Canad. J. Math. Vol. **61** (3), 2009 pp. 566–582

Convex Subordination Chains in Several Complex Variables

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Abstract. In this paper we study the notion of a convex subordination chain in several complex variables. We obtain certain necessary and sufficient conditions for a mapping to be a convex subordination chain, and we give various examples of convex subordination chains on the Euclidean unit ball in $\mathbb{C}^n$. We also obtain a sufficient condition for injectivity of $f(z/\|z\|, \|z\|)$ on $B^n \setminus \{0\}$, where $f(z,t)$ is a convex subordination chain over $(0,1)$.

1 Introduction and Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n, and the unit ball B_1^n is denoted by B^n. The closed unit ball in $\mathbb{C}^n$ is denoted by $\overline{B}^n$, and the boundary of B^n is denoted by ∂B^n. In the case of one variable, B^1 is denoted by U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If $f \in H(B^n)$, we say that f is *normalized* if $f(0) = 0$ and $Df(0) = I_n$. If $f \in H(B^n)$ is normalized, then f has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} A_k(z^k), \quad z \in B^n,$$

where $A_k = \frac{1}{k!}D^k f(0)$ is the k-th Fréchet derivative of f at $z = 0$.

Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n. In the case of one complex variable, the set $S(B^1)$ is denoted by S. Also let $K(B^n)$ be the subset of $S(B^n)$ consisting of convex mappings on B^n. In the case of one complex variable, the set $K(B^1)$ is denoted by K.

If $f \in H(B^n)$, we say that f is locally biholomorphic on B^n if $J_f(z) \neq 0$, $z \in B^n$, where $J_f(z) = \det Df(z)$ and $Df(z)$ is the derivative of f at z.

Received by the editors May 10, 2006; revised August 25, 2006.

Ian Graham's research is partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant A9221. Hidetaka Hamada's research is partially supported by Grant-in-Aid for Scientific Research (C) no. 17540183 from Japan Society for the Promotion of Science, 2006.

AMS subject classification: Primary 32H02; secondary 30C45.

Keywords: biholomorphic mapping, convex mapping, convex subordination chain, Loewner chain, subordination.

https://doi.org/10.4153/CJM-2009-030-x Published online by Cambridge University Press

If $f, g \in H(B^n)$, we say that f is subordinate to g ($f \prec g$) if there exists a Schwarz mapping v (*i.e.,* $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|, z \in B^n$) such that $f = g \circ v$.

A mapping $f\colon B^n \times [0, \infty) \to \mathbb{C}^n$ is called a *Loewner chain* if $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$, $0 \leq s \leq t < \infty$.

The subordination condition is equivalent to the existence of a unique Schwarz mapping $v = v(z, s, t)$, called the *transition mapping* of $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), \quad z \in B^n, \ t \geq s \geq 0.$$

In [19] and [4] the authors obtained the following sufficient condition for a mapping to be a Loewner chain (see also [7, Theorem 8.1.6]; cf. [22]).

Lemma 1.1 *Let $h = h(z, t)\colon B^n \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:*

(i) *$h(\cdot, t)$ is a normalized holomorphic mapping on B^n and* $\operatorname{Re} \langle h(z, t), z\rangle \geq 0$ *for $z \in B^n$, $t \geq 0$.*

(ii) *$h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B^n$.*

Let $f = f(z, t)\colon B^n \times [0, \infty) \to \mathbb{C}^n$ be a mapping such that $f(\cdot, t) \in H(B^n)$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$, and $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Assume that

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t) \quad a.e. \quad t \geq 0, \ \forall z \in B^n.$$

Further, assume that there exists an increasing sequence $\{t_m\}_{m \in \mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$, and

$$\lim_{m \to \infty} e^{-t_m} f(z, t_m) = F(z)$$

locally uniformly on B^n. Then $f(z, t)$ is a Loewner chain.

In this paper we study the notion of a convex subordination chain in several complex variables. We obtain certain necessary and sufficient conditions for a mapping to be a convex subordination chain and we give some examples of convex subordination chains on the Euclidean unit ball in $\mathbb{C}^n$. Other results related to convex mappings can be found in [2].

2 Convex Subordination Chains

We begin this section with the following subordination result, which provides a necessary and sufficient condition for a mapping to be subordinate to a convex mapping. In the case of one complex variable, see [25]. If $g \equiv f$, then the condition (2.1) reduces to the analytical characterization of convexity due to Suffridge (see [26, 27]).

Theorem 2.1 *Let $f\colon B^n \to \mathbb{C}^n$ be a convex mapping and $g \in H(B^n)$ be such that $g(0) = f(0)$. Then $g \prec f$ if and only if*

$$\operatorname{Re} \langle [Df(z)]^{-1}(f(z) - g(u)), z\rangle > 0, \quad \|u\| < \|z\| < 1. \tag{2.1}$$

https://doi.org/10.4153/CJM-2009-030-x Published online by Cambridge University Press

Proof First assume that $g \prec f$. Then there exists a Schwarz mapping $\omega = \omega(z)$ such that $g(z) = f(\omega(z))$ for $z \in B^n$. Let $z, u \in B^n$ be such that $\|u\| < \|z\|$. Using Suffridge's characterization of convexity (see [26], [27]), we have

$$\operatorname{Re} \langle [Df(z)]^{-1}(f(z) - f(w)), z \rangle > 0, \quad \|w\| < \|z\| < 1,$$

and hence

$$\operatorname{Re} \langle [Df(z)]^{-1}(f(z) - g(u)), z \rangle = \operatorname{Re} \langle [Df(z)]^{-1}(f(z) - f(\omega(u))), z \rangle > 0,$$

since $\|\omega(u)\| \le \|u\| < \|z\| < 1$. Therefore the condition (2.1) holds.

We next assume that the condition (2.1) holds and prove that $g \prec f$. Without loss of generality, we may assume that $f(0) = 0$. Suppose $g \not\prec f$. Then $g(B^n) \not\subseteq f(B^n)$. Since there exists some $r \in (0, 1)$ such that $g(B^n_r) \not\subseteq f(B^n_r)$, there exists a point $z_0 \in B^n_r$ such that $g(z_0) \notin f(B^n_r)$. Since $g(0) = f(0) \in f(B^n_r)$, there exists $t_0 \in (0, 1]$ such that $g(t_0 z_0) \in f(\partial B^n_r)$. Hence there exists a point $z_1 \in \partial B^n_r$ such that $g(t_0 z_0) = f(z_1)$. Next, taking into account this equality and the relation (2.1), we obtain for $z = z_1$ and $u = t_0 z_0$ that

$$\begin{aligned} 0 &< \operatorname{Re} \langle [Df(z_1)]^{-1}(f(z_1) - g(t_0 z_0)), z_1 \rangle \\ &= \operatorname{Re} \langle [Df(z_1)]^{-1}(f(z_1) - f(z_1)), z_1 \rangle = 0. \end{aligned}$$

This is a contradiction. Hence we must have $g \prec f$, as desired. This completes the proof. ■

We next introduce the notion of a convex subordination chain. In the case of one complex variable, see [25].

Definition 2.2 Let J be an interval in $\mathbb{R}$. A mapping $f = f(z, t)\colon B^n \times J \to \mathbb{C}^n$ is called a *convex subordination chain* (c.s.c.) over J if the following conditions hold:

(i) $f(0, t) = 0$ and $f(\cdot\,, t)$ is convex (biholomorphic) for $t \in J$.

(ii) $f(\cdot\,, t_1) \prec f(\cdot\,, t_2)$ for $t_1, t_2 \in J$, $t_1 \le t_2$.

We do not assume continuity in t, although this is needed in Theorem 2.9.

Example 2.3 If $f \in K(B^n)$ and $f(z, t) = e^t f(z)$ for $z \in B^n$ and $t \ge 0$, then $f(z, t) = e^t f(z)$ is a c.s.c. over $[0, \infty)$. For example, the mapping $f(z, t) = e^t z/(1 - z_1)$ is a c.s.c. over $[0, \infty)$. Similarly, if $Q\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 such that $\|Q\| \le 1/2$ and $e_1 = (1, 0, \dots, 0) \in \partial B^n$, then $F\colon B^n \to \mathbb{C}^n$ given by

$$F(z) = \frac{z}{1 - z_1} + Q\Big(\frac{\widetilde{z}}{1 - z_1}\Big) e_1, \quad z = (z_1, \widetilde{z}) \in B^n,$$

is convex by a result of Muir and Suffridge (see [17, 18]). Hence, if $F(z, t) = e^t F(z)$, then $F(z, t)$ is a c.s.c. over $[0, \infty)$.

Remark 2.4. If $f_j(z_j,t)$ is a c.s.c. on U over an interval $J \subseteq \mathbb{R}$ for $j = 1,\dots,n$, then

$$f(z,t) = (f_1(z_1,t),\dots,f_n(z_n,t)), \quad z = (z_1,\dots,z_n) \in B^n, t \in J,$$

need not be a c.s.c. on B^n over J for $n \geq 2$. Indeed, if $f_j(z_j,t) = e^t z_j/(1 - z_j)$ for $|z_j| < 1$, $t \geq 0$ and $j = 1,\dots,n$, then $f_j(z_j,t)$ is a c.s.c. over $[0,\infty)$. Moreover,

$$f(z,t) = \Big(\frac{e^t z_1}{1-z_1},\dots,\frac{e^t z_n}{1-z_n}\Big), \quad z = (z_1,\dots,z_n) \in B^n, t \geq 0,$$

is a Loewner chain, but is not a c.s.c. over $[0,\infty)$ for $n \geq 2$. Indeed, the mapping

$$g(z) = \Big(\frac{z_1}{1-z_1},\dots,\frac{z_n}{1-z_n}\Big), \quad z = (z_1,\dots,z_n) \in B^n,$$

is not convex in dimension $n \geq 2$ (see [23, 24]).

On the other hand, if $f_j(z_j,t)$ is a Loewner chain, which satisfies condition (2.2), then we obtain the following.

Example 2.5 Let $f_j(z_j,t)$ be a Loewner chain such that

$$\Big|\frac{z_j f_j''(z_j,t)}{f_j'(z_j,t)}\Big| \leq 1, \quad |z_j| < 1, \quad t \geq 0,\ j = 1,\dots,n. \tag{2.2}$$

Also let

$$f(z,t) = (f_1(z_1,t),\dots,f_n(z_n,t)), \quad z = (z_1,\dots,z_n) \in B^n,\ t \geq 0.$$

Then $f(z,t)$ is a c.s.c. over $[0,\infty)$.

Proof In view of [9, Theorem 3.4] (see also [16, Theorem 4.1]), we deduce that $f(\cdot,t)$ is a convex mapping for $t \geq 0$. On the other hand, since $f_j(z_j,t)$ is a Loewner chain, it is easily seen that $f(z,t)$ is a Loewner chain too. ∎

The next result gives a necessary and sufficient condition for a mapping to be a c.s.c. over an interval $J \subseteq \mathbb{R}$.

Corollary 2.6 *Let $f = f(z,t)\colon B^n \times J \to \mathbb{C}^n$ be a mapping such that $f(\cdot,t)$ is locally biholomorphic on B^n and $f(0,t) = 0$ for $t \in J$. Then $f(z,t)$ is a c.s.c. if and only if*

$$\operatorname{Re}\langle [Df(z,t_2)]^{-1}(f(z,t_2) - f(u,t_1)), z\rangle > 0 \tag{2.3}$$

for $\|u\| < \|z\| < 1$ and $t_1,t_2 \in J$ with $t_1 \leq t_2$.

Proof It suffices to apply Theorem 2.1. Indeed, if $f(z,t)$ is a c.s.c., then $f(\cdot,t)$ is a convex mapping for $t \in J$ and $f(\cdot,t_1) \prec f(\cdot,t_2)$ for $t_1,t_2 \in J$, $t_1 \leq t_2$. Then the condition (2.3) follows in view of (2.1).

Conversely, if the condition (2.3) holds, then

$$\operatorname{Re}\langle [Df(z,t)]^{-1}(f(z,t) - f(u,t)), z\rangle > 0$$

for all $z,u \in B^n$ with $\|u\| < \|z\|$ and $t \in J$. Hence $f(\cdot,t)$ is convex for $t \in J$ by [26]. Finally, it suffices to apply Theorem 2.1 to conclude that $f(\cdot,t_1) \prec f(\cdot,t_2)$ for $t_1,t_2 \in J$, $t_1 \leq t_2$, as desired. ∎

The basic separation theorem in convexity theory gives the following criterion for a mapping to be a c.s.c. over an interval $J \subseteq \mathbb{R}$. For the proof of Theorem 2.7, we use an argument similar to that in the proof of Theorem 2.8.

Theorem 2.7 *Let $f = f(z,t)\colon \overline{B}^n \times J \to \mathbb{C}^n$ be a mapping such that $f(\cdot,t)$ is continuous on $\overline{B}^n$, $f(\cdot,t)$ is convex on B^n and $f(0,t) = 0$ for $t \in J$. Then $f(z,t)$ is a convex subordination chain over J if and only if*

(2.4)
$$\sup_{\|z\|\le 1} \operatorname{Re}\langle f(z,t_1), w\rangle \le \sup_{\|z\|\le 1} \operatorname{Re}\langle f(z,t_2), w\rangle, \quad \forall w \in \partial B^n,\ t_1,t_2 \in J, t_1 \le t_2.$$

One of the aims of this paper is to give a generalization to several complex variables of a theorem of Ruscheweyh on convex subordination chains over the interval $(0,1)$. We give two criteria for a mapping to be a c.s.c. over this interval. The first uses the maximum principle and ideas similar to Theorem 2.7.

Theorem 2.8 *Let $f = f(z,t)\colon \overline{B}^n \times (0,1) \to \mathbb{C}^n$ be a mapping such that $f(\cdot,t)$ is continuous on $\overline{B}^n$, $f(\cdot,t)$ is convex on B^n and $f(0,t) = 0$ for $t \in (0,1)$. Then $f(z,t)$ is a convex subordination chain over $(0,1)$ if and only if for any $w \in \partial B^n$, the function g_w given by*

$$g_w(z) = \operatorname{Re}\Big\langle f\Big(\frac{z}{\|z\|}, \|z\|\Big), w\Big\rangle, z \in B^n \setminus \{0\},$$

satisfies the condition

$$\sup_{\|z\|=t_1} g_w(z) \le \sup_{\|z\|=t_2} g_w(z), \quad 0 < t_1 \le t_2 < 1. \tag{2.5}$$

Proof First, assume that condition (2.5) holds. We need to prove that $f(\cdot,t_1) \prec f(\cdot,t_2)$ for $t_1,t_2 \in (0,1)$, $t_1 \le t_2$. Since $f(\cdot,t)$ is biholomorphic for $t \in (0,1)$, the previous relation is equivalent to $f(B^n,t_1) \subseteq f(B^n,t_2)$ for $t_1,t_2 \in (0,1)$, $t_1 \le t_2$. Suppose that there exist $t_1,t_2 \in (0,1)$, $t_1 < t_2$, such that $f(B^n,t_1) \nsubseteq f(B^n,t_2)$. Then $f(B^n,t_1) \nsubseteq f(\overline{B}^n,t_2)$, and hence there exists a point $z_0 \in B^n \setminus \{0\}$ such that $f(z_0,t_1) \notin f(\overline{B}^n,t_2)$. Let $Y_1 = \{f_{t_1}(z_0)\}$ and let $Y_2 = f_{t_2}(\overline{B}^n)$ where $f_{t_j}(z) = f(z,t_j)$, $j = 1,2$. Then Y_2 is a nonempty closed and convex set in $\mathbb{C}^n$, and since

$$d(Y_1,Y_2) = \min_{z\in\overline{B}^n} \|f_{t_1}(z_0) - f_{t_2}(z)\| > 0,$$

we deduce that there exists some $l \in L(\mathbb{C}^n,\mathbb{C}) \setminus \{0\}$ such that

$$\sup_{z\in\overline{B}^n} \operatorname{Re}[l(f_{t_2}(z))] < \operatorname{Re}[l(f_{t_1}(z_0))] \tag{2.6}$$

(see *e.g.*, [13, p. 81]). Now, since $l \in L(\mathbb{C}^n,\mathbb{C}) \setminus \{0\}$, there exists a point $w \in \mathbb{C}^n \setminus \{0\}$ such that $l(z) = \langle z, w\rangle$, $z \in \mathbb{C}^n$. We may assume that $\|w\| = 1$. Hence, from (2.6) we obtain

$$\sup_{z\in\overline{B}^n} \operatorname{Re}\langle f(z,t_2), w\rangle < \operatorname{Re}\langle f(z_0,t_1), w\rangle,$$

and thus

$$\sup_{\|z\|\le t_2} \operatorname{Re}\left\langle f\left(\frac{z}{t_2}, t_2\right), w\right\rangle < \operatorname{Re}\langle f(z_0, t_1), w\rangle.$$

In particular, we have

$$\sup_{\|z\|=t_2} \operatorname{Re}\left\langle f\left(\frac{z}{\|z\|}, \|z\|\right), w\right\rangle < \operatorname{Re}\langle f(z_0, t_1), w\rangle,$$

and hence

$$\sup_{\|z\|=t_2} g_w(z) < \operatorname{Re}\langle f(z_0, t_1), w\rangle \le \sup_{\|z\|\le 1} \operatorname{Re}\langle f(z, t_1), w\rangle.$$

Since the function $\operatorname{Re}\langle f(\,\cdot\,, t_1), w\rangle$ is pluriharmonic on B^n, and hence harmonic on B^n, and is continuous on $\overline{B}^n$, we deduce in view of the maximum principle for harmonic functions that

$$\sup_{\|z\|\le 1} \operatorname{Re}\langle f(z, t_1), w\rangle = \sup_{\|z\|=1} \operatorname{Re}\langle f(z, t_1), w\rangle.$$

On the other hand, since

$$\sup_{\|z\|=1} \operatorname{Re}\langle f(z, t_1), w\rangle = \sup_{\|z\|=t_1} g_w(z),$$

we deduce from the above relations that

$$\sup_{\|z\|=t_2} g_w(z) < \sup_{\|z\|=t_1} g_w(z).$$

However, this relation is in contradiction to (2.5). Thus we must have $f(B^n, t_1) \subseteq f(B^n, t_2)$ for $t_1 \le t_2$.

Conversely, assume that $f(z, t)$ is a c.s.c. such that $f(\,\cdot\,, t)$ is continuous on $\overline{B}^n$ for $t \in (0, 1)$. Then there exists a Schwarz mapping $v = v(z, t_1, t_2)$ such that

$$f(z, t_1) = f(v(z, t_1, t_2), t_2), z \in B^n, 0 < t_1 \le t_2 < 1.$$

Therefore, we obtain that

$$\begin{aligned}\sup_{\|z\|\le 1} \operatorname{Re}\langle f(z, t_1), w\rangle &= \sup_{\|z\|<1} \operatorname{Re}\langle f(z, t_1), w\rangle \\ &= \sup_{\|z\|<1} \operatorname{Re}\langle f(v(z, t_1, t_2), t_2), w\rangle \\ &\le \sup_{\|z\|\le 1} \operatorname{Re}\langle f(z, t_2), w\rangle,\end{aligned}$$

for all $w \in \partial B^n$ and $0 < t_1 \le t_2 < 1$. Since the function $\operatorname{Re}\langle f(\,\cdot\,, t_j), w\rangle$ is harmonic on B^n and continuous on $\overline{B}^n$ for $j = 1, 2$, we deduce that

$$\sup_{\|z\|\le 1} \operatorname{Re}\langle f(z, t_j), w\rangle = \sup_{\|z\|=1} \operatorname{Re}\langle f(z, t_j), w\rangle, w \in \partial B^n, j = 1, 2,$$

by the maximum principle for harmonic functions. Hence the relation (2.5) follows, as desired. This completes the proof. ∎

The next criterion is a generalization of a one-variable result of Ruscheweyh (see [25, Theorem 2.40]).

Theorem 2.9 *Let $f = f(z,t)\colon \overline{B}^n \times [0,1) \to \mathbb{C}^n$ be a continuous mapping such that $f(\cdot\,,t)$ is convex on B^n for $t \in (0,1)$, and $f(0,t) = f(z,0) = 0$ for $t \in [0,1)$ and $z \in B^n$. For $w \in \partial B^n$, let G_w be the function defined by*

$$G_w(z) = \begin{cases} g_w(z) & z \in B^n \setminus \{0\} \\ 0 & z = 0. \end{cases} \tag{2.7}$$

If either

(i) *G_w has no local maximum in B^n, for all $w \in \partial B^n$, or*
(ii) *G_w has no maximum in B^n_r, for all $r \in (0,1)$ and for all $w \in \partial B^n$,*

then $f(z,t)$ is a convex subordination chain over $(0,1)$.

Proof We will show that $f(B^n,t_1) \subseteq f(B^n,t_2)$ for $0 < t_1 < t_2 < 1$. Suppose on the contrary that there exist t_1, t_2 and a point $z_0 \in B^n$ such that $f(z_0,t_1) \notin f(B^n,t_2)$. Since $f(B^n,t_1)$ is open and $f(B^n,t_2)$ is a bounded convex domain in $\mathbb{C}^n$, by replacing z_0 by a nearby point if necessary, we may assume that $f(z_0,t_1) \notin f(\overline{B}^n,t_2)$. This implies that there exists $z' \in \partial B^n$ such that $f(z',t_1) \notin f(\overline{B}^n,t_2)$, for otherwise convexity would imply that $f(B^n,t_1) \subseteq f(\overline{B}^n,t_2)$. By the basic separation theorem there exists $w \in \partial B^n$ such that

$$\operatorname{Re}\langle f(\xi,t_2),w\rangle < \operatorname{Re}\langle f(z',t_1),w\rangle$$

for $\xi \in \overline{B}^n$ (equivalently, for $\xi \in \partial B^n$). Letting $\tilde{z} = t_1 z'$, we have

$$0 \le \sup_{\|z\|=t_2} \operatorname{Re}\Big\langle f\Big(\frac{z}{\|z\|},\|z\|\Big),w\Big\rangle < \operatorname{Re}\Big\langle f\Big(\frac{\tilde{z}}{\|\tilde{z}\|},\|\tilde{z}\|\Big),w\Big\rangle.$$

Together with the assumption that $f(z,0) = 0$, for $z \in B^n$, this implies that the corresponding function G_w given by (2.7) has a maximum in $B^n_{t_2}$, and hence a local maximum in B^n. This is a contradiction. ∎

Remark 2.10. Let $f = f(z,t)\colon B^n \times (0,1) \to \mathbb{C}^n$ be a mapping such that $f(\cdot\,,t) \in H(B^n)$, $f(0,t) = 0$, $Df(0,t) = a(t)I_n$, $t \in (0,1)$, where $a\colon (0,1) \to \mathbb{C}$ is a continuous function such that $|a(\cdot)|$ is increasing on $(0,1)$ and $a(t) \ne 0$, $t \in (0,1)$. Assume that

$$f(z,t) = a(t)z + \sum_{k=2}^{\infty} A_k(t)(z^k), \quad z \in B^n$$

is the power series expansion of $f(\cdot\,,t)$ on B^n for $t \in (0,1)$. If

$$\sum_{k=2}^{\infty} k^2\|A_k(t)\| \le |a(t)|, \quad t \in (0,1),$$

then $f(\cdot\,,t)$ is convex on B^n by [24] and further, $f(\cdot\,,t)$ extends as a homeomorphism to $\overline{B}^n$ for $t \in (0,1)$ (see [5]; cf. [10, Corollary 4.6]).

We next apply Theorem 2.9 to obtain an example of a c.s.c. over $(0, 1]$ (cf. [25, Theorem 2.41]). This result is one of the motivations for the study of c.s.c. in several complex variables. It would be interesting to see if Theorem 2.11 remains true for any mapping $f \in K(B^n)$, $n \geq 2$. If $n = 1$, the answer is positive (see [25, Theorem 2.41]).

Theorem 2.11 *For a normalized holomorphic mapping f on B^n, let*

$$s_f(z,t) = \frac{1-t^2}{1+t^2} Df(tz)(tz) + f(tz).$$

Also let

$$f(z) = z + \sum_{k=2}^{\infty} A_k(z^k), \quad z \in B^n.$$

If

$$\sum_{k=2}^{\infty} k^2 \|A_k\| \leq 1, \tag{2.8}$$

then $s_f(z,t)$ is a c.s.c. over $(0, 1]$.

Proof Note that condition (2.8) implies that $f \in K(B^n)$ by [24, Theorem 2.1]. Let

$$b_k = \sup_{0<t\leq 1} \{(1-t^2)k + (1+t^2)\} t^{k-1}.$$

Then $b_k = 2$ for all $k \geq 2$. Since

$$s_f(z,t) = \frac{2t}{1+t^2} z + \sum_{k=2}^{\infty} t^k \Big(\frac{1-t^2}{1+t^2} k + 1 \Big) A_k(z^k),$$

we obtain that

$$\sum_{k=2}^{\infty} k^2 t^k \Big(\frac{1-t^2}{1+t^2} k + 1 \Big) \|A_k\| \leq \frac{t}{1+t^2} \sum_{k=2}^{\infty} k^2 b_k \|A_k\| \leq \frac{2t}{1+t^2}.$$

Therefore, $s_f(\,\cdot\,, t)$ is convex on B^n and extends as a homeomorphism to $\overline{B}^n$ for $t \in (0, 1)$ by Remark 2.10. It is clear from the formula for $s_f(z,t)$ that this mapping is continuous on $\overline{B}^n \times [0, 1)$. Let $z, w \in \partial B^n$ and

$$F_{z,w}(\zeta) = \begin{cases} \operatorname{Re} \left\langle s_f\left(\frac{\zeta z}{\|\zeta z\|}, \|\zeta z\| \right), w \right\rangle & \zeta \in U \setminus \{0\} \\ 0 & \zeta = 0. \end{cases} \tag{2.9}$$

Then $F_{z,w}$ is real analytic on U and is a solution of the elliptic equation

$$\frac{\partial^2 F}{\partial \zeta \partial \overline{\zeta}} + \frac{2}{1-|\zeta|^4} \Big(\zeta \frac{\partial F}{\partial \zeta} + \overline{\zeta} \frac{\partial F}{\partial \overline{\zeta}} \Big) = 0.$$

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By Hopf's maximum principle (see *e.g.*, [1]), $F_{z,w}$ cannot have a local maximum on U unless it is constant.

Now suppose that for some $w \in \partial B^n$ and for some $r \in (0,1)$ the function G_w constructed using s_f has a maximum in B^n_r. This maximum cannot occur at 0, for otherwise the function $F_{z,w}$ given by (2.9) would be identically 0 for all z. This would imply $G_w(z)$ is identically 0. Then $G_w(z) \to 0$ as $\|z\| \nearrow 1$. On the other hand, since

$$\|Df(z)(z)\| = \Big\| z + \sum_{k=2}^{\infty} kA_k(z^k)\Big\| \le \|z\| \Big[1 + \sum_{k=2}^{\infty} k\|A_k\| \cdot \|z\|^{k-1}\Big]$$

$$\le 1 + \frac{1}{2}\sum_{k=2}^{\infty} k^2\|A_k\| \le \frac{3}{2}, \quad z \in B^n,$$

by condition (2.8), we deduce that

$$\lim_{\|z\| \nearrow 1} \Big[\frac{1-\|z\|^2}{1+\|z\|^2} \operatorname{Re}\langle Df(z)(z), w\rangle\Big] = 0.$$

Then

$$0 = \lim_{\|z\| \nearrow 1} G_w(z) = \lim_{\|z\| \nearrow 1} \operatorname{Re}\langle f(z), w\rangle,$$

and thus $\operatorname{Re}\langle f(z), w\rangle = 0$ for $z \in \partial B^n$. This relation implies that $\operatorname{Re}\langle f(z), w\rangle \equiv 0$. Therefore

$$0 = \lim_{r \to 0} \operatorname{Re}\Big\langle \frac{f(rw)}{r}, w\Big\rangle = \operatorname{Re}\langle Df(0)(w), w\rangle.$$

However, this is impossible since $\|w\| = 1$.

Hence the maximum of G_w in B^n_r occurs at a point $z_0 \neq 0$ and has a value greater than 0. But now let $\tilde{z} = z_0/\|z_0\|$ and consider the function $F_{\tilde{z},w}$. This function has a local maximum when $\zeta = \|z_0\|$ and is not constant, which is a contradiction. Hence by Theorem 2.9, $s_f(z,t)$ is a c.s.c. over the interval $(0,1)$.

The only remaining step is to show that $s_f(z,t)$ is actually a c.s.c. over the interval $(0,1]$. This may be seen by applying a version of the Carathéodory convergence theorem in several complex variables (see [11, Theorem 2.1]). The proof is complete. ■

In view of Theorem 2.11 we obtain

Example 2.12 Let $A\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be a symmetric bilinear operator such that $\|A\| \le 1/4$. Also let

$$L(z,t) = \frac{2t}{1+t^2} z + \frac{3t^2 - t^4}{1+t^2} A(z^2), \quad z \in B^n,\ t \in (0,1].$$

Then $L(z,t)$ is a c.s.c. over $(0,1]$.

Proof It suffices to apply Theorem 2.11 with $f(z) = z + A(z^2)$. ■

From Theorem 2.11 we obtain the following consequence (compare with [25]):

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Corollary 2.13 *Let $f\colon B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping which satisfies condition* (2.8)*. Then*

$$\frac{1-t^2}{1+t^2}Df(tz)(tz) + f(tz) \prec f(z), \quad z \in B^n,\ t \in (0,1].$$

Proof Indeed, in view of Theorem 2.11, $s_f(z,t)$ is a c.s.c. over $(0,1]$, and hence $s_f(z,t) \prec s_f(z,1)$ for $z \in B^n$. ∎

The following sufficient condition for injectivity is related to Theorem 2.8. Note the strict inequality in (2.10).

Theorem 2.14 *Let $f = f(z,t)\colon \overline{B}^n \times (0,1) \to \mathbb{C}^n$ be a mapping such that $f(\,\cdot\,,t)$ is continuous and injective on $\overline{B}^n$, $f(\,\cdot\,,t)$ is convex on B^n and $f(0,t) = 0$ for $t \in (0,1)$. If for any $w \in \partial B^n$, the function g_w given in Theorem* 2.8 *satisfies the condition*

$$\sup_{\|z\|=t_1} g_w(z) < \sup_{\|z\|=t_2} g_w(z), \quad 0 < t_1 < t_2 < 1, \tag{2.10}$$

then the mapping $F(z) = f\Big(\dfrac{z}{\|z\|}, \|z\|\Big)$ is injective on $B^n \setminus \{0\}$.

Proof By Theorem 2.8, $f(z,t)$ is a c.s.c. over $(0,1)$. Let $v = v(z,s,t)$ be the transition mapping associated to $f(z,t)$. Using arguments similar to those in the proof of Theorem 2.8, we obtain that

$$f(\overline{B}^n, t_1) \subset f(B^n, t_2), \quad 0 < t_1 < t_2 < 1. \tag{2.11}$$

We argue by contradiction. If there exist $t_1, t_2 \in (0,1)$ such that $t_1 < t_2$ and $f(\overline{B}^n, t_1) \not\subset f(B^n, t_2)$, then there exists a point $z_0 \in \overline{B}^n \setminus \{0\}$ such that $f(z_0, t_1) \notin f(B^n, t_2)$. Let $Y_1 = \{f_{t_1}(z_0)\}$ and let $Y_2 = f_{t_2}(B^n)$. Then Y_2 is a nonempty open and convex set in $\mathbb{C}^n$, Y_1 is also a convex set in $\mathbb{C}^n$ and $Y_1 \cap Y_2 = \varnothing$. In view of a separation theorem by hyperplanes (see *e.g.*, [13, p. 179]), we deduce that there exist some $l \in L(\mathbb{C}^n, \mathbb{C})$ and $c \in \mathbb{R}$ such that $\operatorname{Re}[l(f_{t_2}(z))] < c \le \operatorname{Re}[l(f_{t_1}(z_0))]$, $\forall z \in B^n$. Hence $\sup_{z\in\overline{B}^n} \operatorname{Re}[l(f(z,t_2))] \le \operatorname{Re}[l(f(z_0,t_1))]$. Then as in the proof of Theorem 2.8, we obtain a contradiction (to the strictness of the inequality in (2.10)). Hence the condition (2.11) holds. Since f_{t_1} is continuous on $\overline{B}^n$, it follows that

$$v_{t_1,t_2}(z) = v(z,t_1,t_2) = f_{t_2}^{-1}(f_{t_1}(z)), \quad z \in \overline{B}^n,\ 0 < t_1 < t_2 < 1,$$

defines a continuous extension of v_{t_1,t_2} to $\overline{B}^n$ and

$$v_{t_1,t_2}(\overline{B}^n) \subset B^n, \quad 0 < t_1 < t_2 < 1. \tag{2.12}$$

Now, let $z_1, z_2 \in B^n \setminus \{0\}$ be such that $F(z_1) = F(z_2)$. Let $t_j = \|z_j\|$ for $j = 1,2$. We have one of the following possibilities:

(i) $t_1 = t_2$. Then $f(z_1/t_1, t_1) = f(z_2/t_1, t_1)$ and since $f(\,\cdot\,,t_1)$ is injective on $\overline{B}^n$, we deduce that $z_1 = z_2$.

(ii) $t_1 \neq t_2$. Suppose that $t_1 < t_2$. We have

$$f\Big(\frac{z_2}{t_2}, t_2\Big) = f\Big(\frac{z_1}{t_1}, t_1\Big) = f\Big(v\Big(\frac{z_1}{t_1}, t_1, t_2\Big), t_2\Big)$$

and since $f(\,\cdot\,, t_2)$ is injective on $\overline{B}^n$, we deduce that

$$v\Big(\frac{z_1}{t_1}, t_1, t_2\Big) = \frac{z_2}{t_2} \in \partial B^n.$$

However, this is a contradiction to (2.12).

In conclusion, we must have $z_1 = z_2$, as desired. This completes the proof. ■

Taking into account the proof of Theorem 2.14 and in view of Theorem 2.7, we obtain the following.

Theorem 2.15 *Let $f = f(z,t)\colon \overline{B}^n \times (0,1) \to \mathbb{C}^n$ be a mapping such that $f(\,\cdot\,, t)$ is continuous and injective on $\overline{B}^n$, $f(\,\cdot\,, t)$ is convex on B^n and $f(0,t) = 0$ for $t \in (0,1)$. If*

$$\sup_{\|z\| \le 1} \operatorname{Re} \langle f(z,t_1), w\rangle < \sup_{\|z\| \le 1} \operatorname{Re} \langle f(z,t_2), w\rangle, \quad \forall w \in \partial B^n,\ 0 < t_1 < t_2 < 1,$$

then $f(z,t)$ is a convex subordination chain over $(0,1)$ and the mapping

$$F(z) = f\Big(\frac{z}{\|z\|}, \|z\|\Big)$$

is injective on $B^n \setminus \{0\}$.

Combining the results of Theorems 2.9 and 2.14, we obtain the following sufficient condition of injectivity.

Corollary 2.16 *Let $f = f(z,t)\colon \overline{B}^n \times [0,1) \to \mathbb{C}^n$ be a continuous mapping such that $f(\,\cdot\,, t)$ is convex on B^n, $f(\,\cdot\,, t)$ is injective on $\overline{B}^n$ for $t \in (0,1)$, and $f(0,t) = f(z,0) = 0$ for $t \in [0,1)$ and $z \in B^n$. If the function G_w defined in Theorem 2.9 has no maximum in B^n_r, for all $r \in (0,1)$ and for all $w \in \partial B^n$, then $f(z,t)$ is a convex subordination chain over $(0,1)$ and the mapping $F\colon B^n \to \mathbb{C}^n$ given by*

$$F(z) = \begin{cases} f\Big(\frac{z}{\|z\|}, \|z\|\Big) & z \in B^n \setminus \{0\} \\ 0 & z = 0 \end{cases}$$

is injective on B^n.

Proof Since condition (ii) of Theorem 2.9 holds, it is not difficult to see that condition (2.10) holds too. Thus F is injective on $B^n \setminus \{0\}$. Since $F(0) = 0$ and $f(\,\cdot\,, t)$ is injective on $\overline{B}^n$ for $t \in (0,1)$, it follows that $f(z/\|z\|, \|z\|) \neq 0$ for $z \in B^n \setminus \{0\}$. Hence F is injective on B^n. This completes the proof. ■

In view of Theorem 2.11 and Corollary 2.16, we obtain the following consequence.

https://doi.org/10.4153/CJM-2009-030-x Published online by Cambridge University Press

Corollary 2.17 *Let $f\colon B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping which satisfies condition* (2.8)*. Then the mapping $F\colon B^n \to \mathbb{C}^n$ given by*

$$F(z) = \frac{1-\|z\|^2}{1+\|z\|^2}Df(z)(z) + f(z)$$

is injective on B^n.

Proof Taking into account the proof of Theorem 2.11, we deduce that the mapping $s_f(z,t)$ satisfies condition (ii) of Theorem 2.9. Also $s_f(\,\cdot\,,t)$ is convex on B^n and has a continuous and injective extension to $\overline{B}^n$ for $t \in (0,1)$, by the proof of Theorem 2.11. Then it is easy to see that $s_f(z,t)$ is continuous on $\overline{B}^n \times [0,1)$. Hence we deduce from Corollary 2.16 that the mapping F is injective on B^n, as desired. ■

We next obtain another example of a c.s.c. over $(0,1)$.

Example 2.18 Let $A\colon (0,1) \to L(\mathbb{C}^n,\mathbb{C}^n)$ be a continuous mapping such that $\det A(t) \neq 0$, $t \in (0,1)$, and let $f(z,t) = A(t)(z)$ for $z \in B^n$ and $t \in (0,1)$. Then $f(z,t)$ is a c.s.c. over $(0,1)$ if and only if

$$\|A^*(t_1)(w)\| \le \|A^*(t_2)(w)\|, \quad 0 < t_1 \le t_2 < 1,\ w \in \partial B^n, \tag{2.13}$$

where $A^*(t_j)$ is the adjoint operator of $A(t_j)$, $j = 1,2$. In addition, if the strict inequality holds in (2.13) for $0 < t_1 < t_2 < 1$ and $w \in \partial B^n$, then the mapping $F(z) = A(\|z\|)(z/\|z\|)$ is injective on $B^n \setminus \{0\}$.

Proof Clearly $f(\,\cdot\,,t)$ is convex on B^n and is continuous on $\overline{B}^n$ for $t \in (0,1)$. On the other hand, since

$$\begin{aligned}\sup_{\|z\|\le 1} \operatorname{Re}\langle f(z,t),w\rangle &= \sup_{\|z\|\le 1} \operatorname{Re}\langle A(t)(z),w\rangle = \sup_{\|z\|\le 1} \operatorname{Re}\langle z, A^*(t)(w)\rangle \\ &= \|A^*(t)(w)\|,\end{aligned}$$

for $t \in (0,1)$ and $w \in \partial B^n$, we deduce that relation (2.4) reduces to condition (2.13). From Theorem 2.7 we deduce that $f(z,t)$ is a c.s.c. over $(0,1)$ if and only if (2.13) holds, as desired. The second part follows from Theorem 2.15. ■

3 Examples of Convex Subordination Chains Over $[0,\infty)$

We next obtain some examples of c.s.c. over $[0,\infty)$ on B^n by starting with convex subordination chains over $[0,\infty)$ on the unit disc.

Example 3.1 If $f(z_1,t)$ is a c.s.c. over $[0,\infty)$ on the unit disc U such that $f'(0,t) = e^t$, $t \ge 0$, and if

$$\Phi_n(f)(z,t) = \Big(f(z_1,t), \tilde{z}e^{t/2}(f'(z_1,t))^{1/2}\Big), z = (z_1,\tilde{z}) \in B^n, \quad t \ge 0,$$

then $\Phi_n(f)(z,t)$ is a c.s.c. over $[0,\infty)$ on B^n. We choose the branch of the power function such that $(f_t'(z_1))^{1/2}|_{z_1=0} = e^{t/2}$ for $t \ge 0$, where $f_t(z_1) = f(z_1,t)$.

Proof Indeed, since $f(z_1,t)$ is a c.s.c. such that $f'(0,t)=e^t$ for $t\geq 0$, it follows that $g_t=e^{-t}f(\cdot,t)\in K$ for each $t\geq 0$. Fix $t\geq 0$ and let

$$G(z,t)=G_t(z)=\Big(g_t(z_1),\tilde{z}(g_t'(z_1))^{1/2}\Big),z=(z_1,\tilde{z})\in B^n.$$

Then $G_t\in K(B^n)$ by [23] (see also [6] and [3]) and since $\Phi_n(f)(z,t)=e^tG(z,t)$, we deduce that $\Phi_n(f)(\cdot,t)$ is a convex mapping on B^n. Next, since $f(\cdot,t_1)\prec f(\cdot,t_2)$ for $0\leq t_1\leq t_2<\infty$, we deduce that there exists a Schwarz function $v_{t_1,t_2}(z_1)=v(z_1,t_1,t_2)$ such that

$$f(z_1,t_1)=f(v(z_1,t_1,t_2),t_2),z_1\in U,\quad t_2\geq t_1\geq 0.$$

Let $V_{t_1,t_2}\colon B^n\to\mathbb{C}^n$ be given by

$$V_{t_1,t_2}(z)=\Big(v_{t_1,t_2}(z_1),\tilde{z}e^{(t_1-t_2)/2}(v_{t_1,t_2}'(z_1))^{1/2}\Big),\quad z=(z_1,\tilde{z})\in B^n,\ t_2\geq t_1\geq 0.$$

We choose the branch of the power function such that $(v_{t_1,t_2}'(z_1))^{1/2}|_{z_1=0}=e^{(t_1-t_2)/2}$. Then V_{t_1,t_2} is a holomorphic mapping on B^n, $V_{t_1,t_2}(0)=0$, and taking into account the Schwarz–Pick Lemma, we obtain that

$$\begin{aligned}\|V_{t_1,t_2}(z)\|^2&=|v_{t_1,t_2}(z_1)|^2+\|\tilde{z}\|^2e^{t_1-t_2}|v_{t_1,t_2}'(z_1)|\\&\leq|v_{t_1,t_2}(z_1)|^2+\|\tilde{z}\|^2\frac{1-|v_{t_1,t_2}(z_1)|^2}{1-|z_1|^2}<1,\end{aligned}$$

for all $z=(z_1,\tilde{z})\in B^n$ and $t_2\geq t_1\geq 0$. Hence V_{t_1,t_2} is a Schwarz mapping and it is easy to see that

$$\Phi_n(f)(z,t_1)=\Phi_n(f)(V_{t_1,t_2}(z),t_2),z\in B^n,t_2\geq t_1\geq 0.$$

Consequently, $\Phi_n(f)(\cdot,t_1)\prec\Phi_n(f)(\cdot,t_2)$, and thus $\Phi_n(f)(z,t)$ is a c.s.c. over $[0,\infty)$, as desired. ■

Before giving another example of a c.s.c. over $[0,\infty)$, we introduce the following definitions.

Definition 3.2 Let $\mathcal{L}_n$ be the set of all Loewner chains on $B^n\times[0,\infty)$. The set $\mathcal{L}_1$ is denoted by $\mathcal{L}$. Let $\mathcal{L}_n^0$ be the set of all Loewner chains $f(z,t)$ on $B^n\times[0,\infty)$ such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family. (Equivalently, $\mathcal{L}_n^0$ is the set of Loewner chains whose initial element $f(\cdot,0)$ has parametric representation; see [7, 8, 21]). We also denote by $\mathcal{L}C_n$ the set of all convex subordination chains $f(z,t)$ over $[0,\infty)$ on B^n such that $Df(0,t)=e^tI_n$ for $t\geq 0$. The set $\mathcal{L}C_1$ is denoted by $\mathcal{L}C$.

We note that $\mathcal{L}_1^0=\mathcal{L}$ and $\mathcal{L}C_n\subseteq\mathcal{L}_n^0$. Any Loewner chain $f(z,t)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z\in B^n$ (see [7, 8]).

In view of Example 3.1, $\Phi_n(\mathcal{L}C)\subseteq\mathcal{L}C_n$. In [7, 8] it is shown that the operator Φ_n has the property that $\Phi_n(\mathcal{L})\subseteq\mathcal{L}_n^0$.

The following definition is motivated by the recent work of Muir [14].

Definition 3.3 Let $Q\colon \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2. For any function $f \in \mathcal{L}$, define the mapping $\Phi_{n,Q}(f)\colon B^n \times [0,\infty) \to \mathbb{C}^n$ by

$$\Phi_{n,Q}(f)(z,t) = \big(f(z_1,t)+Q(\widetilde{z})f'(z_1,t), \tilde{z}e^{t/2}(f'(z_1,t))^{1/2}\big),\ z = (z_1,\widetilde{z}) \in B^n,\ t \geq 0.$$

We choose the branch of the power function such that $(f'(z_1,t))^{1/2}|_{z_1=0} = e^{t/2}$ for $t \geq 0$.

Remark 3.4. Note that $\Phi_{n,0} = \Phi_n$, where Φ_n is the operator given in Example 3.1. It is easy to see that $e^{-t}\Phi_{n,Q}(f)(\cdot,t) \in S(B^n)$ for $t \geq 0$. Also, since $f(z_1,t)$ is a Loewner chain, $f(z_1,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z_1 \in U$. Then $\Phi_{n,Q}(f)(z,\cdot)$ is also locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$.

Theorem 3.5 *If $Q\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2, then $\Phi_{n,Q}(\mathcal{L}C) \subseteq \mathcal{L}C_n$ if and only if $\|Q\| \leq 1/2$.*

Proof First, assume $\|Q\| \leq 1/2$. Let $f = f(z_1,t) \in \mathcal{L}C$. Also let $F(z,t) = \Phi_{n,Q}(f)(z,t)$ for $z \in B^n$ and $t \geq 0$. Then $F(0,t) = 0$ and since $f'(0,t) = e^t$ and Q is homogeneous of degree 2, it follows that $DF(0,t) = e^t I_n$. It is not difficult to deduce that $F(\cdot,t)$ is biholomorphic on B^n. On the other hand, since $f(z_1,t)$ is a Loewner chain, $f(z_1,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z_1 \in U$, and there is a function $p(z_1,t)$ such that $p(\cdot,t) \in H(U)$, $p(0,t) = 1$, $\operatorname{Re} p(z_1,t) > 0$, $|z_1| < 1$, $t \geq 0$, and

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t)p(z_1,t) \quad a.e. \quad t \geq 0,\ \forall z_1 \in U.$$

Moreover, there is an increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $0 < t_m \to \infty$ and the limit

$$\lim_{m\to\infty} e^{-t_m} f(z_1,t_m) = g(z_1)$$

exists locally uniformly on U (see [20] and [7]). Clearly g is a holomorphic function on U and since $g(0) = 0$, $g'(0) = 1$, we deduce by Hurwitz's theorem that $g \in S$. Then $F(z,\cdot)$ is also locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$ and

$$\lim_{m\to\infty} e^{-t_m} F(z,t_m) = \Phi_{n,Q}(g)(z,0)$$

locally uniformly on B^n. Now, let

$$h(z,t) = \Big(z_1 p(z_1,t) - Q(\widetilde{z}), \frac{\widetilde{z}}{2}\Big(1 + p(z_1,t) + z_1 p'(z_1,t) + Q(\widetilde{z})\frac{f''(z_1,t)}{f'(z_1,t)}\Big)\Big),$$

for all $z \in B^n$ and $t \geq 0$. Then $h(\cdot,t)$ is a normalized holomorphic mapping on B^n for $t \geq 0$ and $h(z,\cdot)$ is measurable on $[0,\infty)$ for all $z \in B^n$. Using elementary computations and the equality, based on Vitali's theorem (see *e.g.,* [20, Chapter 6]),

$$\frac{\partial}{\partial t}\Big(\frac{\partial f}{\partial z_1}\Big)(z_1,t) = \frac{\partial}{\partial z_1}\Big(\frac{\partial f}{\partial t}\Big)(z_1,t) \quad a.e. \quad t \geq 0,\ \forall z_1 \in U,$$

we obtain that

$$\frac{\partial F}{\partial t}(z,t) = DF(z,t)h(z,t) \quad a.e. \quad t \geq 0, \quad \forall z \in B^n.$$

On the other hand, since $e^{-t}f(\,\cdot\,,t) \in K$, $t \geq 0$, it follows that

$$\Big| \frac{1-|z_1|^2}{2} \cdot \frac{f''(z_1,t)}{f'(z_1,t)} - \overline{z}_1 \Big| \leq 1, \quad |z_1| < 1,\ t \geq 0, \tag{3.1}$$

(see *e.g.*. [7]). Next, using the fact that $\|Q\| \leq 1/2$, the above inequality and arguments similar to those in the proof of [8, Theorem 2.1], we obtain that $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$ for $z \in B^n$ and $t \geq 0$. Indeed, if $\widetilde{z} = 0$, then

$$\operatorname{Re}\langle h(z,t), z\rangle = |z_1|^2 \operatorname{Re} p(z_1,t) \geq 0, \quad |z_1| < 1.$$

Next, we assume that $\widetilde{z} \neq 0$. Then it is easy to see that $h(\,\cdot\,,t)$ is holomorphic in a neighborhood of each point $z = (z_1, \widetilde{z}) \in \overline{B}^n$ with $\widetilde{z} \neq 0$. Let us write $z = \lambda Z$ for $Z = (Z_1, \tilde{Z}) \in \partial B^n$ such that $\tilde{Z} \neq 0$ and $0 < |\lambda| \leq 1$. Then the inequality $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$ is equivalent to

$$\operatorname{Re}\Big\langle \frac{h(\lambda Z,t)}{\lambda}, Z\Big\rangle \geq 0.$$

The left-hand side of the above expression is the real part of a holomorphic function of the complex variable $\lambda \in \overline{U}$, and hence is harmonic. Taking into account the minimum principle for harmonic functions, the minimum of the above expression occurs for some $\lambda \in \partial U$, and hence $z \in \partial B^n$. Consequently, it suffices to prove that

$$\operatorname{Re}\langle h(z,t), z\rangle \geq 0, \quad z = (z_1, \widetilde{z}) \in \partial B^n,\ \widetilde{z} \neq 0,\ t \geq 0.$$

Since $p(0,t) = 1$ and $\operatorname{Re} p(z_1,t) > 0$, it follows that (see *e.g.*, [7])

$$|p'(z_1,t)| \leq \frac{2}{1-|z_1|^2} \operatorname{Re} p(z_1,t), \quad |z_1| < 1,\ t \geq 0. \tag{3.2}$$

Fix $t \geq 0$ and let $z = (z_1, \widetilde{z}) \in \partial B^n$ with $\widetilde{z} \neq 0$. In view of relations (3.1) and (3.2), we obtain

$$\begin{aligned}
\operatorname{Re}\langle h(z,t), z\rangle &= \frac{1+|z_1|^2}{2}\operatorname{Re} p(z_1,t) + \frac{1-|z_1|^2}{2}\operatorname{Re}[z_1 p'(z_1,t)] \\
&\quad + \frac{1-|z_1|^2}{2} + \operatorname{Re}\Big[Q(\widetilde{z})\Big\{\frac{1-|z_1|^2}{2} \cdot \frac{f''(z_1,t)}{f'(z_1,t)} - \overline{z}_1\Big\}\Big] \\
&\geq \frac{(1-|z_1|)^2}{2}\operatorname{Re} p(z_1,t) + \frac{1-|z_1|^2}{2} - (1-|z_1|^2)\|Q\| \geq 0,
\end{aligned}$$

whenever $\|Q\| \leq 1/2$. Taking into account Lemma 1.1, we deduce that $F(z,t)$ is a Loewner chain.

Next, let $q_t(z_1) = e^{-t} f_t(z_1)$. Then $q_t \in K$ and since

$$e^{-t}F(z,t) = \Big(q_t(z_1) + Q(\widetilde{z})q_t'(z_1), \widetilde{z}(q_t'(z_1))^{1/2} \Big), \quad z \in B^n,\ t \geq 0,$$

we conclude by [14, Theorem 3.1] that $e^{-t}F(\,\cdot\,,t) \in K(B^n)$, $t \geq 0$. Therefore $\Phi_{n,Q}(f) \in \mathcal{L}C_n$.

For the converse, suppose that $\|Q\| > 1/2$. Let $f(\zeta,t) = e^t\zeta/(1-\zeta)$ for $|\zeta| < 1$ and $t \geq 0$. Then $f(\zeta,t)$ is a c.s.c. over $[0,\infty)$. Also $F(z,t) = e^t G(z)$ where

$$G(z) = \Big(\frac{z_1}{1-z_1} + \frac{Q(\tilde{z})}{(1-z_1)^2}, \frac{\tilde{z}}{1-z_1} \Big), \quad z = (z_1, \tilde{z}) \in B^n.$$

Muir and Suffridge [18] proved that $G \notin K(B^n)$ if $\|Q\| > 1/2$, and hence $F(z,t)$ cannot be a c.s.c. over $[0,\infty)$. This completes the proof. ■

Remark 3.6. Using arguments similar to those in the above proof, it is possible to show that $\Phi_{n,Q}(\mathcal{L}) \subseteq \mathcal{L}_n^0$ if and only if $\|Q\| \leq 1/4$ (see [12]). Note that Theorem 3.5 has also recently been proved by Muir [15].

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