ON THE LOCAL THEORY OF CONTINUOUS INFINITE PSEUDO GROUPS I.*

MASATAKE KURANISHI

Introduction

The local theory of continuous (infinite) pseudo-groups of transformations was originated by S. Lie, and developed by himself, F. Engel, E. Vessiot, E. Cartan, etc. In the beginning, the definition was not clear and we can find several different definitions in the papers of pioneers. In 1902, E. Cartan introduced a definition using his theory of exterior differential systems and made an extensive study in his series of papers [1], [2], and [3]. The writer will adopt his definition in this series of papers. A continuous pseudo-group of transformations is, roughly speaking, a collection of real (or complex) analytic homeomorphisms of domains in a real (or complex) euclidean space, which is closed under the operations of composition and inverse, and which forms the general solutions of a system of partial differential equations. An example is the collection of conformal mappings of domains in a complex plane, considered as a real euclidean space, because the collection forms the general solutions of Cauchy-Riemann equations. A continuous pseudo-group of transformations is called finite, if the underlying system of differential equations is completely integrable, otherwise infinite. Aside from the applications of the theory to the differential geometry and partial differential equations, he was also interested in the analytic-algebraic structure which lies behind the structure of continuous pseudo-group of transformations. Namely, if G is a pseudo-group of transformations and f, g are in G, then the inverse f^{-1} is defined and the composition $f \circ g$ is defined for some pairs (f, g). Thus G forms an algebraic system

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which looks like a group; on the other hand such algebraic system will be related with the analytic properties of the underlying partial differential equations. This is what we called the analytic-algebraic structure. In the finite case, then, the analytic-algebraic structure is the parameter local groups. One of the key points of E. Cartan's theory, in this respect, is his notion of "isomorphisme holoédrique," which gives the definition of the isomorphism of such analytic-algebraic structures. In the finite case, there exists an "isomorphisme holoédrique" between two pseudo-groups of transformations if and only if their parameter local groups are isomorphic. In the general case, we have no notion of parameter local groups. So he had to define the isomorphism of the underlying analytic-algebraic structures without explicitly defining the structure. The main purpose of the present series of papers is to introduce the notion similar to that of parameter local groups in the general case. Our task, then, is to generalize the notion of local Lie groups to the special case of infinite dimensional parameter space so that the generalized notion can be used as parameter structure of continuous pseudo-groups of transformations.

An examination in the finite case will make clear what approach one should take. In this case, general solutions of the underlying systems of partial differential equations depend on, roughly speaking, a number of arbitrary constants. By parameterizing the general solutions by a finite number of constants, and by means of compositions of transformations, we define the multiplication functions of parameter local Lie groups. In the infinite case, the general solutions depend on a finite number of arbitrary functions, in stead of arbitrary constants. However, replacing arbitrary constants by arbitrary functions, we can carry out the same reasoning as in the finite case, and we obtain the multiplications between parameters and finally something like parameter local groups. The first task, then, is to generalize the notion of analytic functions to the case we are intersested in, because the multiplication functions are analytic in the finite case. However, in order to develop the Lie's fundamental theorems in our generalized case, we have to generalize the notion of formal power series to our case as well. We shall call the generalized formal power series the formal analytic mappings. Chapter I is devoted to present their definitions and to prove several porperties which we shall use later. In Chapter II, using formal analytic mappings as multiplications and commutators, we introduce the notion of formal Lie (F)-group and formal Lie (F)-algebra and prove Lie's fundamental theorems in our case. In the subsequent papers, we shall give a description of E. Cartan's theory of continuous pseudo-groups, and study the relations between our approach and E. Cartan's.

As stated before, we are concerned only with the local aspect of the theory. So the global aspect of the theory, which is the subject of several recent interesting works, will be entirely neglected. It should be noted also that we are concerned only in the analytic case, real or complex, namely in the case where transformations considered are analytic. No effort is made to extend the theory to differentiable case, even though it is a very interesting problem.

Chapter I. Formal Analytic Mappings

A field K containing infinitely many elements will be fixed throughout this chapter. So such words as "over K" will be omitted when no confusion can occur.

1. Vector spaces with (F)-structures

Let H be a vector space (over K) of dimension possibly infinite. Let $H^{(l)}$ and $B^{(l)}$, where $l=0, 1, \ldots$, be vector subspaces of H. Assume that the dimension of each $B^{(l)}$ is finite, say d^l. Let $\mathbf{h}^{(l)}=\{h_1^l, \ldots, h_{d^l}^l\}$ be an ordered basis of $B^{(l)}$.

DEFINITION I. 1. *A collection $(H^{(l)}, B^{(l)}, \mathbf{h}^{(l)})$ is called a (F)-structure in H, if the following conditions are satisfied:*

(1°) $H=H^{(0)}\supseteq H^{(1)}\supseteq \cdots \supseteq H^{(l)}\supseteq H^{(l+1)}\supseteq \cdots$;

(2°) *$H^{(l)}$ is the direct sum of $B^{(l)}$ and $H^{(l+1)}$;*

(3°) *For any sequence $(\xi^l)_{l=0,1,\ldots}$, where $\xi^l\in B^{(l)}$, there is a unique element ξ in H such that $\xi-(\xi^0+\xi^1+\cdots+\xi^l)\in H^{(l+1)}$ for any integer l. We denote ξ by the formal sum $\xi^0+\xi^1+\cdots+\xi^l+\cdots$, or $\sum_{l=0}^{\infty}\xi^l$;*

(4°) *We can find integers p, k and a real number m_1 such that, for sufficiently large l, we have*

$$m_1(l-k)^p \leq \dim(H/H^{(l)}) \leq m_1(l+k)^p.$$

Elements of $B^{(l)}$ will be called homogeneous elements of degree l. The

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basis $\mathbf{h}^{(l)}$ will be called the distinguished basis of degree l. Any member of the distinguished bases will be called a distinguished element. Set $n^l = \dim (H/H^{(l+1)}) = d^0 + d^1 + \cdots + d^l$. We introduce an order lexicographically in the set of distinguished elements. Namely, we define h_j $(j=1, 2, \ldots)$, as follows: $h_{n^{l-1}+j} = h_j^l$ for $j=1, \ldots, d^l$ if d^l is not zero. By the definitions, any element ξ in H can be expressed uniquely as $\xi = \sum_j b^j h_j$, where $b^j \in K$. Conversely, for any sequense $(b^j)_{j=1,2,\ldots}$ of elements in K, the formal sum $\sum b^j h_j$ represents an element in H. $\{h_1, h_2, \ldots\}$ will be called the ordered set of distinguished elements of the structure. It is easy to see that p and m_1 such as in (4°) are uniquely determined by the structure. p and $m = p!\, m_1$ will be called the degree and the multiplicity of the structure. The ordered pair (m, p) will be called the characteristic of the structure.

By a (F)-vector space, we mean a vector space in which a definite (F)-structure is imposed. As far as no confusion can occur, we usually denote by the same symbol, say H, a (F)-vector space as well as its underlying vector space. When we want to make explicit the (F)-structure of a (F)-vector space, a (F)-vector space will be denoted by $(H, H^{(l)}, B^{(l)}, \mathbf{h}^{(l)})$.

Let $(H', H'^{(l)}, B'^{(l)}, \mathbf{h}'^{(l)})$ be a (F)-vector space. Then $(\mathbf{h}^{(l)}, 0)$ and $(0, \mathbf{h}'^{(l)})$, in this order, form an ordered basis, say $\mathbf{h}''^{(l)}$, of $B^{(l)} + B'^{(l)}$, the direct sum of $B^{(l)}$ and $B'^{(l)}$. Then we can check that the collection $(H^{(l)} + H'^{(l)}, B^{(l)} + B'^{(l)}, \mathbf{h}''^{(l)})$ forms a (F)-structure in $H+H'$. The (F)-vector space $(H+H', H^{(l)} + H'^{(l)}, B^{(l)} + B'^{(l)}, \mathbf{h}''^{(l)})$ will be called the direct sum of (F)-vector spaces H and H'. As far as any ambiguity may not occur, the direct sum of (F)-vector spaces H and H' will be denoted by $H+H'$.

Proposition I. 1. *Let (m, p) and (m', p') be the characteristics of H and H', respectively. If $p \gneqq p'$, then the characteristic of $H+H'$ is (m, p). If $p = p'$, the characteristic of $H+H'$ is $(m+m', p)$.*

Example 1. Let H be a vector space of dimension finite, say m. Set $H^{(0)} = H$, $H^{(l)} = \{0\}$ for $l \geq 1$, $B^{(0)} = H$, and $B^{(l)} = \{0\}$ for $l \geq 1$. Let $\mathbf{h}^{(0)}$ be an ordered basis of the vector space H. We regard $\mathbf{h}^{(l)}$ as the empty set for $l \geq 1$. The collection $(H, H^{(l)}, B^{(l)}, \mathbf{h}^{(l)})$ is a (F)-vector space of characteristic $(m, 0)$. It is easy to see that any (F)-structure introduced in H is of characteristic $(m, 0)$. Conversely, if a (F)-vector space is of characteristic $(m, 0)$;

then its underlying vector space is a m-dimensional vector space.

EXAMPLE 2. Let H_p be the vector space of formal power series in p indeterminates $x_1, \ldots, x_p$ with coefficients in K. When p is zero, H_p means the 1-dimensional vector space over K. Denote by $H_p^{(l)}$ the vector subspace of all elements in H_p such that their terms of degree less than l, not including terms of degree strictly l, are zero. Denote by $B_p^{(l)}$ the vector space of homogeneous polynomials of degree l. Monomials in $x_1, \ldots, x_p$ of degree l, ordered lexicographically, form an ordered basis $\mathbf{h}_p^{(l)}$ of $B^{(l)}$. Because the dimension of $H_p/H_p^{(l)}$ is equal to $(p!)^{-1}l\cdot(l+1)\cdots(l+p-1)$, $(H_p, H_p^{(l)}, B_p^{(l)}, \mathbf{h}_p^{(l)})$ forms a (F)-vector space of characteristic $(1, p)$, which we shall denote also by H_p for simplicity. Denote by H_p^s the direct sum of s copies of the (F)-vector space H_p. The characteristic of H_p^s is (s, p).

DEFINITION I. 2. *By a system S of characters we mean an ordered set of a finite number of non-negative integers $s_0, s_1, \ldots, s_p$, where $s_p \neq 0$. Denote by $H(S)$ the direct sum of $H_0^{s_0}, H_1^{s_1}, \ldots$, and $H_p^{s_p}$. By Proposition I. 1, $H(S)$ is a (F)-vector space of characteristic (s_p, p).*

DEFINITION I. 3. *Let F be a linear mapping of H into H'. F is called an analytic linear mapping if and only if there is an integer k such that the image of $H^{(l)}$ by F is in $H'^{(l-k)}$ for sufficiently large l.*

Thus, if the degree of H is zero, any linear mapping of H into H' is analytic. Let H'' be another (F)-vector space. If G is an analytic linear mapping of H' into H'', it is clear that the composition $G\circ F$ is again an analytic linear mapping of H into H''. H and H' are said to be isomorphic if there are analytic linear mappings F and G of H into H' and of H' into H, respectively, such that $G\circ F$ and $F\circ G$ are the identity mappings of H and H', respectively. Thus, what is essential in the definition of the (F)-vector space is the filtration $H^{(l)}$ of H. The homogeneous elements and the distinguished elements are added in the definition, in order to make the description of the later development easier.

PROPOSITION I. 2. *Let H and H' be (F)-vector spaces of characteristics (m, p) and (m', p'). Denote by F an analytic linear mapping of H into H'. Assume that F is bijective. Then either $p \gneqq p'$, or $p = p'$ and $m \geqq m'$.*

Proof. Identify the vector space H with the vector space H' by the mapping F. Then $H^{(l)} \subseteq H'^{(l-k)}$. Choosing k sufficiently large, we find that $(l+k)^p/(l-k)^{p'} \geq (p!\,m')/(p'!\,m)$ for large l. Then the conclusion of our proposition follows immediately by letting l tend to ∞.

THEOREM I. 1. *Let H and H' be (F)-vector spaces. Then H and H' are isomorphic if and only if they have the same characteristic.*

Proof. By Proposition I. 2., it is clear that if H and H' are isomorphic then they have the same characteristic. Let $\{h_1, h_2, \ldots\}$ and $\{h'_1, h'_2, \ldots\}$ be the ordered set of distinguished elements of H and H' respectively. We define linear mappings F and G of H into H' and of H' into H, respectively, as follows:

$$F(\sum_{j=1}^{\infty} b^j h_j) = \sum_{j=1}^{\infty} b^j h'_j,\ G(\sum_{j=1}^{\infty} b^j h'_j) = \sum_{j=1}^{\infty} b^j h_j,\ (b^j \in K).$$

Clearly $G \circ F$ and $F \circ G$ are the identity mappings. Because H and H' have the same characteristic, the condition (4°) implies that F and G are analytic.

In particular, if $S = \{s_0, s_1, \ldots, s_p\}$, $s_p \neq 0$, then $H(S)$ is isomorphic to $H_p^{s_p}$. Let F be a linear mapping of $H+H$ onto H defined by the formula: $F((\sum_{n=0}^{\infty} a_n x_1^n, \sum_{n=0}^{\infty} b_n x_1^n)) = \sum_{n=0}^{\infty} (a_n x_1^{2n} + b_n x_1^{2n+1})$. F is a bijective and analytic linear mapping. However, F^{-1} is not analytic.

DEFINITION I. 4. *A decreasing sequence $\{C^{(l)}\}$ of vector subspaces of H is called an admissible filtratisn of H if there is an integer k such that $H^{(l-k)} \supseteq C^{(l)} \supseteq H^{(l+k)}$ for sufficiently large l.*

PROPOSITION I. 3. *Let F be an analytic linear mapping of H into H'. Set $C^{(l)} = \{\xi \in H;\ F(\xi) \in H'^{(l)}\}$. If there is an analytic linear mapping F' of H' into H such that $F' \circ F$ is the identity mapping of H, then $\{C^{(l)}\}$ is an admissible filtration of H.*

The proof is easy.

A method of constructing a (F)-vector space is as follows: Let $A^{(l)}$, $(l = 0, 1, \ldots)$, be a sequence of finite dimensional vector spaces. Set $d'^{(l)} = \dim A^{(l)}$, $n'^{(l)} = d'^{(0)} + \cdots + d'^{(l)}$. Assume that we can find integers p', k' and a real number m'_1 such that

$$(1) \qquad m'_1(l-k')^{p'} \leq n'^{(l)} \leq m'_1(l+k')^{p'}.$$

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Let H' be the set of all sequences $\alpha=\{a^l\}_{l=0,1,\ldots}$ such that $a^l\in A^{(l)}$. H' forms a vector space by the obvious addition and multiplication by scalars, and $A^{(l)}$ will be identified, by the obvious injection, with the vector subspace of all sequences $\alpha=\{a^i\}$ such that $a^i=0$ for $i\neq l$. Let $H'^{(l+1)}$ be the vector subspace of all sequences $\alpha=\{a^i\}$ such that $a^i=0$ for $i=0,\ldots,l$. We set $H'^{(0)}=H'$. If we choose an ordered basis $\mathbf{h}'^{(l)}$ of $A^{(l)}$ for each l, $(H', H'^{(l)}, A^{(l)}, \mathbf{h}'^{(l)})$ forms a (F)-vector space of characteristic $(p'!\, m_1', p')$. We call this the (F)-vector space generated by $A^{(l)}$ and $\mathbf{h}'^{(l)}$. Now, we apply the above construction in the following case: For a given $(H, H^{(l)}, B^{(l)}, \mathbf{h}^{(l)})$ of characteristic $(p!\, m_1, p)$, take a copy $B^{(l),r}$ of $B^{(l)}$ for each strictly positive integer r, and let $A^{(l)}$ be the direct sum of $B^{(l-r),r}$ for $r=1,2,\ldots,l$, provided $l\geq 1$. We set $A^{(0)}=\{0\}$. We will show that the requirement (1) is satisfied with $m_1'=(p+1)^{-1}m_1$, $p'=p+1$, and with sufficiently large k'. Since $A^{(l)}$ is isomorphic with $H/H^{(l)}$, we have the inequality $m_1(l-k)^p\leq d'^{(l)}\leq m_1(l+k)^p$ for large l. Hence for sufficiently large l, we have $m_1(\sum_{j=0}^{l-(k+1)} j^p)\leq n'^{(l)}\leq m_1(\sum_{j=0}^{l+(k+1)} j^p)$. Because of the inequalities: $C^j_{j-p}\leq(p!)^{-1}j^p$ for $j\geq p$, $(p!)^{-1}j^p\leq C^{j+p}_j$, $\sum_{j=0}^{l} C^{j+p}_j=C^{l+p+1}_{p+1}$, where C^i_j are binomial coefficients, we have $p!\, m_1 C^{l-k}_{p+1}\leq n'^{(l)}\leq p!\, m_1 C^{l+k+p+2}_{p+1}$. Therefore, there exists k' such that $m_1'(l-k')^{p+1}\leq n'^{(l)}\leq m_1'(l+k')^{p+1}$ for large l. Thus (1) is satisfied. Denote by $\mathbf{h}^{(l),r}$ the copy of $\mathbf{h}^{(l)}$ in $B^{(l),r}$. Let $\mathbf{h}'^{(l)}$ be the ordered basis $\mathbf{h}^{(l-1),1}, \mathbf{h}^{(l-2),2},\ldots,\mathbf{h}^{(0),l}$, in this order, of $A^{(l)}$. Denote by ${}^cH$ the (F)-vector space generated by $A^{(l)}$ and $\mathbf{h}'^{(l)}$. In the following, we will constantly use the following notations to express elements in ${}^cH$: Introducing an indeterminate t, we denote by ξt^r the copy of $\xi\in B^{(l)}$ in $B^{(l),r}$. $B^{(l),r}$ being identified with the vector subspace of ${}^cH$, any element α in ${}^cH$ can be uniquely expressed as a formal sum: $\alpha=\sum_l(\sum_{r=1}^{l}\xi^{l-r,r}t^r)$, where $\xi^{l,r}\in B^{(l)}$. Because any element in H can be expressed uniquely as a formal sum of homogeneous elements, we may also express α without ambiguity as $\alpha=\sum_{r=1}^{\infty}\xi_r t^r$, where $\xi_r=\sum_l\xi^{l,r}\in H$. If $\mathbf{h}^{(l)}=\{h_1^l,\ldots,h^l_{d^l}\}$ and $\xi^{l,r}=\sum_{i=1}^{d^l}b^{l,r,i}h_i^l$, where $b^{l,r,i}\in K$, then we may also use the expression $\alpha=\sum_{l,i}\alpha^{i,l}h_i^l$, where $\alpha^{i,l}=\sum_{r=1}^{\infty}b^{l,r,i}t^r$ are formal power series in t without constant terms. Conversely, expressions as $\sum_{r=1}^{\infty}\xi_r t^r$, $\sum_{l,i}\alpha^{i,l}h_i^l$, and $\sum_j\alpha^j h_j$, where $\xi_r\in H$, $\alpha^{i,l}$ and α^j are formal power series in t

without constant terms, and where $\{h_1, h_2, \dots\}$ is the ordered set of distinguished elements in H, represent elements in ${}^cH$.

DEFINITION I. 5. *${}^cH$ will be called the (F)-vector space of formal curves in the (F)-vector space H. If we use indeterminate t to express elements in ${}^cH$ as above, t will be called a parameter of curves in H. For any $\alpha = \sum_{r=1}^{\infty} \xi_r t^r$ in ${}^cH$, we set $(\partial\alpha/\partial t)_{t=0} = \xi_1 \in H$.*

We proved already the following:

PROPOSITION I. 4. *If H is of characteristic (m, p), ${}^cH$ is of characteristic $(m, p+1)$.*

Let us use the same parameter t for curves in H, H', and $H+H'$. Then elements in ${}^cH$, ${}^cH'$, and ${}^c(H+H')$ will be expressed as $\sum_{r=1}^{\infty} \xi_r t^r$, $\sum_{r=1}^{\infty} \zeta_r t^r$, and $\sum_{r=1}^{\infty} (\xi_r, \zeta_r)\, t^r$, where $\xi_r \in H$ and $\zeta_r \in H'$, respectively. Then the mapping $\sum_{r=1}^{\infty} (\xi_r, \zeta_r)\, t^r \to (\sum_{r=1}^{\infty} \xi_r t^r, \sum_{r=1}^{\infty} \zeta_r t^r)$ of ${}^c(H+H')$ to ${}^cH + {}^c(H')$ is bijective and is the isomorphism of the (F)-vector spaces. As far as no confusion can occur, we will identify ${}^c(H+H')$ with ${}^cH + {}^cH'$ by the above mapping.

For any analytic linear mapping F of H into H', we associate a linear mapping ${}^cF$ of ${}^cH$ into ${}^cH'$ as follows:

$$ {}^cF(\sum_{r=1}^{\infty} \xi_r t^r) = \sum_{r=1}^{\infty} F(\xi_r) t^r, $$

where we use the same parameter t to express elements in ${}^cH$ and ${}^cH'$. It is easy to see that ${}^cF$ is also an analytic linear mapping.

2. Formal analytic mapping

Let H be a (F)-vector space. Let $\{h_1, h_2, \dots\}$ be the ordered set of distinguished elements of H. If the degree of h_j is l, we set $|j|_H = l$. Let us introduce an indeterminate a_H^j for each h_j. Denote by $I(H)$ the set of all a_H^j. Denote by $K[H]$ the ring of polynomials in the indeterminates in $I(H)$. The polynomials in the indeterminates a_H^j, where $|j|_H \leq l$, form a subring $K[H, l]$ and $K[H]$ is the sum of $K[H, l]$, $l = 0, 1, \dots$. The above notations will be used throughout the present paper and we may drop index H if no confusion can occur. Let H' be another (F)-vector space.

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Definition I. 6. *By a formal analytic mapping F of H into H', we understand that, for each integer $m = 1, 2, \ldots$, and for each indeterminate $a^i_{H'}$ in $I(H')$, there is assigned a polynomial $F^i_m \in K[H]$ satisfying the following conditions for an integer k: F^i_m is a homogeneous polynomial of degree m and of weight $\leq |i|_{H'} + km$, i.e. F^i_m is a linear combination of terms $a^{j_1}_H \cdots a^{j_m}_H$, where $|j_1|_H + \cdots + |j_m|_H \leq |i|_{H'} + km$. The integer k will be called a degree of F.*

Thus F may be considered as a collection $\{F^i_m\}$ of elements in $K[H]$, satisfying the above conditions. We may use expressions such as $F = \{F^i_m ; a^i_{H'} \in I(H)\}$, or $F = \{F^i_m\}$ to denote a formal analytic mapping of H into H'. Let $M \in K[H]$. For any $\xi = b^1 h_1 + b^2 h_2 + \cdots$, $b^j \in K$, in H, denote by $M(\xi)$ the value of the polynomial M at $a^j_H = b^j$. Let $\{h'_1, h'_2, \ldots\}$ be the ordered set of distinguished elements in H'. What the writer has in mind, by saying that $F = \{F^i_m\}$ is a formal analytic mapping of H into H', is the formal mapping:

$$\xi \to \sum_i \Big(\sum_{m=1}^{\infty} F^i_m(\xi)\Big) h'_i, \tag{2}$$

without caring whether or not the summations in the parenthesises have any meanings. In the case K has a topological structure, we say that F is defined at ξ (with respect to the topological structure), if $\sum_{m=1}^{\infty} F^i_m(\xi)$ converges to a limit, say $F^i(\xi)$, for each i. In this case, $\sum_i F^i(\xi) h'_i$ is called the value of F at ξ, and will be denoted by $F(\xi)$. F is defined everywhere with respect to the discrete topology of K, if, for each i, F^i_m is the zero polynomial for sufficiently large m. In this case, F is completely determined by the mapping: $\xi \to F(\xi)$. So there will be no confusion even if we mean by F, when $F^i_m = 0$ for sufficiently large m for each i, either the collection $\{F^i_m\}$ of polynomials or the mapping: $\xi \to F(\xi)$, according to the context. We hope that this convention does not arouse any confusion.

Let us consider the case when H is of characteristic $(s, 0)$ and H' of $(1, 0)$. Then the associated system $I(H)$ of indeterminates consists of s elements $X^1 = a^1_H, \ldots, X^s = a^s_H$, and $I(H')$ consists of a single element. Hence a formal analytic mapping F of H into H' is a sequence of homogeneous polynomials $F_m(X^1, \ldots, X^s)$, $m = 1, 2, \ldots$, of degree m. Conversely, it is easy to see that any such sequence comes from a formal analytic mapping of H into H'. In this way, F can be identified with the formal power series $\sum_{m=1}^{\infty} F_m(X^1, \ldots,$

X^s) without the constant term.

Definition I. 7. *A formal analytic mapping* $F=\{F^i_m\}$ *of* H *into* H' *is called linear if* $F^i_m=0$ *for* $m \gneqq 1$.

In this case F is defined everywhere in H with respect to the discrete topology of K and, as a mapping, is an analytic linear mapping of H into H'. Conversely, for any analytic linear mapping F' of H into H', it is easy to see that there is a unique linear formal analytic mapping F of H into H' such that F, considered as a mapping, is equal to F'. In this sense, the notion of analytic linear mappings is equal to that of linear formal analytic mappings.

Let M be an element of $K[H]$. For any $\alpha=\sum_j \alpha^j h_j$ in ${}^cH$, where α^j are formal power series in t without the constant term, and where $\{h_1, h_2, \ldots\}$ is the ordered set of distinguished elements, denote by $M(\alpha)$ the formal power series in t obtained by replacing the indeterminates a^j_H by α^j. Let $F=\{F^i_m\}$ be a formal analytic mapping of H into H'. We now associate a mapping $({}^cF)'$ of ${}^cH$ into ${}^c(H')$ as follows: The value of $({}^cF)'$ at α is equal to

$$\sum_i \Big(\sum_{m=1}^{\infty} F^i_m(\alpha)\Big) h'_i, \tag{3}$$

where $\{h'_1, h'_2, \ldots\}$ is the ordered set of distinguished elements in H'. By Definition I. 6., $F^i_m(\alpha)$ are divisible by t^m because α^j are divisible by t. Therefore the summations in the parenthesises in (3) are defined as formal power series in t without the constant terms, and so (3) represents a unique elements in ${}^c(H')$. It is not hard to see that there is a unique formal analytic mapping ${}^cF$ of ${}^cH$ into ${}^c(H')$ such that ${}^cF$ is defined everywhere in ${}^cH$ with respect to the discrete topology of K and such that ${}^cF$, as a mapping, is equal to $({}^cF)'$. According to the convention made before, $({}^cF)'$ will be also denoted by ${}^cF$. ${}^cF$ is called the mapping of curves in H associated with the formal analytic mapping F. If F' is a formal analytic mapping of H into H' such that ${}^cF={}^c(F')$, then it is clear that $F=F'$. Thus ${}^cF$ completely determines F.

Let H'' be another (F)-vector space. Let G be a formal analytic mapping of H' into H''. We claim that there is a unique formal analytic mapping L of H into H'' such that ${}^cL$, as a mapping, is equal to the composition of the mappings ${}^cF$ and ${}^cG$. L may be defined by constructing formally the composition of the formal mappings (2). L will be called the composition of F and

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G, and will be denoted by $G\circ F$. In order to define L explicitly, we introduce the following notation: Let M be a polynomial in indeterminates $x_1, \ldots, x_r, t$, where r may be zero. M can be expressed as $M=\sum_n M_n t^n$, where M_n are polynomials in $x_1, \ldots, x_r$. We set $\kappa_n^t[M]=M_n$. Now, L is defined by the following formula: For each $a_{H''}^j$ in $I(H'')$ and for each $m\geq 1$, set

$$L_m^j=\sum_{n=1}^m \kappa_m^t[G_n^j(\ldots, \sum_{q=1}^m t^q F_q^i, \ldots)], \tag{4}$$

where the formula inside [ ] means the polynomial obtained by replacing the indeterminates $a_{H'}^i$ in G_n^j by $\sum_{q=1}^m t^q F_q^i$. Let k' be a degree of G. Then L_m^j is a linear combination with coefficients in K of terms

$$F_{q_1}^{i_1}\ldots F_{q_n}^{i_n}, \tag{5}$$

where $|i_1|_{H'}+\cdots+|i_n|_{H'}\leq |j|_{H''}+k'n$, $1\leq n\leq m$, and where $q_1+\cdots+q_n=m$. Then it is easy to verify that $\{L_m^j\}$ forms a formal analytic mapping of H into H'' of degree $\leq k+k'$. We can verify by direct calculation that the equality: ${}^cL={}^cG\circ{}^cF$ holds. If F and G are defined everywhere in H and H', respectively, with respect to the discrete topology of K, then so is $G\circ F$, and $G\circ F$ considered as a mapping is equal to the composition of the mappings F and G.

Denote by i_1 and i_2 the canonical injection of H to the first and the second components of $H+H$, respectively. Then $\{i_1h_1, i_1h_2, \ldots, i_2h_1, i_2h_2, \ldots\}$ is the complete set of distinguished elements in $H+H$. Denote by a'^j, a''^j the indeterminates associated with i_1h_j, i_2h_j, respectively. Hence elements M of $K[H+H]$ are polynomials in $\ldots, a'^j, \ldots, \ldots, a''^j, \ldots$. Now we associate for any formal analytic mapping F of H into H' a formal analytic mapping dF of $H+H$ into H' as follows: For each $a_{H'}^i$ in $I(H')$ and for each $m\geq 1$, set

$$(dF)_m^i=\sum_j (\partial F_m^i/\partial a_H^j)' a''^j \tag{6}$$

where ( )$'$ means that we substitute a'^j for a_H^j. It is easy to see that the collection $dF=\{(dF)_m^i\}$ forms a formal analytic mapping of $H+H$ into H'. If F is defined everywhere with respective to the discrete topology of K, so is dF. Therefore, in this case, dF is also considered as a mapping. Identifying ${}^c(H+H)$ with ${}^cH+{}^cH$ by the canonically isomorphism, we have the equality:

${}^c dF = d^c F$. Let G be a formal analytic mapping of H' into H''. Then for any α and β in ${}^c H$,

(7) $$[{}^c d(G \circ F)](\alpha, \beta) = [{}^c dG]({}^c F(\alpha), [{}^c dF](\alpha, \beta)).$$

Denote by $({}^c dF)_0$ the mapping of ${}^c H$ into ${}^c(H')$ defined by the formula: $({}^c dF)_0(\alpha) = [{}^c dF](0, \alpha)$. Then it is easily seen that $({}^c dF)_0$ is the mapping of curves in H associated with a formal analytic mapping $(dF)_0$ of H into H'. Namely, $(dF)_0 = L$ is the collection $\{L^j_m\}$ such that $L^j_1 = F^j_1$, $L^j_m = 0$ for $m \geq 2$. Thus $(dF)_0$ is linear.

DEFINITION I. 8. *dF is called the differential of F. $(dF)_0$ is called the differential of F at the origin.*

3. Jacobians and differential equations

THEOREM I. 2. *Let F be a formal analytic mapping of H into itself. Assume that $(dF)_0$ is the identity mapping of H. Then there is a unique formal analytic mapping G of H into itself such that both $G \circ F$ and $F \circ G$ are equal to the identity mapping of H.*

Proof. For each $a^j \in I(H)$, we set $G^j_1 = a^j \in K[H]$. Assuming that $G^j_1, \ldots, G^j_{m-1} \in K[H]$ are constructed, we set

$$G^j_m = -\sum_{n=1}^{m} \kappa^t_m [F^j_n(\ldots, \sum_{q=1}^{m-1} t^q G^i_q, \ldots)]$$

First we shall show that the collection $G = \{G^j_m;\ a^j \in I(H),\ m = 1, 2, \ldots\}$ forms a formal analytic mapping of H into itself. Since $F = \{F^j_m\}$ is a formal analytic mapping, we remark that G^j_m, $m \geq 2$, is a linear combination of terms:

(8) $$G^{i_1}_{q_1} \cdots G^{i_n}_{q_n}$$

where

(9) $$|i_1|_H + \cdots + |i_n|_H \leq |j|_H + kn,$$
$$q_1 + \cdots + q_n = m,\ 1 \leq q_r \leq m-1,\ r = 1, \ldots, n.$$

Hence, n in (9) must be ≥ 2. We proceed by induction on m. It is clear that G^j_m is a homogeneous polynomial of degree m. We can assume without loss of generality that $k \geq 0$. Assuming that G^i_q is of weight $\leq |i|_H + 2k(q-1)$ for $q < m$ and for any a^i in $I(H)$, we claim that G^j_m is of weight $= N \leq |j|_H$

$+2k(m-1)$. Namely, (8) and (9) imply that $N \le \sum_{r=1}^{n}(|i_r|_H + 2k(q_r - 1)) \le |j|_H + kn + 2k(m-n) = |j|_H + 2km - kn \le |j|_H + 2k(m-1)$, because $2 \le n$. Thus G is a formal analytic mapping of a degree $2k$. By explicitly calculating $F \circ G$, it is easy to see that $F \circ G$ is the identity mapping. Since $(dG)_0$ is the identity mapping, there is a formal analytic mapping F' such that $G \circ F'$ is the identity mapping. Therefore ${}^cG$ is a bijective endomorphism of ${}^cH$ and both ${}^cF$ and ${}^cF'$ must be the inverse mapping of ${}^cG$. Hence $F = F'$. Thus we find that $G \circ F$ is also the identity mapping.

THEOREM I. 3. *Let Y be a formal analytic mapping of $H' + H$ into H. Assume that ${}^cY(\alpha, \beta)$, where $\alpha \in {}^c(H')$ and $\beta \in {}^cH$, is linear with respect to the variable β. Assume also that ${}^cY(0, \beta) = \beta$ for any β in ${}^cH$. Then there is a formal analytic mapping F of $H' + H$ into H such that ${}^cY(\alpha, {}^cF(\alpha, \beta)) = {}^cF(\alpha, {}^cY(\alpha, \beta)) = \beta$ for any α in ${}^c(H')$ and for any β in ${}^cH$.*

Proof. We claim that there is a formal analytic mapping Y^* of $H' + H$ into itself such that ${}^c(Y^*)(\alpha, \beta) = (\alpha, {}^cY(\alpha, \beta))$. Namely, denote by G the everywhere defined formal analytic mapping $(\xi, \eta) \in H' + H \to (\xi, \xi, \eta) \in H' + H' + H$, and by G' the direct sum of the identity mapping of H' and Y, then $Y^* = G' \circ G$ is the required one. It is easy to verify that $(dY^*)_0$ is the identity mapping. By Theorem I. 2, there is the inverse F^* of Y^*. Let F' be the canonical projection of $H' + H$ to H. F' is an everywhere defined formal analytic mapping. Then $F' \circ F^* = F$ is the required one.

In the finite dimensional case, the following theorem is equivalent to the existence theorem of solutions for the system of ordinary differential equations of the first order depending on parameters.

THEOREM I. 4. *Assume that the characteristic of K is zero. Let Y be a formal analytic mapping of $H' + H$ into H'. Assume that ${}^cY(\alpha, 0) = 0$ for any α in ${}^c(H')$. Then there is a unique formal analytic mapping F of $H' + H$ into H' such that*

$$d^cF((\alpha, \beta), (0, \beta)) = {}^cY({}^cF(\alpha, \beta), \beta),$$
$${}^cF(\alpha, 0) = \alpha$$

for any α in ${}^c(H')$ and β in ${}^cH$.

Proof. We naturally identify $I(H)$ and $I(H')$ with subsets of $I(H' + H)$,

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Thus, $K[H]$, $K[H'] \subseteq K[H'+H]$. Now we define polynomials $F^i_{v,w}$, where $a^i \in I(H')$ and v, w are non-negative integers such that $v+w \geq 1$, in $K[H'+H]$ by means of induction on $m = v+w$ as follows:

$$F^i_{1,0} = a^i, \quad F^i_{0,1} = Y^i_1, \quad F^i_{m,0} = 0 \quad \text{for } m \geq 2,$$

$$(10) \qquad F^i_{v,w} = \text{the coefficient of } u^v t^w \text{ in}$$

$$\sum_{n=1}^{m} \frac{1}{w} Y^i_n(\ldots, \sum_{q+r=1}^{m-1} u^q t^r F^j_{q,r}, \ldots; \ldots, tb^l, \ldots)$$

for $v+w \geq 2$, $w \geq 1$, where $a^j \in I(H')$ and $b^l \in I(H)$. Set $F^i_m = \sum_{v+w=m} F^i_{v,w}$. Since ${}^cY(\alpha, 0) = 0$, Y^i_m are in the ideal generated by $I(H)$. In particular, Y^i_1 depend only in the indeterminates in $I(H)$, since it is linear. Therefore we may replace $\sum_{n=1}^{m}$ in (10) by $\sum_{n=2}^{m}$ without affecting the results. Then the similar arguments as in the proof of Theorem I. 2, imply that the collection $F = \{F^i_m\}$ is a formal analytic mapping of $H'+H$ into H. Because Y^i_n are in the ideal generated by $I(H)$, we can replace $\sum_{q+r=1}^{m-1}$ in (10) by $\sum_{q+r=1}^{m}$. Then it is easy to verify that F is the required and unique one.

Similarly, we have the following.

THEOREM I. 5. *Under the same assumption as in Theorem* I. 4, *there is a unique formal analytic mapping F of H into H' such that*

$$d^cF(\beta, \beta) = {}^cY({}^cF(\beta), \beta)$$

for any β in ${}^cH$.

Assume that the characteristic of K is zero. Let $F = \{F^i_m\}$ be a formal analytic mapping of H into H'. For any u in K and α, β in ${}^cH$, ${}^cF(\alpha+u\beta) = \sum_i (\sum_{m=1}^{\infty} F^i_m(\alpha+u\beta))h'_i$, where $\{h'_1, h'_2, \ldots\}$ is the ordered set of distinguished elements of H'. Then $F^i_m(\alpha+u\beta) = F^i_m(\alpha) + u \cdot F^i_{m,1}(\alpha, \beta) + \cdots + u^r F^i_{m,r}(\alpha, \beta) + \cdots + u^m F^i_{m,m}(\beta)$. We set $\partial/\partial u\ {}^cF(\alpha+u\beta) = \sum_i (\sum_{m=1}^{\infty} (\sum_{r=1}^{m} r u^{r-1} F^i_{m,r}(\alpha, \beta)))h'_i$. By definition, $(\partial/\partial u\ {}^cF(\alpha+u\beta))_{u=0} = {}^cdF(\alpha, \beta) = d^cF(\alpha, \beta)$. The usual rules of partial derivatives hold for this operator $\partial/\partial u$. For instance, if $H = H_1 + H_2$, and $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, then $(\partial/\partial u\ {}^cF(\alpha_1+u\beta_1, \alpha_2+u\beta_2))_{u=v} = (\partial/\partial u\ {}^cF(\alpha_1+u\beta_1, \alpha_2+v\beta_2))_{u=v} + (\partial/\partial u\ {}^cF(\alpha_1+v\beta_1, \alpha_2+u\beta_2))_{u=v}$. If $F^i_m = G^i_m + L^i_m$, where $G = \{G^i_m\}$ and $L = \{L^i_m\}$ are supposed to be formal analytic mappings, then $\partial/\partial u\ {}^cF(\alpha+u\beta) = \partial/\partial u\ {}^cG(\alpha+u\beta) + \partial/\partial u\ {}^cL(\alpha+u\beta)$. If $\partial/\partial u\ {}^cF(\alpha+u\beta) = 0$ for any u in K, then ${}^cF(\alpha+u\beta) = {}^cF(\alpha)$ for any u in K. In particular, if $\partial/\partial u\ {}^cF(u\alpha)=0$ then ${}^cF(\alpha)=0$.

LEMMA I. 1. *Assume that the characteristic of K is zero. Take (F)-vector spaces H_1, H_2, and H_3. Denote by F', G' formal analytic mappings of H_1+H_2 into H_3. Let Y be a formal analytic mapping of $H_3+H_1+H_2+H_2$ into H_3. Fix elements α and β in ${}^cH_1$ and in ${}^cH_2$, respectively. Assume that we have equalities: $\partial/\partial u\ {}^cF'(\alpha, u\beta)={}^cY({}^cF'(\alpha, u\beta), \alpha, u\beta, \beta)$ and $\partial/\partial u\ {}^cG'(\alpha, u\beta)={}^cY({}^cG'(\alpha, u\beta), \alpha, u\beta, \beta)$. If, moreover, ${}^cF'(\alpha, 0)={}^cG'(\alpha, 0)$, then ${}^cF'(\alpha, u\beta)={}^cG'(\alpha, u\beta)$ for any $u\in K$.*

Proof. As we can see easily by the arguments at the beginning of the preceding paragraph, ${}^cF'(\alpha, u\beta)=\sum_{l=0}^{\infty} u^l F'_l(\alpha, \beta)$, where $F'_0(\alpha, \beta)={}^cF'(\alpha, 0)$ and $F'_l(\alpha, \beta)$ are divisible by t^l, t^l being the parameter of curves in H_1+H_2. By the definition, $\partial/\partial u\ {}^cF'(\alpha, u\beta)=\sum_{l=1}^{\infty} lu^{l-1}F'_l(\alpha, \beta)$. Then the equality: $\partial/\partial u\ {}^cF'(\alpha, u\beta)={}^cY({}^cF'(\alpha, u\beta), \alpha, u\beta, \beta)$ implies that $F'_l(\alpha, \beta)$ is determined inductively, starting from $F'_0(\alpha, \beta)$, by the formulas entirely determined by Y, α, and by β. Therefore, ${}^cF'(\alpha, u\beta)={}^cG'(\alpha, u\beta)$ for any $u\in K$.

By the similar argument, we prove the following:

LEMMA I. 2. *H_1, H_2, H_3, F' and G' being as in Lemma I. 1, denote by Z a formal analytic mapping of $H_3+H_1+H_2$ into H_3. Fix $\alpha\in{}^cH_1$ and $\beta\in{}^cH_2$. Assume that we have the equalities: $u\ \partial/\partial u\ {}^cF'(\alpha, u\beta)+a{}^cF'(\alpha, u\beta)={}^cZ({}^cF'(\alpha, u\beta), \alpha, u\beta)$ and $u\ \partial/\partial u\ {}^cG'(\alpha, u\beta)+a{}^cG'(\alpha, u\beta)={}^cZ({}^cG'(\alpha, u\beta), \alpha, u\beta)$ for a strictly positive integer a. Assume further that there is a formal analytic mapping Z' of $H_3+H_1+H_2+H_2$ into H_3 such that ${}^cZ(\delta, \alpha, u\beta)=u\ {}^cZ'(\delta, \alpha, u\beta, \beta)$ for any δ in ${}^cH_3$. If, moreover, ${}^cF'(\alpha, 0)={}^cG'(\alpha, 0)$, then ${}^cF'(\alpha, u\beta)={}^cG'(\alpha, u\beta)$ for any u in K.*

Let Y be a formal analytic mapping of $H'+H+H$ into H'. Then it is easy to see that there are formal analytic mappings W and X of $H'+H+H+H$ and of $H'+H'+H+H$ into H' respectively, such that

$$ {}^cW(\alpha, \beta, \gamma, \delta)=[\partial/\partial u({}^cY(\alpha, \beta+u\gamma, \delta))]_{u=0}, $$
$$ {}^cX(\alpha, \alpha', \beta, \gamma)=[\partial/\partial u({}^cY(\alpha+u\alpha', \beta, \gamma))]_{u=0} $$

for any α, α' in ${}^c(H')$ and β, γ, δ in ${}^cH$. Under these notations we have the following:

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THEOREM I. 6. *Assume that we have the following equality:*

$$
\begin{aligned}
(11) \qquad & {}^cW(\alpha, \beta, \gamma, \delta) + {}^cX(\alpha, {}^cY(\alpha, \beta, \gamma), \beta, \delta) \\
& = {}^cW(\alpha, \beta, \delta, \gamma) + {}^cX(\alpha, {}^cY(\alpha, \beta, \delta), \beta, \gamma), \\
& \qquad\qquad (\alpha \in {}^c(H'), \beta, \gamma, \delta \in {}^cH).
\end{aligned}
$$

Suppose that ${}^cY(\alpha, \beta, \gamma)$ is linear with respect to the variable γ. Assume also that the characteristic of K is zero. Then there is a unique formal analytic mapping F of $H' + H$ into H' such that

$$
\begin{gathered}
d^cF((\alpha, \beta), (0, \gamma)) = {}^cY({}^cF(\alpha, \beta), \beta, \gamma) \\
{}^cF(\alpha, 0) = \alpha.
\end{gathered}
$$

Proof. By Theorem I. 4 there is a formal analytic mapping F of $H' + H$ into H' such that $d^cF((\alpha, \beta), (0, \beta)) = {}^cY({}^cF(\alpha, \beta), \beta, \beta)$ and such that ${}^cF(\alpha, 0) = \alpha$. We will show that F is the required one. Differentiating the equality: $d^cF((\alpha, \beta + u\gamma), (0, \beta + u\gamma)) = {}^cY({}^cF(\alpha, \beta + u\gamma), \beta + u\gamma, \beta + u\gamma)$ with respect to u, we find that

$$
\begin{aligned}
& d^cF((\alpha, \beta), (0, \gamma)) - {}^cY({}^cF(\alpha, \beta), \beta, \gamma) \\
(12) \qquad & = -[\partial/\partial u(d^cF((\alpha, \beta + u\gamma), (0, \beta)))]_{u=0} + {}^cW({}^cF(\alpha, \beta), \beta, \gamma, \beta) \\
& \quad + {}^cX({}^cF(\alpha, \beta), d^cF((\alpha, \beta), (0, \gamma)), \beta, \beta).
\end{aligned}
$$

Set $H_1 = H' + H$, $H_2 = H$, and $H_3 = H'$. Denote by F' the formal analytic mapping of $H_1 + H_2$ into H_3 such that ${}^cF'(\alpha, \gamma, \beta) = d^cF((\alpha, \beta), (0, \gamma)) - {}^cY({}^cF(\alpha, \beta), \beta, \gamma)$ for any (α, γ) in ${}^cH_1$ and β in ${}^cH_2$. Substituting ${}^cW({}^cF(\alpha, \beta), \beta, \gamma, \beta)$ in (12) by a sum obtained by (11), we find by the definition of F that

$$
\begin{aligned}
& d^cF'((\alpha, \gamma, \beta), (0, 0, \beta)) + {}^cF'(\alpha, \gamma, \beta) \\
& \qquad = {}^cX({}^cF(\alpha, \beta), {}^cF'(\alpha, \gamma, \beta), \beta, \beta).
\end{aligned}
$$

Hence,

$$
u\, \partial/\partial u\ {}^cF'((\alpha, \gamma, u\beta)) + F'((\alpha, \gamma, u\beta)) = {}^cZ({}^cF'(\alpha, \gamma, u\beta), \alpha, \gamma, u\beta),
$$

where Z is the formal analytic mapping of $H_3 + H_1 + H_2$ into H_3 such that ${}^cZ(\delta, \alpha, \gamma, \beta) = {}^cX({}^cF(\alpha, \beta), \delta, \beta, \beta)$. Since ${}^cY(\alpha, \beta, \gamma)$ is linear with respect to γ, so is ${}^cX(\alpha, \alpha', \beta, \gamma)$. Hence there is Z' such as in the assumption in Lemma 1. 2. Since $d\,{}^cF((\alpha, u\beta), (0, \beta)) - {}^cY({}^cF(\alpha, u\beta), u\beta, \beta) = 0$, it follows that ${}^cF'((\alpha, \gamma, 0)) = 0$. On the other hand, denoting by G' the zero mapping of $H_1 + H_2$ into H_3, we find that $u\, \partial/\partial u\ {}^cG'((\alpha, \gamma, u\beta)) + G'((\alpha, \gamma, u\beta)) = {}^cZ({}^cG'$

$(\alpha, \gamma, u\beta)$, $\alpha, \gamma, u\beta)$. Therefore, by Lemma 1. 2, we see that ${}^cF'(\alpha, \gamma, \beta) = 0$, i.e. $d^cF((\alpha, \dot{\beta}), (0, \gamma)) = {}^cY({}^cF(\alpha, \beta), \beta, \gamma)$.

4. Germs of analytic mappings

Throughout this section except in the last part we assume that K is the field of complex numbers with the topology of the euclidean distance. A formal power series is called convergent if its radius of convergence is strictly positive. Denote by $\mathscr{H}_p$ the vector space of convergent power series in $x_1, \ldots, x_p$. $\mathscr{H}_p$ is considered as a subspace of H_p. We set $\mathscr{H}_0 = K$. Let $\mathscr{H}_p^s$ be the direct sum of s copies of $\mathscr{H}_p$. For a system of characters $S = \{s_0, \ldots, s_p\}$, $s_p \neq 0$, (cf. Def. I. 2), set $\mathscr{H}(S) = \mathscr{H}_0^{s_0} + \cdots + \mathscr{H}_p^{s_p}$. For each $\lambda = s_0 + 1, \ldots, s_0 + s_1 + \cdots s_p$, the λ-th component ξ_λ of ξ in $H(S)$ is a convergent power series in $x_1, \ldots, x_{p(\lambda)}$. If $s_0 > 0$, for $\lambda = 1, \ldots, s_0$, ξ_λ is in K and we set $p(\lambda) = 0$. For any strictly positive numbers u and v, denote by $\mathscr{H}(S; u, v)$ the set of all elements ξ in $\mathscr{H}(S)$ such that, for a sufficiently small $\varepsilon > 0$ and for each λ, (i) the radius of convergence of $\xi_\lambda \geq u + \varepsilon$ and (ii) $|\xi_\lambda(x_1, \ldots, x_{p(\lambda)})| \leq v - \varepsilon$ for any $|x_r| < u$. Of course, ε depends on ξ. When $\mathscr{H}_p^s = \mathscr{H}(S)$, i.e. $s_0 = \cdots = s_{p-1} = 0$ and $s_p = s$, we set $\mathscr{H}_p^s(u, v) = \mathscr{H}(S; u, v)$. Let $C(a)$ be the disk in the complex plane of radius a with the origin as its center. A mapping f of $C(a)$ into $\mathscr{H}(S; u, v)$ is said to be a regular curve in $\mathscr{H}(S; u, v)$, if, for each $\lambda = 1, 2, \ldots, s_0 + s_1 + \cdots + s_p$, the function $h(x_1, \ldots, x_{p(\lambda)}, t) = [f_\lambda(t)](x_1, \ldots, x_{p(\lambda)})$ is holomorphic for $|x_r| < u$, $|t| < a$.

DEFINITION I. 9. *A mapping $\mathscr{F}$ of $\mathscr{H}(S; u, v)$ into $\mathscr{H}(S'; u', v')$ is called regular if $\mathscr{F}(0) = 0$ and if for any regular curve f in $\mathscr{H}(S; u, v)$, $\mathscr{F} \circ f$ is again a regular curve in $\mathscr{H}(S'; u', v')$.*

PROPOSITION I. 5. *Let $\mathscr{F}$ be a regular mapping of $\mathscr{H}(S; u, v)$ into $\mathscr{H}(S'; u', v')$. Then for any $\varepsilon > 0$ and $c < 1$, the image of $\mathscr{H}(S; u, cv)$ is in $\mathscr{H}(S'; u', c(v' + \varepsilon))$.*

Proof. Take ξ in $\mathscr{H}(S; u, v)$. For any fixed $x = (x_1, \ldots, x_{p(\lambda)})$, $|x_r| < u'$, $f(z) = [F(z\xi)]_\lambda(x)$ is holomorphic for $|z| < 1$, because $z \to z\xi$ is a regular curve. And $|f(z)| < v'$ for $|z| < 1$ and $f(0) = 0$. Hence by a theorem in the theory of complex functions, $|f(z)| \leq cv'$ for $|z| < c < 1$, For any ξ in $\mathscr{A}(S;\ u, v)$, set

$$|\xi|_u = \sup\{|\xi_\lambda(x)|;\ |x_r|<u,\ r\le p(\lambda),\ \lambda=1,\ldots,s_0+\cdots+s_p\}$$

PROPOSITION I. 6. *Let $\mathscr{F}$ be a regular mapping of $\mathscr{A}(S;\ u, v)$ into $\mathscr{A}(S';\ u', v')$. Then, if ξ, η are in $\mathscr{A}\left(S;\ u, \frac{v}{4}\right)$,*

$$|F(\xi)-F(\eta)|_{u'}\le 4\frac{v'}{v}\,|\xi-\eta|_u.$$

Proof. Set $\zeta=\xi-\eta$. Then $\xi-z\zeta\in\mathscr{A}(S;\ u, v)$ for $|z|<R=v\cdot\left(\frac{4}{3}|\zeta|_u\right)^{-1}$. Hence, if $|x_r|<u'$, then $[F(\xi-z\zeta)]_\lambda(x)=f(z)$ is holomorphic for $|z|<R$, $|f(z)|<v'$, and $f(0)=F(\xi)_\lambda(x)$. Since $\zeta\in\mathscr{A}\left(S;\ u, \frac{v}{2}\right)$, $1<R$. Therefore $|f(1)-f(0)|\le 4\frac{v'}{v}\,|\zeta|_u$.

We remark that $\mathscr{A}(S;\ u'', v'')\subseteq\mathscr{A}(S;\ u, v)\cap\mathscr{A}(S;\ u', v')$ for $u''\ge u$, u', and for $v''\le v$, v'. Let $\mathscr{F}_r$ be a regular mapping of $\mathscr{A}(S;\ u_r, v_r)$ into $\mathscr{A}(S';\ u'_r, v'_r)\subseteq\mathscr{A}(S')$, $(r=1, 2)$. We say that $\mathscr{F}_1$ and $\mathscr{F}_2$ are equivalent if there are $u\ge u_1$, u_2 and $v\le v_1$, v_2 such that the restrictions of $\mathscr{F}_1$ and $\mathscr{F}_2$ to $\mathscr{A}(S;\ u, v)$ are equal. Clearly this is an equivalence relation.

DEFINITION I. 10. *An equivalence class of regular mappings under the above relation is called a germ of regular mappings of $\mathscr{A}(S)$ into $\mathscr{A}(S')$.*

PROPOSITION I. 7. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be regular mappings of $\mathscr{A}(S;\ u, v)$ into $\mathscr{A}(S';\ u', v')$. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are equivalent, then $\mathscr{F}_1=\mathscr{F}_2$.*

Proof. Assume that $\mathscr{F}_1$ and $\mathscr{F}_2$ coincide on $\mathscr{A}(S;\ u^*, v^*)$, where $u^*\ge u$ and $v^*\le v$. Take ξ in $\mathscr{A}(S;\ u, v)$. Then there is $\varepsilon>0$ such that the radius of convergence of $\xi_\lambda\ge(1+2\varepsilon)u$ and $(1+2\varepsilon)|\xi|_u<v$. Let $f(z)$ be the mapping of $C(1+\varepsilon)$ in $\mathscr{A}(S)$ defined by the formula: $[f_\lambda(z)](x_1,\ldots,x_{p(\lambda)}) = z\cdot\xi_\lambda(zx_1,\ldots,zx_{p(\lambda)})$. $f(z)$ is a regular curve in $\mathscr{A}(S;\ u, v)$ and $f(z)$ is in $\mathscr{A}(S;\ u^*, v^*)$ for z sufficiently near the origin, say, $|z|<\delta$. Then $\mathscr{F}_1(f(z))=\mathscr{F}_2(f(z))$ for $|z|<\delta$. Hence by the theorem of coincidence $\mathscr{F}_1(f(1))=\mathscr{F}_2(f(1))$, i.e. $\mathscr{F}_1(\xi)=\mathscr{F}_2(\xi)$.

Let $\mathbf{F}$ be a germ of regular mappings of $\mathscr{A}(S)$ into $\mathscr{A}(S')$. We say that $\mathbf{F}$ is defined at ξ in $\mathscr{A}(S)$ if there is a representative $\mathscr{F}$ of the class $\mathbf{F}$ which is a regular mapping of $\mathscr{A}(S;\ u, v)$ containing ξ; $\mathscr{F}(\xi)$ is called the value of $\mathbf{F}$ at ξ and will be denoted by $\mathbf{F}(\xi)$. By Proposition I. 7 the value of **F** at ξ is independent of the choice of $\mathscr{H}(S;\ u,\ v)$ as above. Denote by $\mathscr{D}(\mathbf{F})$ the set of all ξ at which **F** is defined. Then **F** can be regarded as a mapping of $\mathscr{D}(\mathbf{F})$ into $\mathscr{H}(S')$.

Let **F** and **G** be germs of regular mappings of $\mathscr{H}(S)$ into $\mathscr{H}(S')$ and of $\mathscr{H}(S')$ into $\mathscr{H}(S'')$, respecrively. If there is a representative $\mathscr{F}$ of **F** which maps $\mathscr{H}(S;\ u,\ v)$ into $\mathscr{H}(S';\ u',\ v')$ and if there is a representative $\mathscr{G}$ of **G** which maps $\mathscr{H}(S';\ u',\ v')$ into $\mathscr{H}(S'';\ u'',\ v'')$, then we say that the composition of **F** and **G** is defined. It is clear that $\mathscr{G}\circ\mathscr{F}$ is a regular mapping of $\mathscr{H}(S;\ u,\ v)$ into $\mathscr{H}(S'';\ u'',\ v'')$; the germ which contains $\mathscr{G}\circ\mathscr{F}$ is called the composition of **F** and **G** and will be denoted by $\mathbf{G}\circ\mathbf{F}$. It is easy to verify that, if $\mathbf{G}\circ\mathbf{F}$ can be defined, $\mathbf{G}\circ\mathbf{F}$ does not depend on the choice of $\mathscr{F}$ and $\mathscr{G}$ such as above.

Definition I. 11. *A germ* **F** *of regular mappings of* $\mathscr{H}(S)$ *into* $\mathscr{H}(S')$ *is called a germ of analytic mappings of* $\mathscr{H}(S)$ *into* $\mathscr{H}(S')$, *if there are strictly positive numbers* u^*, v, v', w, *and an integer* k *such that for any* $u<u^*$ *there is a representative* $\mathscr{F}_u$ *of* **F** *which is a regular mapping of* $\mathscr{H}(S;\ u,\ u^kv)$ *into* $\mathscr{H}(S';\ wu,\ v')$. *The integer* k *is called a degree of* **F**.

Remark: When the degrees of $\mathscr{H}(S)$ and $\mathscr{H}(S')$ are zero, i.e. when $\mathscr{H}(S)$ and $\mathscr{H}(S')$ are finite dimensional, $\mathscr{H}(S;\ u,\ v)$ and $\mathscr{H}(S;\ u',\ v')$ are domains in $\mathscr{H}(S)$ and $\mathscr{H}(S')$, respectively, and regular mappings in our sense coincide with the usual regular mappings. In this case, any germ of regular mappings is a germ of analytic mappings and our definition coincides with the classical definition of germs of analytic mappings.

Proposition I. 8. *Let* **F** *and* **G** *be germs of analytic mappings of* $\mathscr{H}(S)$ *into* $\mathscr{H}(S')$ *and of* $\mathscr{H}(S')$ *into* $\mathscr{H}(S'')$, *respectively. Then the composition of* **F** *and* **G** *are always defined, and is a germ of analytic mappings of* $\mathscr{H}(S)$ *into* $\mathscr{H}(S'')$.

Proof. Keep the notations in Definition I. 11. Let u_1^*, v_1, v_1', w_1, and k_1 be constants for **G** such as in Definition I. 11. If w satisfies the conditions, then any $w'\leq w$ satisfies the same conditions. Similary we can replace k by any $k'\geq k$. So we can assume that $k_1\geq 0$ and $(u^*w)^{k_1}v_1\cdot(2v')^{-1}<1$. Set $v_2=w^{k_1}v_1v(2v')^{-1}$. Then by Proposition I. 5 there is a representative $\mathscr{F}'_u$ of **F** which is a regular mapping of $\mathscr{H}(S;\ u,\ u^{k+k_1}v_2)$ into $\mathscr{H}(S';\ wu,\ (wu)^{k_1}v_1)$ for $u<u^*$. Hence, if $u<u_2^*=\mathrm{Min}\,(u^*,\ w^{-1}\cdot u_1^*)$, the composition of $\mathscr{F}'_u$ and $\mathscr{G}_{wu}$ maps $H(S;\ u,\ u^{k+k_1}v_2)$ into $H(S'';\ ww_1u,\ v_1')$.

A regular mapping $\mathscr{F}$ of $\mathscr{H}(S;\ u,\ v)$ into $\mathscr{H}(S';\ u',\ v')$ is said to be linear when the following condition is satisfied: If ξ, η and $\alpha\xi+\beta\eta$ are in $\mathscr{H}(S;\ u,\ v)$, where α, β are complex numbers, then $\mathscr{F}(\alpha\xi+\beta\eta)=\alpha\mathscr{F}(\xi)+\beta\mathscr{F}(\eta)$. Set $\mathscr{H}(S;\ u)=\cup\{\mathscr{H}(S;\ u,\ v);\ v>0\}$. $\mathscr{H}(S;\ u)$ is a vector space. For any $\xi\in\mathscr{H}(S;\ u)$ and α such that $|\alpha|<|\xi|_u^{-1}\cdot v$, $\alpha^{-1}\cdot\mathscr{F}(\alpha\xi)$ does not depend on the choice of such α. Set $\mathscr{F}'(\xi)=\alpha^{-1}\cdot\mathscr{F}(\alpha\xi)$. Then $\mathscr{F}'=\mathscr{F}$ on $\mathscr{H}(S;\ u,\ v)$ and $\mathscr{F}'$ is a linear transformation of $\mathscr{H}(S;\ u)$ into $\mathscr{H}(S';\ u')$. Moreover, for any v_1, $\mathscr{F}'|H(S;\ u,\ v_1)$ is a regular mapping. Thus, in this case, the germ represented by $\mathscr{F}$ is defined on $\mathscr{H}(S;\ u)$. A germ **F** of regular mappings is said to be linear if it has a representative which is a linear regular mapping. Then the similar argument as in the proof of Proposition I. 7 shows that any representative of **F** is linear. Assume further that **F** is analytic. For any ξ in $\mathscr{H}(S)$, $\xi\in(S;\ u)$ for sufficiently small u. Since **F** has a representative defined on $\mathscr{H}(S;\ u,\ v_1)$ for sufficiently small u and v_1, the above remarks show that **F** is defined at any ξ in $\mathscr{H}(S)$ and is, as a mapping, a linear transformation of $\mathscr{H}(S)$ into $\mathscr{H}(S')$. Thus

PROPOSITION I. 9. *Let* **F** *be a germ of analytic mappings of* $\mathscr{H}(S)$ *into* $\mathscr{H}(S')$. *If* **F** *is linear,* **F** *is defined everywhere and, as a mapping, is a linear transformation.*

Set $2S=(2s_0,\ldots,2s_p)$. Then we can identify $\mathscr{H}(S)+\mathscr{H}(S)$ and $\mathscr{H}(S;\ u,\ v)+\mathscr{H}(S;\ u,\ v)$ with $\mathscr{H}(2S)$ and $\mathscr{H}(2S;\ u,\ v)$, respectively, in the obvious way. So we can speak of regular mappings of $\mathscr{H}(S;\ u,\ v)+\mathscr{H}(S;\ u,\ v)$, etc. Keep the notations in Definition I. 11. For any ξ and η in $\mathscr{H}(S;\ u,\ 2^{-1}u^kv)$, set $\zeta=(\partial/\partial z\ F_u(\xi+z\eta))_{z=0}$. By the definition of $\mathscr{H}(S:\ u,\ v)$, there is u_1, $u<u_1<u^*$, such that ξ and η are in $\mathscr{H}(S;\ u_1,\ 2^{-1}u_1^kv)$. Hence $\mathscr{F}_{u_1}(\xi+z\eta)$ is in $\mathscr{H}(S';\ wu_1,\ v')$ for $|z|\leq 1$. Hence by Cauchy's integral formula ζ is in $\mathscr{H}(S';\ wu,\ v')$. Thus $d\mathscr{F}_u:\ (\xi,\ \eta)\to\zeta$ is a regular mapping of $\mathscr{H}(S;\ u,\ 2^{-1}u^kv)+\mathscr{H}(S;\ u,\ 2^{-1}u^kv)$ into $\mathscr{H}(S';\ wu,\ v')$. It is clear that, $d\mathscr{F}_u$ and $d\mathscr{F}_{u'}$ are equivalent. Hence $d\mathscr{F}_u$ defines a unique germ of analytic mappings $d\mathbf{F}$ of $\mathscr{H}(S)+\mathscr{H}(S)$ into $\mathscr{H}(S')$. $d\mathbf{F}$ is called the

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differential of $\mathbf{F}$. For any η in $\mathscr{A}(S;\ u,\ 2^{-1}u^k v)$, set $(d\mathscr{F}_u)_0 = d\mathscr{F}_u(0,\ \eta)$. Then $(d\mathscr{F}_u)_0$ represents a germ of analytic mappings $(d\mathbf{F})_0$ of $\mathscr{A}(S)$ into $\mathscr{A}(S')$. $(d\ \mathbf{F})_0$ is called the differential of $\mathbf{F}$ at the origin and is linear. If $\mathbf{G}$ is a germ of analytic mappings of $\mathscr{A}(S')$ into $\mathscr{A}(S'')$, we have the formula:

$$(d(\mathbf{G}\circ\mathbf{F}))_0 = (d\mathbf{G})_0\circ(d\mathbf{F})_0 \tag{13}$$

THEOREM I. 7. *Let* $\mathbf{F}$ *be a germ of analytic mappings of* $\mathscr{A}(S)$ *into* $\mathscr{A}(S')$. *Assume that there is a representative of* $\mathbf{F}$ *which is a regular mapping of* $\mathscr{A}(S;\ u,\ u^k v)$ into $\mathscr{A}(S';\ wu,\ v')$ *for each* $u<u^*$. *Then there is a unique formal analytic mapping* F of $H(S)$ *into* $H(S')$ *such that, for any* $u<u^*$, F *is defined on* $\mathscr{A}(S;\ u,\ u^k v)$ *with respect to the topology of the euclidean distance and* $F(\xi) = \mathbf{F}(\xi)$ *for any* ξ *in* $\mathscr{A}(S;\ u,\ u^k v)$.

Proof. Let $\{h_1,\ h_2,\ \dots\}$ be the ordered set of distinguished elements in $H(S)$. Components of h_i are zero except one component, say $\lambda(i)$-th, which is a monomial $x_{i_1}\cdots x_{i_l}$ in $x_1,\ \dots,\ x_{p(\lambda(i))}$, where $l = |i|_{H(S)}$. For any ξ in $H(S)$, denote by $a^i(\xi)$ the coefficient of $x_{i_1}\cdots x_{i_l}$ in the $\lambda(i)$-th component of ξ. Similarly, letting $\{h'_1,\ h'_2,\ \dots\}$ be the ordered set of distinguished elements in $H(S')$, h'_j has components zero except one, say $\sigma(j)$-th, which is a monomial $x_{j_1}\cdots x_{j_n}$ in $x_1,\ \dots,\ x_{p'(\sigma(j))}$ where $n = |j|_{H(S')}$. Now, let ${}^N H(S)$ be the set of all elements ξ in $H(S)$ such that each component of ξ is a polynomial of degree $\le N$. Then there are convergent power series ${}^N F_\sigma^{j_1\cdots j_n}$ in indeterminates $a^i_{H(S)}$ for $|i|_{H(S')}\le N$ such that

$$\mathbf{F}_\sigma(\xi) = \sum_{j_1\cdots j_n} {}^N F_\sigma^{j_1\cdots j_n}(\dots,\ a^i(\xi),\ \dots)\, x_{j_1}\cdots x_{j_n}$$

for any $u<u^*$ and any $\xi\in {}^N H(S)\cap\mathscr{A}(S;\ u,\ u^k v)$. Let ${}^N F^j_m$ be the homogeneous part of degree m in ${}^N F^j = {}^N F^{j_1\cdots j_n}_{\sigma(j)}$, and set ${}^N F^j_m(\xi) = {}^N F^j_m(\dots,\ a^i(\xi),\ \dots)$, i.e. the value of ${}^N F^j_m$ at $a^i_{H(S)} = a^i(\xi)$. Then, for a constant $v_{n,\,m}$

$$ {}^N F^j_m(\xi) = v_{n,\,m}\left(\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_n}}\frac{\partial^m}{\partial z^m}\mathbf{F}_{\sigma(j)}(z\xi)\right)_{x=z=0} \tag{14}$$

for ξ in ${}^N H(S)\cap\mathscr{A}(S;\ u,\ u^k v)$. By Cauchy's integral formula, the mapping $\mathscr{G}^j_m:\ \xi\to v_{n,\,m}\left(\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_n}}\frac{\partial^m}{\partial z^m}\mathbf{F}_{\sigma(j)}(z\xi)\right)_{z=0}$ maps $\mathscr{A}(S;\ u,\ u^k v)$ into $\mathscr{A}_{p'(\sigma(j))}(w'u,\ u^{-n}v'')$ for $u<u^*$, where w' and v'' are constants, and it is

regular when restricted to each $\mathscr{A}(S;\ u,\ u^k v)$. By (14), ${}^N F^j_m(\xi)$ is the value of $F^j_m(\xi)$ at the origin.

As remarked before, we can assume that $k\geq 0$. We shall show that, for $N'\geq N\geq n+km$, ${}^N F^j_m = {}^{N'}F^j_m$, i.e. they are the same homogeneous polynomial of degree m. Take ξ and η in ${}^N H(S)$ and ${}^{N'}H(S)$, respectively. Assume that $a^i(\eta)=0$ for $|i|_{H(S)}\leq N$, and that $\xi\in\mathscr{A}(S;\ u,\ 4^{-1}u^k v)$ for a real number $u<u^*$. Assume further that the coefficients of components of η are so small that $\eta\in\mathscr{A}(S;\ u',\ 4^{-1}(u')^{N+1}\cdot v)$ for any $u'\leq u$. This is possible, because $a^i(\eta)=0$ for $|i|_{H(S)}\leq N$. Since $z\xi\in {}^N H(S)\cap H(S;\ u', 4^{-1}(u')^k v)$ for $|z|<(u'/u)^k$, applying Proposition I. 6 for $\mathscr{F}^{\cdot j}_m$, we find that $|\mathscr{F}^{\cdot j}_m(z\xi+\eta)-\mathscr{F}^{\cdot j}_m(z\xi)|_{w'u'}$ $\leq (u')^{-k-n}\cdot 4v''|\eta|_{u'}$ for $|z|<(u'/u)^k$. Hence by the choice of η we have, for a constant v_1,

$$|{}^{N'}F^j_m(z\xi+\eta)-{}^N F^j_m(z\xi)|\leq u'^{\,N+1-k-n}v_1 \tag{15}$$

for any $|z|\leq(u'/u)^k$. Since $P(z)={}^{N'}F^j_m(z\xi+\eta)-{}^N F^j_m(z\xi)$ is a polynomial in z of degree $\leq m-1$, if it is not zero, then $P(z)=z^l Q(z)$, where $0\leq l\leq m-1$, $Q(0)\neq 0$. Then by (15) we find that $kl\geq N+1-k-n\geq k(m-1)+1)$. Since $k\geq 0$ and $m-1\geq l\geq 0$, it is a contradiction. Hence $P(z)$ is a zero polynomial, and so ${}^{N'}F^j_m={}^N F^j_m$ for $N'\geq N\geq n+km$. Now, we set $F^j_m={}^N F^j_m$, $N\geq n+km$. For any ξ in $H(S)$, denote by ${}^N\xi$ the element in ${}^N H(S)$ such that $a^i({}^N\xi)=a^i(\xi)$ for $|i|_{H(S)}\leq N$. Then there is a constant $v_2<\frac{v}{4}$ such that ${}^N\xi$ are in $\mathscr{A}(S;\ u,\ 4^{-1}u^k v)$ for ξ in $\mathscr{A}(S;\ 2u,\ (2u)^k v_2)$. Applying Proposition I. 6, we find that $\sum_{m=1}^{\infty}F^j_m(\xi)$ converges to $F^j(\xi)=F^{j_1\cdots j_n}_{\sigma(j)}(\xi)$ for $\xi\in H(S;\ u,\ u^k v_2)$, where $u<2^{-1}u^*$ and that $F_\sigma(\xi)=\sum_{j_1\cdots j_n}F^{j_1\cdots j_n}_\sigma(\xi)x_{j_1}\cdots x_{j_n}$. Thus the only thing left to prove is the fact that $\{F^j_m\}$ forms a formal analytic mapping. By definition F^j_m is a homogeneous polynomial of degree m. Take ξ in ${}^N H(S)$. If there is $u<u^*$ such that $|a^i(\xi)|=u^{k-l}v_3$, for a sufficiently small constant v_3, where $l=|i|_{H(S)}$, then $\xi\in\mathscr{A}(S;\ 2^{-1}u,\ 2^{-k}u^k v)$, and so $|{}^N F^j_m(\xi)|\leq|\mathscr{F}^j_m(\xi)|_{2^{-1}w'u}\leq u^{-n}$ $(2^n v'')$, where $n=|j|_{H(S')}$. Since ${}^N F^j_m$ is a homogeneous polynomial of degree m, the above inequality implies that $|{}^N F^j_m(\xi)|\leq u^{-km-n}v''_1$ for $|a^i(\xi)|\leq u^{-l}$, where v'' is a constant. This shows that F^j_m is of weight $\leq|j|_{H(S')}+km$. Thus the theorem is proved.

The above F will be called the formalization of $\mathbf{F}$. If a formal analytic

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mapping F is a formalization of a germ of analytic mappings, we shall say that F is convergent.

PROPOSITION I. 10. *Assume that the characteristics of $H(S)$ and of $H(S')$ are equal. Then there are everywhere defined germs of linear analytic mappings* **F** *and* **G** *of $\mathscr{A}(S)$ and $\mathscr{A}(S')$ into $\mathscr{A}(S')$ and $\mathscr{A}(S)$, respectively, such that* **F**∘**G** *and* **G**∘**F** *are indentity mappings.*

Proof. (1°) First we prove the case when $s'_r = s_r + s_{r+1} + \cdots + s_p$, $(r = 0, 1, \ldots, p)$, where $S = (s_0, s_1, \ldots, s_p)$, $S' = (s'_0, s'_0, \ldots, s'_p)$. ξ in $\mathscr{A}(S)$ has s_r components which are functions of $x_1, \ldots, x_r$; we denote these by $\xi^r_1, \ldots, \xi^r_{s_r}$. Similar notations will be used for elements in $\mathscr{A}(S')$. We define a linear mapping F of $\mathscr{A}(S)$ into $\mathscr{A}(S')$ by the formula: For $\xi \in \mathscr{A}(S)$,

$$F(\xi)^0_{s_0+\cdots+s_{q-1}+\sigma} = \xi^q_\sigma(0) \qquad (q = 0, \ldots, p-1;\ 1 \leq \sigma \leq s_q \text{ if } s_q \neq 0),$$

$$F(\xi)^r_{s_r+\cdots+s_{q-1}+\sigma} = \frac{\partial}{\partial x_r}\xi^q_\sigma$$

$$(r = 1, \ldots, p;\ q = r, \ldots, p-1;\ 1 \leq \sigma \leq s_q \text{ if } s_q \neq 0).$$

Of course $s_r + \cdots + s_{q-1} + \sigma$ means σ when $q = r$. We define a linear mapping G of $\mathscr{A}(S')$ into $\mathscr{A}(S)$ by the formula: For $\eta \in \mathscr{A}(S')$,

$$G(\eta)^r_\sigma = \eta^0_{s_0+\cdots+s_{r-1}+\sigma} + \int_0^{x_1} \eta^1_{s_1+\cdots+s_{r-1}+\sigma}\,dx_1 + \int_0^{x_2} \eta^2_{s_2+\cdots+s_{r-1}+\sigma}\,dx_2 + \cdots + \int_0^{x_r} \eta^r_\sigma\,dx_r.$$

Then it is not hard to check that they represent germs of analytic mappings and $F\circ G$, $G\circ F$ are identity mappings.

(2°) General case: We write $\mathscr{A}(S) \approx \mathscr{A}(S')$ when **F** and **G** such as in Proposition I. 10 exist. It is sufficient to show that $\mathscr{A}(S) \approx \mathscr{A}^{s_p}_p$. By (1°), we can assume that $s_0 \neq 0$, $s_1 \neq 0, \ldots, s_p \neq 0$. Therefore it is sufficient to prove the following statement: If $s_r > 0$, $s_{r+1} > 0, \ldots, s_p > 0$, $p > r$, then $\mathscr{A}(0, \ldots, 0, s_r, s_{r+1}, \ldots, s_p) \approx \mathscr{A}(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p)$. In fact, making use of (1°) twice, $\mathscr{A}(0, \ldots, 0, s_r, s_{r+1}, \ldots, s_p) \approx \mathscr{A}(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p) + \mathscr{A}_r \approx \mathscr{A}(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p) + (\mathscr{A}_0 + \mathscr{A}_1 + \cdots + \mathscr{A}_r) \approx \mathscr{A}(0, \ldots, 0, s_r - 1, s_{r+1} - 1, \ldots, s_p) + (\mathscr{A}_0 + \cdots + \mathscr{A}_{r+1}) \approx \mathscr{A}(0, \ldots, 0, s_r - 1, s_{r+1} - 1, \ldots, s_p) + \mathscr{A}_{r+1} \approx \mathscr{A}(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p)$.

Let $\mathscr{A}^R(S)$, $H^R(S)$ be the set of elements ξ in $\mathscr{A}(S)$, $H(S)$ such that components of ξ are power series with real coefficients respectively. Set $\mathscr{A}^R(S;\ u, v) = \mathscr{A}^R(S) \cap \mathscr{A}(S;\ u, v)$. A mapping $\mathscr{F}$ of $\mathscr{A}^R(S;\ u, v)$

into $\mathscr{A}^R(S';\ u',\ v')$ is called regular, if there is a regular mapping $\mathscr{F}_C$ of $\mathscr{A}(S;\ u,\ v)$ into $\mathscr{A}(S';\ u',\ v')$ such that $\mathscr{F}$ is the restriction of $\mathscr{F}_C$ to $\mathscr{A}^R(S;\ u,\ v)$. By the theorem of coincidence, it is clear that such $\mathscr{F}_C$ is unique. By the similar procedure as above, we define germs of analytic mappings of $\mathscr{A}^R(S)$ into $\mathscr{A}^R(S')$. They can be regarded as germs of analytic mappings of $\mathscr{A}(S)$ into $\mathscr{A}(S')$, representatives of which map intersections with $\mathscr{A}^R(S)$ into $\mathscr{A}^R(S')$. The formalizations of germs of analytic mappings of $\mathscr{A}^R(S)$ into $\mathscr{A}^R(S')$ are formal analytic mappings of $H^R(S)$ into $H^R(S')$.

Remark. Theorems in §3 do not hold for convergent formal analytic mappings. For instance, for a fixed real number $R>1$, let $\mathbf{F}$ be a germ of analytic mappings of $\mathscr{A}_1$ into itself defined by the formula:

$$[\mathbf{F}(\xi)](z)=\xi(z)+(Rz)\cdot\xi(z)\cdot\xi(Rz).$$

Then $(d\mathbf{F})_0$ is the identity mapping. By Theorem I. 2, there is the inverse formal analytic mapping G of the formalization of $\mathbf{F}$. However, G is not convergent.

5. Examples

(A) We can consider $\int_0^{x_r}$ and $\partial/\partial x_r$ as germs of analytic mappings. They are linear and everywhere defined.

(B) Let $A_\lambda(y_1,\ldots,y_s,\ x_{p+1},\ldots,x_{p+l})$, $\lambda=1,\ldots,s'$, be analytic functions defined for $|y_\sigma|\leq v$, $|x_{p+\mu}|\leq u^*$, $(\sigma=1,\ldots,s;\ \mu=1,\ldots,l)$. Assume that $A_\lambda(0,\ldots,0)=0$ and that $|A_\lambda(y,\ x)|<v'$ in the above domain. Then we have a mapping $\mathscr{F}$ of $\mathscr{A}_p^s(u,\ v)$ into $\mathscr{A}_{p+l}^{s'}(u,\ v')$ for $u<u^*$ defined by the formula: $[\mathscr{F}(\xi)]_\lambda(x_1,\ldots,x_{p+l})=A_\lambda(\xi_1(x_1,\ldots,x_p),\ldots,\xi_s(x_1,\ldots,x_p),$ $x_{p+1},\ldots,x_l)$. Clearly they are representatives of a germ of analytic mappings of $\mathscr{A}_p^s$ into $\mathscr{A}_{p+l}^{s'}$.

(C) By a homeomorphism element in p-dimensional complex euclidean space C^p at the origin, we mean a homeomorphic analytic mapping of a domain containing the origin onto a domain in C^p. By identifying two homeomorphism elements at the origin which coincide on a neighborhood of the origin, we define germs of homemorphism elements of C^p at the origin. Denote by $C^p(u)$ the domain: $|x_r|<u$, $(r=1,\ldots,p)$. For any ξ in $H_p^p(u,\ 2^{-1}u)$, let f^ξ be the mapping: $f_r^\xi(x_1,\ldots,x_p)=x_r+\xi_r(x_1,\ldots,x_p)$ of $C^p(u)$ into C^p, f^ξ maps

$C^p(u)$ into $C^p(2u)$, and $C^p(2^{-1}u)$ into $C^p(u)$. Hence if η is another element in $\mathscr{A}_p^p(u, 2^{-1}u)$, then the composition $f^\eta \circ f^\xi$ is defined on $C^p(2^{-1}u)$ and its image is in $C^p(2u)$. Therefore there is a unique element γ in $\mathscr{A}_p^p(2^{-1}u, 3u)$ such that the germ of f^γ at the origin is equal to that of $f^\eta \circ f^\xi$. Putting $\gamma = \mathscr{M}_u(\eta, \xi)$, we have a mapping $\mathscr{M}_u$ of $\mathscr{A}_p^p(u, 2^{-1}u) + \mathscr{A}_p^p(u, 2^{-1}u)$ into $\mathscr{A}_p^p(2^{-1}u, 3u)$. One finds easily that the r-th component of $\mathscr{M}_u(\xi, \eta)$ is $\eta_r(x) + \xi_r(x_1 + \eta_1(x), \ldots, x_p + \eta_p(x))$, where $x = (x_1, \ldots, x_p)$. Clearly $\mathscr{M}_u$ are representatives of a germ of analytic mappings of $\mathscr{A}_p^p + \mathscr{A}_p^p$ into $\mathscr{A}_p^p$, which may be called the multiplication in the general infinite pseudo-group of p-dimensional homeomorphism elements at the origin. $\mathscr{M}_u$ map the intersections with $\mathscr{A}_p^{Rp} + \mathscr{A}_p^{Rp}$ into $\mathscr{A}_p^{Rp}$, so they also represent a germ of real analytic mappings.

(D) If ξ is in $\mathscr{A}_p^p(2u, \delta u^2)$, where $u<1$, $\delta < 4^{-1}$, then $|f_r^\xi(x)| > u - 4^{-1}u = 4^{-1}3u$ for $|x_r| = u$ and $|f^\xi(0)| < 4^{-1}u$. Hence the image of $C^p(u)$ by f^ξ contains $C^p(2^{-1}u)$. Moreover for x in $C^p(u)$, $|\partial f_r^\xi/\partial x_r| \geq 1 - \delta$ and $|\partial f_r^\xi/\partial x_s| < \delta$, $(r \neq s)$. Therefore there is $\delta > 0$, independent of $u<1$, such that the inverse $(f^\xi)^{-1}$ is defined on $C^p(2^{-1}u)$ and maps it into $C^p(u)$. Thus there is a unique γ in $\mathscr{A}_p^p(2^{-1}u, 2u)$ such that the germ of f^γ at the origin is equal to that of $(f^\xi)^{-1}$. Putting $\gamma = \mathscr{J}_u(\xi)$, we define a mapping $\mathscr{A}_p^p(u, \delta u^2)$ into $\mathscr{A}_p^p(2^{-1}u, 2u)$ for $u<1$. It is clear that $\mathscr{J}_u$ are representatives of a germ of analytic mappings of $\mathscr{A}_p^p$ into $\mathscr{A}_p^p$. $\mathscr{J}_u$ preserve $\mathscr{A}_p^{Rp}$, so they are also representatives of a germ of real analytic mappings. The germ may be called the inverse operation in the infinite pseudo-group of p-dimensional homeomorphism elements at the origin.

(E) Let A_λ, $\lambda = 1, \ldots, s$, be analytic functions in variables $x_1, \ldots, x_{p-1}$, $y_1, \ldots, y_s$, and y_μ^r, where $\mu = 1, \ldots, s$ and $r = 1, \ldots, p$. Assume that A_λ are defined when the absolute values of each variables are less than, say, a. Now we consider a system of partial differential equations:

$$(\Sigma) \qquad \frac{\partial y_\lambda}{\partial x_{p+1}} = A_\lambda\left(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots\right), \quad (\lambda = 1, \ldots, s).$$

Let ξ be in $\mathscr{A}_p^s$ and assume that $|\xi_\lambda(0)|$, $|(\partial \xi_\lambda/\partial x_r)_{x=0}| < a$. An element η in $\mathscr{A}_{p-1}^s$ is called a solution of (Σ) with the initial condition ξ, if $y_\lambda = \eta_\lambda$ is a solution of (Σ) and if $\eta_\lambda(x_1, \ldots, x_p, 0) = \xi_\lambda(x_1, \ldots, x_p)$. The classical theorem of Cauchy-Kowalewski can be sharpened to the following:

THEOREM I. 8. *Assume that each A_λ is zero when all the variables, except $x_1, \ldots, x_{p+1}$, are zero. Then there is a unique germ of analytic mappings* F *of $\mathscr{A}_p^s$ into $\mathscr{A}_{p+1}^s$ with the following properties: If $\mathscr{F}: \mathscr{A}_p^s(u, v) \to \mathscr{A}_{p+1}^s(u', v')$ is a representative of* F *and if v, $u^{-1}v < a$, then for any ξ in $\mathscr{A}_p^s(u, v)$ $\mathscr{F}(\xi)$ is the unique solution of (Σ) with the intial condition ξ.*

Proof. By the classical theorem, there are positive real numbers u_1, v_1, u_1', v_1', $(u_1, u_1^{-1}v_1 < a)$, and a regular mapping $\mathscr{F}^1$ of $\mathscr{F}_p^s(u_1, v_1)$ into $\mathscr{A}_{p+1}^s(u_1', v_1')$ such that, for any ξ in $\mathscr{A}_p^s(u_1, v_1)$, $\mathscr{F}^1(\xi)$ is the unique solution of (Σ) with the initial condition ξ. Let $\mathscr{F}$ be a regular mapping of $\mathscr{A}_p^s(u, v)$ into $\mathscr{A}_{p+1}^s(u', v')$ equivalent to $\mathscr{F}^1$ and assume that v, $u^{-1}v < a$. Then by the similar method as in the proof of Proposition I. 7 we see that $\mathscr{F}(\xi)$ is a solution of (Σ) with the initial condition ξ, and the classical theorem assures us that $\mathscr{F}(\xi)$ is the unique one. It remains to show that $\mathscr{F}^1$ is a representative of a germ of analytic mappings. We divide the proof in several steps.

(I) The case when $A_\lambda = B_\lambda(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s) + \sum_{\mu, r} B_{\lambda r}^\mu(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s)y_\mu^r$. We use the same notations introduced in the beginning part of the proof of Theorem I. 7, except we will write $\mu(i)$ instead of $\lambda(i)$, and $|\ \ |_{s,p}$ instead of $|\ \ |_{H^s_p}$.

(I. i) For ξ in H_p^s and for i such that $|i|_{s,p} > 0$, denote by $\partial_i\xi$ the partial derivative $\partial^l\xi_{\mu(i)}/\partial x_{i_1}\cdots\partial x_{i_l}$. For $n = 0, 1, \ldots$ let B^n be a collection consisting of analytic functions $A_{\lambda, n}$, $A_{\lambda, n}^{i(1)\cdots i(q)}$ in the variables $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_s)$ defined in a common domain containing the origin and independent of n, where $h_{i(1)}, \ldots, h_{i(q)}$ run through distinguished elements of H_p^s such that $|i(t)|_{s,p} > 0$ and $|i(1)|_{s,p} + \cdots + |i(q)|_{s,p} \le n$, $q = 1, \ldots, n$. In what follows, we regard $A_{\lambda, n}$ as the case $q = 0$ of $A_{\lambda, n}^{i(1)\cdots i(q)}$. So B^n consists of $A_{\lambda, n}^{i(1)\cdots i(q)}$, the case $q = 0$ included. Now we will show that we can choose B^n with the following properties: For any ξ in $\mathscr{A}_p^s$ such that $|\xi_\lambda(0)| < a$, if

$$\eta_\lambda = T_\lambda(\xi) = \sum_{n=0}^{\infty} (\sum A_{\lambda, n}^{i(1)\cdots i(q)}(x, \xi(x))\,(\partial_{i(1)}\xi)\cdots(\partial_{i(q)}\xi))x_{p+1}^n$$

are convergent then $\eta = (\eta_1, \ldots, \eta_s)$ is the unique solution of (Σ) with the initial condition ξ, and vice versa. In the above expression, the termes $q = 0$

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mean $A_{\lambda, n}(x, \xi(x))$. We define B^n by induction on n. B° consists of $A_{\lambda, 0} = y_\lambda$. We assume that $B^\circ, \ldots, B^{n-1}$ are already defined. Then we can find $A_{\lambda, n}^{i(1)\cdots i(q)}$ by formally differentiating the equation (Σ) with respect to x_{p+1} $n-1$ times and by remarking that we are considering the special case mentioned above.

(I. ii) Assume that the members of B^n are defined for $|x_r| < u_2$, $|y_\mu| < u_2$. Then there exist positive real numbers $u_3 < u_2$ and v_3 such that, for $|x_r|$, $|y_\mu| < u_3$,

$$|A_{\lambda, n}^{i(1)\cdots i(q)}(x, y)| \leq (u_3)^{-n-q}(v_3)^{l(1)+\ldots+l(q)+1},$$

where $l(t) = |i(t)|_{s, p}$.

Proof. Let $\mathscr{S}^1$, u_1, v_1, u_1', v_1' be such as stated in the beginning of the proof of our Theorem. We can assume that v_1, $u_1 < u_2$ and that $u_1' \leq u_1$. Then if ξ is in $\mathscr{H}_p^s(u_1, v_1)$, $T_\lambda(\xi) = \mathscr{S}_\lambda^1(\xi) \in \mathscr{H}_{p+1}^1(u_1', v_1')$, and so

$$(16) \qquad |\sum A_{\lambda, n}^{i(1)\cdots i(q)}(x, \xi(x))(\partial_{i(1)}\xi)\cdots(\partial_{i(q)}\xi)| \leq (u_1')^{-n} v_1'$$

for $|x_r| < u_1'$. Take a sufficiently large real number b such that $b^p \leq 4(b-1)^p$. Then, for any $|x_r^\circ| < u_1$, $|y_\mu^\circ| < 4^{-1}v_1$, and $|y_i^\circ| < (2u_1 b)^{-l}4^{-1}v_1$ where $l = |i|_{s, p}$, there is ξ in $H_p^s(u_1, v_1)$ such that $\xi_\mu(x^\circ) = y^\circ$ and $[\partial_i \xi](x^\circ) = y_i^\circ$. Therefore, by (16), if $|x_r| < u_1'$ and $|y_\mu| < \mathrm{Min}\,(4^{-1}v_1, u_1')$ then

$$|A_{\lambda, n}^{i(1)\cdots i(q)}(x, y)| \leq (u_1')^{-n}(4^{-1}v_1)^{-q}(2u_1 b)^{l(1)+\ldots+l(q)} v_1'.$$

Hence it is sufficient to put $u_3 = \mathrm{Min}\,(4^{-1}v_1, u_1', 1)$ and $v_3 = \mathrm{Max}\,(v_1', 2u_1 b, 1)$.

(I. iii) Introduce an indeterminate Y_i for each distinguished element h_i such that $|i|_{s, p} > 0$. Set $f_n(Y) = \sum Y_{i(1)}\cdots Y_{i(q)}$ where the summation is with respect to all $Y_{i(1)}\cdots Y_{i(q)}$, $(q = 1, 2, \ldots)$, such that $|i(1)|_{s, p} + \cdots + |i(q)|_{s, p} = n$. Denote by f_n the value of $f_n(Y)$ when $Y_i = 2^{-l}$, $l = |i|_{s, p}$. We set $f_0 = 1$. Then $f^{(n)} = f_0 + f_1 + \cdots + f_n \leq 2^{sp(n+1)}$.

Proof. Set $g_n(Y) = \sum Y_i$, where the summation is with respect to all $|i|_{q, p} = n$. Set $g(Y) = 1 + g_1(Y) + \cdots + g_n(Y) + \cdots$. Then $g(Y)^n = \sum_{q \leq n} a^{i(1)\ldots i(q)} Y_{i(1)}\cdots Y_{i(q)}$, where $a^{i(1)\ldots i(q)} > 0$. Therefore, when we denote by $h^n(t)$ the formal power series in t obtaind by putting $Y_i = (t/2)^l$, $l = |i|_{s, p}$, in $f_n(Y)$, we see that the coefficient of t^n in $h^n(t)$ is less than that in $(1-(t/2))^{-spn}$. Because f_n is equal to the coefficient of t^n in $h^n(t)$, it follows that $f_n \leq 2^{spn}$.

(I. iv) By the choice made in (ii), $u_3 \leq 1$ and $v_3 \geq 1$. Take ξ in $\mathscr{A}_p^s(u, v)$, $(u, v < u_3)$. Since $|\partial_i \xi| \leq (u/2)^{-l} v$ for $|x_r| < (u/2)$, using (ii), we find that, for $|x_r| < (u/2)$.

$$\begin{aligned}
&|\sum A_{\lambda, n}^{i(1)\cdots i(q)}(x, \xi(x))\,(\partial_{i(1)}\xi)\cdots(\partial_{i(q)}\xi)| \\
&\leq u_3^{-n} v_3 (\sum u_3^{-q} v_3^{l(1)+\cdots+l(q)} (u/2)^{-(l(1)+\cdots+l(q))} v^q) \\
&\leq (v_3^{-1}\cdot u_3^2)^{-n} v_3 (\sum (u/4)^{-(l(1)+\cdots+l(q))} 2^{-(l(1)+\cdots+l(q))}) \\
&\leq ((4v_3)^{-1}\cdot(u_3^2 u))^{-n} v_3 f^{(n)} \leq (wu)^{-n} v'
\end{aligned}$$

where $w = (2^{sp+2} v_3)^{-1} u_3^2$ and $v' = 2^{sp} v_3$. Therefore $T_\lambda(\xi) \in \mathscr{A}_{p+1}(w'u, 2v')$, $w' = (w/2)$. This means that for $u, v < u_3$ $\mathscr{S}: \mathscr{A}_p^s(u, v) \ni \xi \to (T_1(\xi), \ldots, T_s(\xi)) \in \mathscr{A}_{p+1}^s(w'u, 2v')$ is a regular mapping equivalent to $\mathscr{S}^1$. Thus our Theorem is proved in our special case.

(II) The general case. A_λ being given, consider the following system of partial differential equations:

$$(\Sigma')\qquad \begin{aligned}
\frac{\partial y_\lambda}{\partial x_{p+1}} &= A_\lambda(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s, \ldots, y_\mu^r, \ldots), \\
\frac{\partial y_\lambda^r}{\partial x_{p+1}} &= \frac{\partial A_\lambda}{\partial x_r} + \sum_{\mu=1}^{s} \frac{\partial A_\lambda}{\partial y_\mu}\cdot y_\mu^r + \sum_{\mu=1}^{s}\sum_{q=1}^{p} \frac{\partial A_\lambda}{\partial y_\mu^q}\cdot\frac{\partial y_\mu^r}{\partial x_q}
\end{aligned}$$

where $y_1, \ldots, y_s, \ldots, y_\mu^r, \ldots$ are unknown functions. Since partial derivatives of unknown functions appear only linearly in (Σ'), we know by (I) that there is a germ $\mathbf{F}_1$ of analytic mappings of $\mathscr{A}_p^{sp+s}$ into $\mathscr{A}_{p+1}^{sp+s}$ such that representatives of $\mathbf{F}_1$ give the solutions with initial conditions. Denote by $\mathbf{F}_0$, $\mathbf{F}_2$ the germs of everywhere defined analytic mappings: $\mathscr{A}_p^s \ni \zeta \to (\zeta, \ldots, \partial\zeta_\mu/\partial x_r, \ldots) \in \mathscr{A}_p^{sp+s}$, $\mathscr{A}_p^{sp+s} \ni (\ldots, \eta_\lambda, \ldots, \ldots, \eta_\lambda^r, \ldots) \to (\eta_1, \ldots, \eta_s) \in \mathscr{A}_p^s$, respectively. Now, let $\mathscr{S}^1$ be the regular mapping of $\mathscr{A}_p^s(u_1, v_1)$ into $\mathscr{A}_{p+1}^s(u', v')$ such as in the beginning of our proof. Then, because of the uniqueness of solutions of (Σ') under initial conditions, $\mathscr{S}^1$ is a representative of the germ $\mathbf{F}_2\circ\mathbf{F}_1\circ\mathbf{F}_0$. This finishes the proof of Theorem I. 8.

(F) Let B_λ, $\lambda = 1, \ldots, s$, be analytic functions in variables $x_1, \ldots, x_{p+1}$, $y_1, \ldots, y_s, \ldots, y_\mu^r, \ldots, z_1, \ldots, z_{s'}$, where $\mu = 1, \ldots, s$ and $r = 1, \ldots, p$. Assume that B_λ are defined when the absolute values of all the variables are less than a. Assume also that the values of B_λ are zero when all variables, except $x_1, \ldots, x_{p+1}$, are zero. Now, for any ζ in $\mathscr{A}_{p+1}^{s'}$ such that $|\zeta_\sigma(0)| < a$ for $\sigma = 1, \ldots, s'$, consider the following system of partial differential equations:

$$(\Sigma_\xi)\qquad \frac{\partial y_\lambda}{\partial x_{p+1}} = B_\lambda(x_1, \ldots, x_{p-1}, y_1, \ldots, y_s, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots, \xi_1(x_1, \ldots, x_{p-1}), \ldots, \xi_{s'}(x_1, \ldots, x_{p-1}))$$

THEOREM I. 9. *There is a unique germ* **F** *of analytic mappings of* $\mathscr{H}^s_p + \mathscr{H}^{s'}_{p+1}$ *into* $\mathscr{H}^s_{p+1}$ *with the following properties: For any regular mapping* $\mathscr{F}$ *of* $\mathscr{H}^s_p(u, v) + \mathscr{H}^{s'}_{p+1}(u, v)$ *into* $\mathscr{H}^s_{p+1}(u', v')$, *representing* **F**, *and such that* $v, u^{-1}v < a$, $\mathscr{F}(\xi, \zeta)$ *is the unique solution of* (Σ_ζ) *with the initial condition* ξ, *where* $\xi \in \mathscr{H}^s_p(u, v)$ *and* $\zeta \in \mathscr{H}^{s'}_{p+1}(u, v)$.

Proof. Introducing a new variable t, consider the following system of partial differential equations:

$$(\Sigma'')\qquad \begin{aligned} \frac{\partial y_\lambda}{\partial x_{p+1}} &= B_\lambda(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots, z_1, \ldots, z_{s'}), \\ \frac{\partial z_\sigma}{\partial x_{p+1}} &= \frac{\partial z_\sigma}{\partial t}, \qquad (\lambda = 1, \ldots, s;\ \sigma = 1, \ldots, s'), \end{aligned}$$

where $x_1, \ldots, x_p, t, x_{p+1}$ are independent variables and $y_1, \ldots, y_s, z_1, \ldots, z_{s'}$ are unknown functions. The unique solution of the equation: $\partial z/\partial x_{p+1} = \partial z/\partial t$ with the initial condition: $(z(x_1, \ldots, x_p, t, x_{p+1}))_{x_{p+1}=0} = \zeta(x_1, \ldots, x_p, t)$ is equal to $\zeta(x_1, \ldots, x_p, t + x_{p+1})$. Therefore for $\xi_\lambda(x_1, \ldots, x_p)$, $\zeta_\sigma(x_1, \ldots, x_p, t)$ such that $|\xi_\lambda(0)|$, $|(\partial\xi_\lambda/\partial x_r)_{x=0}|$, $|\zeta_\sigma(0)| < a$, the solution of (Σ'') with the initial condition: $(y_\lambda)_{x_{p+1}=0} = \xi_\lambda$, $(z_\sigma)_{x_{p+1}=0} = \zeta_\sigma$ must be necessarily of the form: $y_\lambda = \alpha_\lambda(x_1, \ldots, x_p, t, x_{p+1})$, $z_\sigma = \zeta_\sigma(x_1, \ldots, x_p, t + x_{p+1})$. Hence $y_\lambda = \alpha_\lambda(x_1, \ldots, x_p, 0, x_{p+1})$ is the solution of (Σ_ζ) with the initial condition ξ. Thus, applying Theorem I. 8 to (Σ''), we can easily prove Theorem I. 9.

Chapter II. Formal Lie (F)-groups and (F)-algebras

Using the formal analytic mappings, we define the notion of formal Lie (F)-groups and formal Lie (F)-algebras and establish the one-to-one correspondence between the isomorphic classes of formal Lie (F)-groups and of formal Lie (F)-algebras. If the parameter (F)-vector spaces are of finite dimension, formal Lie (F)-algebras are usual Lie algebras. If, moreover, the multiplications are convergent, a formal Lie (F)-groups are local Lie groups. Thus our theory generalizes the classical theory of correspondence between local Lie groups and Lie algebras. Since our arguments closely follow that of the classical theory, only sketches of proofs will be given. *A* field K of characteristic zero will be fixed throughout this chapter.

1. Definitions

DEFINITION II. 1. *Let H be a (F)-vector space. A formal analytic mapping G of $H+H$ into H is called a formal Lie (F)-group in the parameter space H, if it satisfies the following conditions:*

(1°) The multiplication ${}^cG$ of ${}^cH$ is associative and 0 is the unit element. We shall write $\alpha\cdot\beta$ in stead of ${}^cG(\alpha, \beta)$, where $\alpha, \beta\in{}^cH$;

(2°) There is a formal analytic mapping J of H into itself such that, for and α in ${}^cH$, ${}^cJ(\alpha)$ is the inverse of α. We shall write α^{-1} instead of ${}^cJ(\alpha)$.

We shall use the same letter ${}^cG$ to denote the group structure defined in ${}^cH$ by the above multiplication.

Let G_1 and G_2 be formal Lie (F)-groups in parameter spaces H_1 and H_2, respectively. If there is a formal analytic mapping F of H_1 into H_2 such that ${}^cF$ is a homomorphism of the group ${}^cG_1$ into ${}^cG_2$, then ${}^cF$ is called a representation of G_1 into G_2. Let G_3 be a formal Lie (F)-group. If F' is a representation of G_2 into G_3, then it is clear that $F'\circ F$ is a representation of G_1 into G_3. G_1 and G_2 are said to be isomorphic if there are representations F and F' of G_1 into G_2 and of G_2 into G_1, respectively, such that $F'\circ F$ and $F\circ F'$ are the identity mappings.

Let G be a formal Lie (F)-group in the parameter space H. Let us use letters $\alpha, \beta, \gamma, \ldots$ to denote elements in ${}^cH$. In the remainder of this section, we shall keep G fixed and shall use the same notations. Then there are formal analytic mappings V and W of $H+H$ into H such that

$$ {}^cV(\alpha, \beta) = [\partial/\partial u(\alpha\cdot u\beta)]_{u=0}, $$
$$ {}^cW(\alpha, \beta) = [\partial/\partial u(\alpha^{-1}\cdot(\alpha+u\beta))]_{u=0}. $$

(For the meaning of the above $\partial/\partial u$, consult the sentences following Theorem I. 5). We have

(17) $$\alpha\cdot u\beta \equiv \alpha + u\,{}^cV(\alpha, \beta) \qquad (\mathrm{mod}\ u^2),$$

(18) $$\alpha + u\beta \equiv \alpha\cdot(u\,{}^cW(\alpha, \beta)) \qquad (\mathrm{mod}\ u^2),$$

(19) $$[\partial/\partial u(\alpha\cdot(\beta+u\gamma))]_{u=0} = {}^cV(\alpha\cdot\beta, {}^cW(\beta, \gamma)),$$

From these formulae one finds easily that

(20) $$ {}^cV(\alpha, {}^cW(\alpha, \beta)) = {}^cW(\alpha, {}^cV(\alpha, \beta)) = \beta. $$

For any element $\alpha = \xi_1 t + \xi_2 t^2 + \cdots + \xi_l t^l + \cdots$ in ${}^cH$, where ξ_r are in H for $r = 1, 2, \ldots,$ and for any u in K, denote by $\alpha \cdot u$ the element $(u\xi_1)t + \cdots + (u^l \xi_l)t^l + \cdots$ in ${}^cH$. A curve α in H, that is, an element in ${}^cH$, is called a one-parameter subgroup of G if for any u_1 and u_2 in K we have the equality:

$$ (\alpha \cdot u_1) \cdot (\alpha \cdot u_2) = \alpha \cdot (u_1 + u_2). $$

A formal Lie (F)-group is said to be under a canonical coordinate system, if for any ξ in H, ${}^c\xi$ is a one-parameter subgroup, where ${}^c\xi$ is the curve ξt.

Theorem II. 1. *Let G be a formal Lie (F)-group. Then there is a formal Lie (F)-group G' which is isomorphic to G and which is under a canonical coordinate system.*

Proof. By Theorem I. 5 there is a formal analytic mapping F of H into itself such that

$(*)$ $$ d{}^cF(\alpha, \alpha) = {}^cV({}^cF(\alpha), \alpha) $$

Set $H_1 = H_2 = H_3 = H$. Denote by F' and F'' the formal analytic mapping of $H_1 + H_2$ into H_3 such that ${}^cF'(\alpha, \beta) = {}^cF(\alpha + \beta)$ and ${}^cF''(\alpha, \beta) = {}^cF(\alpha) \cdot {}^cF(\beta)$, respectively. Fix ξ in H and set $\alpha = {}^c\xi \cdot u_1 = u_1 {}^c\xi$, $\beta = {}^c\xi$. Then, by $(*)$ we find that $\partial/\partial u\ {}^cF'(\alpha, u\beta) = {}^cV(F'(\alpha, u\beta), \beta)$, and by $(*)$ and (19) that $\partial/\partial u\ {}^cF''(\alpha, u\beta) = {}^cV({}^cF''(\alpha, u\beta), \beta)$. Since ${}^cF'(\alpha, 0) = {}^cF''(\alpha, 0) = {}^cF(\alpha)$, Lemma I. 1 implies that ${}^cF'(\alpha, u\beta) = {}^cF''(\alpha, u\beta)$ for any u in K. Hence ${}^cF({}^c\xi)$ is a one-paramenter subgroup of G for any ξ in H. Since ${}^cV(0, \alpha) = \alpha$ for any α in ${}^cH$, $(dF)_0$ is the identity mapping. By Theorem I. 2, F is an isomorphism of H onto itself. Then it is clear that there is a formal Lie (F)-group G' in the parameter space H such that F is an isomorphism of G' onto G. Then it is easy to see that G' is under a canonical coordinate system.

One finds easily the following:

Proposition II. 1. *If G is under a canonical coordinate system, then for any α in ${}^cH$,*

$$ {}^cV(\alpha, \alpha) = {}^cW(\alpha, \alpha) = \alpha, $$

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Definition II. 2. *A formal analytic mapping L of H+H into H is called a formal Lie (F)-algebra in the parameter space H if, by the multiplication* ${}^cL$ *in* ${}^cH$, ${}^cH$ *forms a Lie algebra. We shall write, as customary,* $[\alpha, \beta]$ *instead of* ${}^cL(\alpha, \beta)$.

Because L is bilinear, one can see easily that L is everywhere defined with respect to the discrete topology of K, so we identify L with the bilinear mapping of $H+H$ into H. We shall write $[\xi, \eta]$ instead of $L(\xi, \eta)$ for any ξ, η in H. We define representation and isomorphisms of formal Lie (F)-algebras in the obvious way.

A formal Lie (F)-group G being given, we consider a formal analytic mapping V_1 of $H+H$ into H such that

$$ {}^cV_1(\alpha, \beta) = [\partial/\partial u\, {}^cV(u\alpha, \beta)]_{u=0}. $$

We define a formal analytic mapping L of $H+H$ into H by the formula:

$$ {}^cL(\alpha, \beta) = -{}^cV_1(\alpha, \beta) + {}^cV_1(\beta, \alpha). \tag{21} $$

We shall show that L is a formal Lie (F)-algebra in the parameter space H. L is called the formal Lie (F)-algebra associated with the formal Lie (F)-group G.

Since ${}^cV(0, \alpha) = \alpha$, it is clear that $(u\alpha)\cdot(u\beta) = u\alpha + u\beta + u^2\, {}^cV_1(\alpha, \beta) + \cdots$. Therefore we can write

$$ (u\alpha)\cdot(u\beta) \equiv u\alpha + u\beta + u^2\, {}^cV_1(\alpha, \beta) + u^3 Y(\alpha, \beta) \qquad (\text{mod } u^4). \tag{22} $$

Lemma II. 1. *For any* α, β, *and* γ in ${}^cH$,

$$ {}^cV_1(\alpha, {}^cV_1(\beta, \gamma)) - {}^cV_1({}^cV_1(\alpha, \beta), \gamma) $$
$$ = Y(\alpha, \beta) - Y(\beta, \gamma) + Y(\alpha+\beta, \gamma) - Y(\alpha, \beta+\gamma). $$

Proof. Calculating the both side of the equality: $(u\alpha)\cdot((u\beta)\cdot(u\gamma)) = ((u\alpha)\,(u\beta))\cdot(u\gamma)$ modulo u^4 by the formula (22) and equating the coefficients of u^3, we obtain the formula.

Theorem II. 2. *The formal analytic mapping L defined by* (21) *is a formal Lie* (F)*-algebra.*

Proof. It is clear that L is bilinear and skew-symmetric. Therefore it is sufficient to prove Jacobi's identity:

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$$[[\alpha, \beta], \gamma]+[[\beta, \gamma], \alpha]+[[\gamma, \alpha], \beta]=0.$$

for any α, β, and γ in ${}^cH$. Now set $Z(\alpha, \beta, \gamma)={}^cV_1(\alpha, {}^cV_1(\beta, \gamma))+{}^cV_1(\beta, {}^cV_1(\gamma, \alpha))+{}^cV_1(\gamma, {}^cV_1(\alpha, \beta))-{}^cV_1({}^cV_1(\alpha, \beta), \gamma)-{}^cV_1({}^cV_1(\beta, \gamma), \alpha)-{}^cV_1({}^cV_1(\gamma, \alpha), \beta)$. Then by Lemma II. 1 one find that $Z(\alpha, \beta, \gamma)=(Y(\alpha+\beta, \gamma)-Y(\gamma, \alpha+\beta))+(Y(\beta+\gamma, \alpha)+Y(\alpha+\gamma, \beta))-(Y(\alpha, \beta+\gamma)+Y(\beta, \alpha+\gamma))$. Therefore we see that $Z(\alpha, \beta, \gamma)=Z(\beta, \alpha, \gamma)$. Now if we express the left hand side of Jacobi's identity in terms of the function ${}^cV_1$, then we find that it is equal to $-Z(\alpha, \beta, \gamma)+Z(\beta, \alpha, \gamma)=0$.

2. The fundamental theorems

THEOREM II. 3. *Let G_r be a formal Lie (F)-group, $(r=1, 2)$. Denote by L_r the formal Lie (F)-algebra associated with G_r. Let F be a representation of G_1 into G_2. Then $(dF)_0$ is a representation of L_1 into L_2.*

Proof. Since ${}^cF$ is a homomorphism of the groups, it is easy to see that ${}^cV({}^cF(\alpha'), (d^cF)_0(\beta))=d^cF(\alpha', {}^cV(\alpha', \beta))$. Putting $\alpha'=u\alpha$ and differentiating with respect to u, one finds that ${}^cV_1((d^cF)_0(\alpha), (d^cF)_0(\beta))=(d^cF)_0({}^cV_1(\alpha, \beta))+[\partial/\partial u(d^cF(u\alpha, \beta))]_{u=0}$. Then the theorem follows, because $[\partial/\partial u(d^cF(u\alpha, \beta))]_{u=0}=[\partial/\partial u(d^cF(u\beta, \alpha))]_{u=0}=[\partial^2/\partial u\partial v({}^cF(u\alpha+v\beta))]^{u=0}_{v=0}$.

Let G be a formal Lie (F)-group in the parameter space H. Denote by $[\alpha, \beta]$ the commutator in the associated formal Lie (F)-algebra L. Denote by W' a formal analytic mapping of $H+H+H$ into H defined by the formula:

$${}^cW'(\alpha; \beta, \gamma)=[\partial/\partial u\, {}^cW(\alpha+u\beta, \gamma)]_{u=0}.$$

LEMMA II. 2. *For any α, β, and γ in ${}^cH$,*

$$-{}^cW'(\alpha; \beta, \gamma)+{}^cW'(\alpha; \gamma, \beta)=[{}^cW(\alpha, \gamma), {}^cW(\alpha, \beta)].$$

Proof. Set $Y(u, v)=\alpha^{-1}\cdot(\alpha+u\beta+v\gamma)$. Then by (19), $[\partial/\partial v\, Y(u, v)]_{v=0}={}^cV(u\,{}^cW(\alpha, \beta), {}^cW(\alpha, \gamma)+u\,{}^cW'(\alpha; \beta, \gamma))$ modulo u^2. Therefore $[\partial/\partial u\, \partial/\partial v\, Y(u, v)]_{u=v=0}={}^cV_1({}^cW(\alpha, \beta), {}^cW(\alpha, \gamma))+{}^cW'(\alpha; \beta, \gamma)$. Because $\partial/\partial v(\partial/\partial u\, Y(u, v))=\partial/\partial u(\partial/\partial v\, Y(u, v))$, we have the required equality.

Putting $\alpha=0$ in Lemma II. 2, we find the formula:

$$[\alpha, \beta]={}^cW_1(\alpha, \beta)-{}^cW_1(\beta. \alpha). \tag{23}$$

where ${}^cW_1(\alpha, \beta)={}^cW'(0; \alpha, \beta)$.

PROPOSITION II. 2. *Set* ${}^cZ(\alpha) = {}^cW(\alpha, \alpha)$. *Then*

$$d^cW((\alpha, \beta), (\alpha, \beta)) = -[{}^cW(\alpha, \beta), {}^cZ(\alpha)] + d^cZ(\alpha, \beta).$$

Proof. Calculate $[\partial/\partial u\, {}^cZ(\alpha + u\beta)]$ and apply Lemma II. 2. In particular we have

PROPOSITION II. 2′. *If G is under a canonical coordinate system, then*

$$(**) \qquad d^cW((\alpha, \beta), (\alpha, \beta)) = -[{}^cW(\alpha, \beta), \alpha] + \beta.$$

Proof. Apply Proposition II. 1 to the above formula.

THEOREM II. 4. *Let G_r be a formal Lie (F)-group ($r = 1, 2$). Denote by L_r the formal Lie (F)-algebra associated with G_r. Let F be a representation of L_1 into L_2. If both G_1 and G_2 are under canonical coordinate systems, then F is also a representation of G_1 into G_2.*

Proof. Denote by H and H' the parameter space of G_1 and G_2, respectively. Let A and B be formal analytic mappings of $H + H$ into H' such that ${}^cA(\alpha, \beta) = {}^cF({}^cW(\alpha, \beta))$ and ${}^cB(\alpha, \beta) = {}^cW({}^cF(\alpha), {}^cF(\beta))$, respectively. Operating ${}^cF$ on both side of $(**)$, we find that $d^cA((\alpha, \beta), (\alpha, \beta)) = -[{}^cA(\alpha, \beta), {}^cF(\alpha)] + {}^cF(\beta)$. Replacing α and β in $(**)$ by ${}^cF(\alpha)$ and ${}^cF(\beta)$, respectively, we see that $d^cB((\alpha, \beta), (\alpha, \beta)) = -[{}^cB(\alpha, \beta), {}^cF(\alpha)] + {}^cF(\beta)$. By Theorem I. 5, ${}^cA(\alpha, \beta) = {}^cB(\alpha, \beta)$, i.e. ${}^cF({}^cW(\alpha, \beta)) = {}^cW({}^cF(\beta))$. Operating ${}^cV({}^cF(\alpha), \cdot)$ on both sides of the just proved formula and replacing β by ${}^cV(\alpha, \gamma)$, we find that ${}^cF({}^cV(\alpha, \gamma)) = {}^cV({}^cF(\alpha), {}^cF(\gamma))$. Then by (19) and Theorem I. 6, one sees easily that F is a representation of G_1 into G_2.

THEOREM II. 5. *Let L be a formal Lie (F)-algebra. Then there is a formal Lie (F)-group G such that L is the formal Lie (F)-algebra associated with G.*

Proof. Let H be the parameter space of L. By Theorem I. 5 there is a formal analytic mapping W of $H + H$ into H such that

$$(24) \qquad d^cW((\alpha, \beta), (\alpha, \beta)) = [{}^cW(\alpha, \beta), \alpha] + \beta.$$

By construction ${}^cW(0, \alpha) = \alpha$ and ${}^cW(\alpha, \beta)$ is linear with respect to the variable β. Set ${}^cW'(\alpha; \beta, \gamma) = [\partial/\partial u\, {}^cW(\alpha + u\beta, \gamma)]_{u=0}$. Replacing α in (24) by $\alpha + u\gamma$ and differentiating by u, we find that

(25) $$[\partial/\partial u\, {}^cW'(\alpha+u\alpha\,;\ \gamma,\ \beta)]_{u=0} = -2\,{}^cW'(\alpha\,;\ \gamma,\beta)+[{}^cW'(\alpha\,;\ \gamma,\beta),\alpha]+[{}^cW(\alpha,\beta),\gamma].$$

Putting $\alpha=0$, we see that $[\beta,\gamma]=2\,{}^cW_1(\gamma,\beta)$, where ${}^cW_1(\alpha,\beta)={}^cW'(0\,;\ \alpha,\beta)$. Since the commutator is skew-symmetric, it follows that

(26) $$[\alpha,\beta]={}^cW_1(\beta,\alpha)-{}^cW_1(\alpha,\beta).$$

Set $H_1=H+H$, $H_2=H$, and $H_3=H$. Let F' be the formal analytic mapping of H_1+H_2 into H_3 such that ${}^cF'(\alpha,\gamma,\beta)={}^cW'(\alpha\,;\ \beta,\gamma)-{}^cW'(\alpha\,;\ \gamma,\beta)-[{}^cW(\alpha,\gamma),{}^cW(\alpha,\beta)]$ for any $(\alpha,\gamma)\in H_1$ and $\beta\in H_2$. Then (25), (26), Jacobi's identity, and the definition of W imply that

$$d^cF'((\alpha,\gamma,\beta),(0,0,\beta))=-2^cF'(\alpha,\gamma,\beta)+[{}^cF'(\alpha,\gamma,\beta),\beta],$$

that is,

$$u\partial/\partial u\,{}^cF'(\alpha,\gamma,u\beta)+2\,{}^cF'(\alpha,\gamma,u\beta)=[{}^cF'(\alpha,\gamma,u\beta),u\beta].$$

Moreover ${}^cF'(\alpha,\gamma,0)=0$ by (26). On the other hand, denoting by G' the zero mapping of H_1+H_2 into H_3, we see that ${}^cG'(\alpha,\gamma,u\beta)$ satisfies the same differential equation as ${}^cF'(\alpha,\gamma,u\beta)$. Therefore, by Lemma I. 2., we find that $F'(\alpha,\gamma,\beta)=0$, i.e.

(27) $${}^cW'(\alpha\,;\ \beta,\gamma)-{}^cW'(\alpha\,;\ \gamma,\beta)=[{}^cW(\alpha,\gamma),{}^cW(\alpha,\beta)].$$

By Theorem I. 3 there is a formal analytic mapping V of $H+H$ into H such that

(28) $$\beta={}^cV(\alpha,{}^cW(\alpha,\beta))={}^cW(\alpha,{}^cV(\alpha,\beta)).$$

Set ${}^cV'(\alpha\,;\ \beta,\gamma)=[\partial/\partial u\,{}^cV(\alpha+u\beta,\gamma)]_{u=0}$. Replacing α by $\alpha+u\,{}^cV(\alpha,\gamma)$ in the first equality in (28), differentiaing with respect to u, and replacing β in the resulting equality by ${}^cV(\alpha,\beta)$, we find that ${}^cV'(\alpha\,;\ {}^cV(\alpha,\gamma),\beta)=-{}^cV(\alpha,{}^cW'(\alpha\,;\ {}^cV(\alpha,\beta),{}^cV(\alpha,\gamma))$. Then by (27), we have the formula

(29) $${}^cV'(\alpha\,;\ {}^cV(\alpha,\beta),\gamma)-{}^cV'(\alpha\,;\ {}^cV(\alpha,\gamma),\beta)={}^cV(\alpha,[\beta,\gamma]).$$

Let Y be the formal analytic mapping of $H+H+H$ into H such that ${}^cY(\alpha,\beta,\gamma)={}^cV(\alpha,{}^cW(\beta,\gamma))$. Then (27) and (29) imply that Y satisfies the conditions in Theorem I. 6. Therefore there is a formal analytic mapping G of $H+H$ into H such that

$$d^cG((\alpha, \beta), (0, \gamma)) = {}^cY({}^cG(\alpha, \beta), \beta, \gamma)$$
$$= {}^cV({}^cG(\alpha, \beta), {}^cW(\beta, \gamma)),$$
$${}^cG(\alpha, 0) = \alpha.$$

Then we find that G is a formal Lie (F)-group under a canonical coordinate system such that the given L is the formal Lie (F)-algebra associated with G.

A Formal Lie (F)-group G, or a formal Lie (F)-algebra L, is said to be convergent if the formal analytic mapping G, or L, is convergent. If G is convergent, then the associated formal Lie (F)-algebra is convergent. However, the converse is not always valid.

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Mathematical Institute
Nagoya University