i_1) with only (i_1, i_3) being red and with one edge less than the assumed shortest circuit of this type. Thus, (i_1, i_3) must be blue and, therefore, $k = 3$. Now consider the triangle formed by $(i_1, i_2) \in R$ and (i_2, i_3), $(i_1, i_3) \in B$. By Lemma 1, (V, R) is connected and we must have

††Anticipating the result of Lemma 4, we refer to R as "the" $\tilde{\Gamma}$-class for which $V(R) = V$. The uniqueness of such a $\tilde{\Gamma}$-class in every Γ-connected graph is an immediate consequence of Lemma 4.

a red edge which meets vertex i_3, say (i_3, r), and it is clear that r is distinct from both i_1 and i_2. Since all the edges of R are equivalent under $\tilde{\Gamma}$, there must be a Γ-chain (i.e., a sequence of Γ-related edges) leading from (i_1, i_2) to (i_3, r). If there are other triangles in (V, E) which violate the lemma, assume that the one under consideration was chosen among all such triangles to have the shortest possible Γ-chain from the red edge to the vertex shared by the two blue ones. Consider the edge next to (i_1, i_2) in a shortest Γ-chain leading to (i_3, r). This edge is either (i_1, s) or (i_2, s) for some $s \in V$. The two cases are completely symmetric and we may assume $(i_1, i_2)\ \Gamma\ (i_2, s)$ to be the first link in the chain. Therefore, $(i_1, s) \notin E$, and since (i_2, i_3) is blue and (i_2, s) is red we must have $(i_3, s) \in E$. Furthermore, since (i_1, i_3) is blue, also (i_3, s) must be blue because $(i_1, s) \notin E$ implies $(i_1, i_3)\ \Gamma\ (i_3, s)$. Thus, $s \neq r$ and we have now another violating triangle with one red edge (i_2, s) and two blue edges (i_2, i_3) and (i_3, s) sharing the same vertex i_3. For this triangle, however, we have a Γ-chain from (i_2, s) to (i_3, r) which consists of only a proper part of the Γ-chain for the original triangle, namely, the part which follows edge (i_1, i_2). This contradicts the minimality assumption for the length of the original Γ-chain and therefore no circuit which violates the lemma can be found in a Γ-connected graph.

Lemma 4 implies that if (V, E) is Γ-connected and $R \in \tilde{E}$ is such that $V(R) = V$, then (V, B), $B = E - R$, is *not connected* because, otherwise, adding any edge from R back to (V, B) would create a circuit which will violate the lemma. In view of Lemma 1, the fact that (V, B) is not connected means that for every Γ-connected graph there exists exactly one $\tilde{\Gamma}$-class R which meets all the graph vertices as asserted earlier.

Consider a Γ-connected graph (V, E). As shown above, the "blue part" (V, B) of this graph consists of $p \geqq 2$ pieces; that is, maximal connected subgraphs, some of which may be isolated vertices. Let V_i, $i = 1, 2, \ldots, p$, be the subset of V which spans the ith piece of (V, B) and let $U = \{V_i\}$ denote the set of these p subsets of V.

LEMMA 5. *If* (V, E) *is* Γ-*connected, then for every* $V_i \in U$, $E(V_i) \subset B$.

Proof. If there were a red edge joining two vertices belonging to the same $V_i \in U$, we would have a circuit violating Lemma 4, because between any two vertices which belong to the same piece of (V, B) there exists a path consisting of blue edges only.

LEMMA 6. *If* (V, E) *is* Γ-*connected, then every* $V_i \in U$ *is uniformly hinged on some set* $Y_i \subset V - V_i$.

Proof. Since (V, E) is Γ-connected and each $V_i \in U$ is properly contained in V, we must have a red edge $(x, y) \in R$ with $x \in V_i$ and $y \in V - V_i$. Let Y_i be the set of all vertices in $V - V_i$ which are joined by an edge to at least one vertex of V_i. To prove that V_i is uniformly hinged on Y_i we have to show that for every $y \in Y_i$, $(y, V_i) \subset E$. To this end, let $y \in Y_i$ and let $X = \{x \in V_i | (x, y) \in R\}$. The set X is not empty because all the edges joining the vertices of V_i to a vertex of $V - V_i$ must be red and Y_i was constructed to contain all the vertices outside of V_i which are adjacent to some vertex of V_i. If $X \neq V_i$, there must be a blue edge $(x', z) \in E(V_i)$ with $x' \in X$ and $z \in V_i - X$. Since (x', y) is red and (x', z) is blue, we must have $(y, z) \in E$ and this edge cannot be blue because $z \in V_i$ and $y \notin V_i$ (and also because of Lemma 4). Thus, (y, z) must be red, which contradicts the assumption that $X \neq V_i$.

We turn now to TRO graphs and the special role of uniformly hinged vertex sets in such graphs.

Consider a graph (V, E) and let $X \subset V$ be uniformly hinged on $Y \subset V$. A directed image $(V, E^{\rightarrow})$ of (V, E) is said to be *regular* with respect to X if for $y \in Y$ either $[y, X] \subset E^{\rightarrow}$ or $[X, y] \subset E^{\rightarrow}$.

LEMMA 7. *If (V, E) is TRO and $X \subset V$ is uniformly hinged on $Y \subset V$, then there exists a TRD image $(V, E^{\rightarrow})$ of (V, E) which is regular with respect to X.*

Proof. Assume that (V, E) is TRO and let $(V, (E^{\rightarrow})^*)$ be a TRD image of (V, E). If X is uniformly hinged on Y and $(V, (E^{\rightarrow})^*)$ is not regular with respect to X, let Z be the subset of Y which consists of all $z \in Y$ for which there exist x_1 and x_2 in X such that $x_1 \rightarrow z$ and $z \rightarrow x_2$ in $(V, (E^{\rightarrow})^*)$. First, we observe that transitivity in $(V, (E^{\rightarrow})^*)$ and the fact that X is hinged on Y imply that no vertex outside of $X \cup Y$ may be adjacent to a vertex of Z. That is,

$$z \in Z \text{ and } (v, z) \in E \text{ imply } v \in X \cup Y. \tag{15}$$

Secondly, since $x \in X, y \in Y - Z$, and $[x, y] \in (E^{\rightarrow})^*$ imply $[X, y] \subset (E^{\rightarrow})^*$, and since for every $z \in Z$ we have $[z, x] \in E^{\rightarrow}$ for some $x \in X$, transitivity in $(V, (E^{\rightarrow})^*)$ requires that

$$x \in X, y \in Y - Z \text{ and } [x, y] \in (E^{\rightarrow})^* \text{ imply } [Z, y] \subset (E^{\rightarrow})^*. \tag{16}$$

Similarly, we have

$$x \in X, y \in Y - Z \text{ and } [y, x] \in (E^{\rightarrow})^* \text{ imply } [y, Z] \subset (E^{\rightarrow})^*. \tag{17}$$

Now, let

$$A = \{[x, z] \in (E^{\rightarrow})^* | \, x \in X \text{ and } z \in Z\}$$

and let $(V, E^{\rightarrow})$ be the directed graph obtained from $(V, (E^{\rightarrow})^*)$ by reversing the orientation of all, and only those, edges which belong to A. The new directed image of (V, E) is regular with respect to X with $[Z, X] \subset E^{\rightarrow}$, and to complete the proof we have to show that $(V, E^{\rightarrow})$ is TRD. To this end, assume that there exist i, j, and k in V such that $[i,j]$, $[j,k] \in E^{\rightarrow}$ and $[i,k] \notin E^{\rightarrow}$. This situation is equivalent to the following alternatives:

(18) $$[i,j] \in E^{\rightarrow},\ [j,k] \in E^{\rightarrow} \quad \text{and} \quad [k,i] \in E^{\rightarrow}$$

or

(19) $$[i,j] \in E^{\rightarrow},\ [j,k] \in E^{\rightarrow} \quad \text{and} \quad (i,k) \notin E.$$

Since $(V, (E^{\rightarrow})^*)$ is TRD and the only edges whose orientation might have been changed are connected between X and Z, it follows that at least one of i, j, k belongs to X, at least another one belongs to Z and, from (15), the third of i, j, k must belong to $X \cup Y$. This last vertex is further restricted to be contained in $Y - Z$ because $[Z, X] \subset E^{\rightarrow}$ and if any two of i, j, k belong to either X or Z, neither of the assumed alternatives (18) and (19) can hold. Now, let x, y, and z be the elements from $\{i, j, k\}$ which belong to X, $Y - Z$, and Z, respectively. Evidently, $(x, y) \in E$ and the orientation of (x, y) in $(V, E^{\rightarrow})$ is the same as it was in $(V, (E^{\rightarrow})^*)$. From (16) and (17) we have either

(20) $$[x, y] \in (E^{\rightarrow})^* \quad \text{and} \quad [z, y] \in (E^{\rightarrow})^*$$

or

(21) $$[y, x] \in (E^{\rightarrow})^* \quad \text{and} \quad [y, z] \in (E^{\rightarrow})^*.$$

Since also the orientation of (y, z) is the same in both $(V, (E^{\rightarrow})^*)$ and $(V, E^{\rightarrow})$, it follows that one of the alternatives (20) or (21) must hold with regard to $E^{\rightarrow}$. However, it is easy to verify that neither of these alternatives is compatible with any of the assumed alternatives (18) or (19), and hence $(V, E^{\rightarrow})$ is TRD.

LEMMA 8. *If (V, E) is TRO and $X \subset V$ is uniformly hinged on $Y \subset V$, then $(V, E - E(X))$ is also TRO.*

Proof. If (V, E) is TRO and X is uniformly hinged on Y, then by Lemma 7 there exists a TRD image $(V, E^{\rightarrow})$ of (V, E) which is regular with respect to X. Consider $(V, E^{\rightarrow} - E^{\rightarrow}(X))$. We claim that $(V, E^{\rightarrow} - E^{\rightarrow}(X))$ is a TRD image of $(V, E - E(X))$. This can readily be established by observing that if $[i, j]$ and $[j, k]$ belong to $E^{\rightarrow} - E^{\rightarrow}(X)$, then i and k cannot both belong to X because then j must belong to Y, which contradicts the regularity of the original graph $(V, E^{\rightarrow})$ with respect to X. Thus, at least one of i and k belongs to $V - X$, and since the existence of $[i, k]$ in $E^{\rightarrow}$ is guaranteed by the transitivity of $(V, E^{\rightarrow})$, $[i, k]$ must also be present in $E^{\rightarrow} - E^{\rightarrow}(X)$. Hence, $(V, E^{\rightarrow} - E^{\rightarrow}(X))$ is TRD or $(V, E - E(X))$ is TRO as asserted.

LEMMA 9. *If (V, E) is TRO, then for every $X \subset V$ the vertex section $(X, E(X))$ of (V, E) is TRO.*

The reader may readily verify this lemma by considering a TRD image of (V, E) and the vertex section $(X, E^{\rightarrow}(X))$ thereof.

LEMMA 10. *If* (V, E) *is TRO and* Γ*-connected, then* (V, R) *is TRO, where* $R \in \tilde{E}$ *is the* $\tilde{\Gamma}$*-class of red edges.*

Proof. The lemma is trivially true if the set $B = E - R$ of blue edges is empty. If B is not empty, then by Lemma 6 every set $V_i \in U$ which spans a piece of (V, B) is uniformly hinged on some $Y_i \subset V$. By Lemma 8, the graph $(V, E - E(V_i))$ is TRO for every $V_i \in U$ and by Lemma 5 it contains all the original red edges of (V, E). It is also clear that $(V, E - E(V_i))$ is Γ-connected with the same set of edges R forming the red $\tilde{\Gamma}$-class for this graph as for (V, E). The lemma follows now by induction on the number of pieces in (V, B) which contain at least one edge.

LEMMA 11. *If* (V, E) *is TRO, then for every* $\tilde{\Gamma}$*-class* $A \in \tilde{E}$ *the edge-section* (V, A) *is TRO.*

Proof. Consider the vertex section of (V, E) which is spanned by $V(A) \subset V$ for an arbitrary $A \in \tilde{E}$. Each such vertex section is a Γ-connected graph with A being the set of its red edges. Thus, if (V, E) is TRO, then by Lemmas 9 and 10, $(V(A), A)$ is also TRO for every $A \in \tilde{E}$. Since (V, A) consists of $(V(A), A)$ and the isolated vertices belonging to $V - V(A)$, (V, A) is also TRO.

LEMMA 12. *If* (V, E) *is TRO, then* $(V, E - A)$ *is TRO for every* $A \in \tilde{E}$.

Proof. If (V, E) is TRO, then by Lemma 2, $V(A)$ is uniformly hinged on some $Y \subset V$ and by Lemma 7 there exists a TRD image of (V, E) which is regular with respect to $V(A)$. Let $(V, E^{\rightarrow})$ be such a TRD image of (V, E) and let $C^{\rightarrow} = E^{\rightarrow} - A^{\rightarrow}$. Suppose that $i \rightarrow j$ and $j \rightarrow k$ in $(V, C^{\rightarrow})$. Since $(V, E^{\rightarrow})$ is TRD, we must have $i \rightarrow k$ in $(V, E^{\rightarrow})$. If $(i, k) \in C$, there is nothing to prove. If $(i, k) \in A$, then i and k belong to $V(A)$ and, by the regularity of $(V, E^{\rightarrow})$ with respect to $V(A)$, also j must belong to $V(A)$. Thus, the vertex section of (V, E) which is spanned by $V(A)$ contains a triangle formed by $(i, k) \in A$ and (i, j), $(j, k) \in C = E - A$. This, however, violates Lemma 4 because the vertex-section spanned by $V(A)$ is a Γ-connected graph with A being the set of its red edges. Hence, we must have $(i, k) \in C$ and, therefore, $i \rightarrow k$ in $(V, C^{\rightarrow})$.

Proof of Theorem 2. The proof is by induction on the number of edges. The theorem is trivially true if (V, E) is a TRO graph with only one edge. Suppose that the theorem holds for all TRO graphs with k or less edges and consider a TRO graph (V, E) with $k + 1$ edges. Let $A \in \tilde{E}$ be the Γ-class of edges that are oriented during the first phase and let $(V, A^{\rightarrow})$ be the resulting directed image of (V, A). By Lemmas 11 and 12, (V, A) and $(V, E - A)$ are both TRO. If $A \neq E$, then (V, A) has k or less edges and, by the inductive hypothesis, $(V, A^{\rightarrow})$ will pass the TRD test and the algorithm will reach its second phase with the graph $(V, E - A)$. Since this TRO graph certainly contains less than $k + 1$ edges, the algorithm will terminate in state (S). To complete the proof we have to consider the case in which $A = E$. But, if (V, E) is TRO and every pair of its edges are Γ-related, then the direction of any edge in a TRD image of (V, E) uniquely determines the direction of all other edges in accordance with the Γ-implied orientation rule. Since the graph obtained by reversing the direction of every edge in a TRD graph is also TRD, the direction of the first edge can always be chosen arbitrarily, without effecting the decision reached by the algorithm.

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