On the Reduction of Singular Matrix Pencils

By H. W. TURNBULL.

(*Received* 13*th March*, 1934. *Read* 7*th December*, 1934.)

Introduction.

The following rational method of dealing with the reduction of a singular matrix pencil to canonical form has certain advantages. It is based on the principle of vector chains, the length of the chain determining a minimal index. This treatment is analogous to that employed by Dr A. C. Aitken and the author in *Canonical Matrices* (1932) 45-57, for the nonsingular case. In Theorems 1 and 2 tests are explicitly given for determining the minimal indices. Theorem 2 gives a method of discovering the lowest row (or column) minimal index. Theoretically it should be possible to state a corresponding theorem for each of these indices, not necessarily the lowest, and prior to any reduction of the pencil. This extension still awaits solution.

Theorem 3 is logically equivalent to the arguments used by Kronecker (who was the first to discuss the singular case, *Berlin Sitzungsberichte* (1890), 1375 and (1891), 9, 33) and subsequently by Dickson (*Trans. American Math. Soc.*, **29** (1927), 239-253). For a geometrical treatment see Segre, *Atti Acc. Torino*, **19** (1884), 878.

§ 1. Let

$$\Lambda = rA + sB = [ra_{ij} + sb_{ij}] \tag{1}$$

be a matrix pencil, where the elements a_{ij} and b_{ij} all belong to a field $\mathcal{F}$, while r, s are independent variables. Each matrix A, B, Λ is assumed to have n rows and n' columns, while neither A nor B is a scalar multiple of the other. It is proposed to reduce Λ to a canonical form

$$P\Lambda Q, \qquad |P| \neq 0, \qquad |Q| \neq 0, \tag{2}$$

where P and Q are nonsingular constant matrices with elements in $\mathcal{F}$, P having n rows and columns, and Q having n'.

Let ρ be the rank of Λ in r and s: that is, let ρ be the highest order among the minor determinants of Λ *which do not vanish* *identically for all r and s.* Then obviously there exist nonnegative integers μ and μ' such that

$$\mu = n - \rho \geqq 0, \qquad \mu' = n' - \rho \geqq 0. \tag{3}$$

Two cases arise, the singular and the nonsingular. In the latter both μ and μ' are zero: in the former at least one, μ say, is nonzero. It will be proved that in the singular case μ is the number of ways, linearly independent in $\mathcal{F}$, in which the rows of Λ are linearly related, while μ' similarly relates the columns.

Such a row relation $\sum_{i=1}^{n} \theta_i \text{ row}_i = 0$ can be conveniently written as a matrix product

$$\theta\Lambda \equiv [\theta_1, \theta_2, \ldots, \theta_n](rA + sB) = 0 \tag{4}$$

as appears at once when written out in full. The coefficients θ_i of this relation here appear as the n components of a row vector θ which is said to annihilate Λ.

By Smith's Theorem[1] the matrix Λ can be reduced in $\mathcal{F}$ to a diagonal form D such that

$$H\Lambda K = \text{diag}\,(E_1, E_2, \ldots, E_\rho, 0, \ldots, 0) = D \tag{5}$$

where H and K are nonsingular matrices each of whose elements are homogeneous polynomials in r, s, divided possibly by a power of s, whereas the determinants $|H|$ and $|K|$ are independent of r. The ρ nonzero elements E are the invariant factors. This allows relation (4) to take a simpler form: thus $0 = \theta\Lambda = \theta H^{-1} DK^{-1}$. Hence

$$\phi D = 0, \qquad \text{where } \phi = \theta H^{-1}. \tag{6}$$

This ϕ, so found, is also a row vector: and clearly it can annihilate D if and only if its first ρ components are zero. Thus

$$\phi = [0, 0, \ldots, 0, \phi_{\rho+1}, \phi_{\rho+2}, \ldots, \phi_n] \tag{7}$$

where the last $\mu\ (= n - \rho)$ components are arbitrary functions of r and s. Let the unit matrix of n rows and columns be written

$$I = \{i_1, i_2, \ldots, i_n\} \tag{8}$$

where $i_1 = [1, 0, \ldots, 0]$, $i_2 = [0, 1, \ldots, 0]$, $\ldots$, $i_n = [0, \ldots, 0, 1]$. Then (7) can be written

$$\phi = \phi_{\rho+1}\, i_{\rho+1} + \phi_{\rho+2}\, i_{\rho+2} + \ldots + \phi_n\, i_n \tag{9}$$

[1] Turnbull and Aitken: *Canonical Matrices* (1932), 23.

which shows that the most general condition (4) is a consequence of μ linearly independent conditions

$$i_{\rho+1} D = 0, \quad i_{\rho+2} D = 0, \; \ldots, \quad i_n D = 0. \tag{10}$$

Each of these conditions is of the form $i_h H \Lambda K = 0$, from which the nonsingular K and any scalar common factor of the components can be deleted and the powers of s in the denominators cleared, the result being called $\theta\Lambda = 0$. This proves the following theorem.

THEOREM I. *If the pencil $\Lambda = rA + sB$ possesses row dependence, there are exactly μ distinct conditions*

$$\theta\Lambda = 0$$

where θ is a row-vector whose components are homogeneous polynomials in r and s with coefficients in $\mathcal{F}$, $(n - \mu)$ being the rank of Λ in r and s.

Correlatively: *there are exactly μ' distinct relations of column dependence*

$$\Lambda\theta' = 0$$

where θ' is a column vector homogeneous in r and s.

There is no necessary connection between θ and θ'.

§ 2. Let the relations just found be arranged in ascending degree in r and s, as

$$\theta_1 \Lambda = 0, \; \theta_2 \Lambda = 0, \; \ldots, \; \theta_\mu \Lambda = 0, \tag{1}$$

where the degree of the *vector* θ_i is m_i, so that

$$0 \leqq m_1 \leqq m_2 \leqq \ldots \leqq m_\mu. \tag{2}$$

These are in fact the Kronecker minimal indices of row dependence characterising a singular pencil. A like set $[m_i']$, μ' in number, refers to column dependence. These sets $[m_i]$ and $[m_i']$ are numerical invariants under nonsingular transformation from Λ to the type $P\Lambda Q$ above, and also under nonsingular linear transformation from r, s to r', s'. The proof of these statements is immediate: in either case we have only to suppose the contrary and then obtain an identity of less than minimal order by applying the reciprocal transformation; which involves an absurdity.

These minimal indices together with the set of invariant factors $E_1, E_2, \ldots, E_\rho$ of § 1 (5) completely characterise the pencil Λ under such transformations, as Kronecker originally proved. A direct method will now be given for finding these minimal indices.

§ 3. Consider the following matrices

$$M_1 = [A, B],\ M_2 = \begin{bmatrix} A & B & . \\ . & A & B \end{bmatrix},\ M_3 = \begin{bmatrix} A & B & . & . \\ . & A & B & . \\ . & . & A & B \end{bmatrix}, \ldots ,$$

$$N_1 = \begin{bmatrix} A \\ B \end{bmatrix},\ N_2 = \begin{bmatrix} A & . \\ B & A \\ . & B \end{bmatrix},\ N_3 = \begin{bmatrix} A & . & . \\ B & A & . \\ . & B & A \\ . & . & B \end{bmatrix}, \ldots , \tag{1}$$

the M consisting of n, $2n$, $3n$, ... rows, and the N of n', $2n'$, $3n'$, ... columns respectively. Let ρ_i, ρ_j' denote their respective ranks. Then, if $\mu_i = in - \rho_i$, $\mu_j' = jn' - \rho_j'$, we have

$$\mu_1 = n - \rho_1 \geqq 0,\ \mu_2 = 2n - \rho_2 \geqq 0,\ \mu_3 = 3n - \rho_3 \geqq 0, \ldots , \tag{2}$$

and

$$\mu_1' = n' - \rho_1' \geqq 0,\ \mu_j' \geqq 0. \tag{3}$$

THEOREM 2. *If μ_{m+1} is the first nonzero integer in the sequence μ_1, μ_2, ..., then m is the value of the smallest minimal index of row dependence, while μ_{m+1} is the number of such indices which are equal. Column dependence is given similarly by $\mu'_{m'+1}$.*

Proof. By Smith's Theorem, if $\mu_1 > 0$, exactly μ_1 distinct relations $\sum_i \lambda_i \,\text{row}_i = 0$ exist between the rows of M_1, where the λ_i are $2n'$ constants which are not all zero in $\mathcal{F}$. On introducing the row vector

$$u = [\lambda_1, \lambda_2, \ldots , \lambda_n] \neq 0$$

we may put such a relation in the form of a matrix product

$$u\,[A, B] = 0, \tag{4}$$

that is $uA = 0$, $uB = 0$: so that $u\,[rA + sB] = 0$ for all r, s. But u is a nonzero constant vector in $\mathcal{F}$. We have therefore secured a minimal index $m_1 = 0$: and the number of such is $\mu_1\ (\neq 0)$.

Next if $\mu_1 = 0$, $\mu_2 > 0$, then a row vector consisting of $2n$ components exists such that

$$[u_1, u_2] \begin{bmatrix} A & B & . \\ . & A & B \end{bmatrix} = 0, \qquad [u_1, u_2] \neq 0. \tag{5}$$

Here u_1 is a set of n components, u_2 is a further set, and in all there are $2n$ components. Hence

$$u_1 A = 0,\ u_1 B + u_2 A = 0,\ u_2 B = 0,$$

whence

$$[u_1 r + u_2 s]\,[rA + sB] = 0, \tag{6}$$

https://doi.org/10.1017/S0013091500008002 Published online by Cambridge University Press

for all values of r, s. But this is explicitly a minimal relation $\theta\Lambda = 0$, where $\theta = ru_1 + su_2$ is a vector of index *unity*. There are μ_2 such distinct relations, while there are none of the zero index type, since $u[A, B] \neq 0$ if $\mu_1 = 0$, for all nonzero constant vectors u.

Next if $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 > 0$, then (4) and (5) are impossible, but three vectors u_1, u_2, u_3 each of n components exist such that

$$[u_1, u_2, u_3]\, M_3 = 0, \qquad [u_1, u_2, u_3] \neq 0. \tag{7}$$

Hence

$$u_1 A = 0,\ u_1 B + u_2 A = 0,\ u_2 B + u_3 A = 0,\ u_3 B = 0;$$

that is

$$[u_1 r^2 + u_2 rs + u_3 s^2]\,[rA + sB] = 0 \tag{8}$$

for all r, s. This gives μ_3 distinct relations of index 2. The general case is now evident: it also applies to columns by means of the expressions $\begin{bmatrix} A \\ B \end{bmatrix} \{u_1'\}$, $\begin{bmatrix} A & . \\ B & A \\ . & B \end{bmatrix} \{u_1', u_2'\}$ etc., where $\{u_1', u_2'\}$ denotes a column of $2n$ elements. This proves the theorem.

It should be remarked that the matrices N are not the transposed of the M: the elements within A (and B) maintain their same relative positions. Also, while the method discovers the initial index m_1 or m_1' it does not at once discover higher indices, if any.

For example:

$$\Lambda = \begin{bmatrix} r & . & . & . \\ s & r & . & . \\ . & s & r & . \\ . & . & s & . \\ s & . & . & r \\ . & . & . & s \end{bmatrix}, \quad A = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & . \\ . & . & . & 1 \\ . & . & . & . \end{bmatrix}, \quad B = \begin{bmatrix} . & . & . & . \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ 1 & . & . & . \\ . & . & . & 1 \end{bmatrix}. \tag{9}$$

Here the ranks of M_1, M_2, M_3, M_4 are 6, 12, 16, 20 respectively so that $\mu_3 = 2$ is the first nonzero μ. This implies two minimal indices each equal to $3 - 1 = 2$.

§4. In the singular case let m denote the smallest minimal index, so that the corresponding minimal relation $\theta\Lambda = 0$ can be written more explicitly as

$$[u_0 s^m - u_1 s^{m-1} r + u_2 s^{m-2} r^2 - \ldots + (-)^m u_m r^m]\,(rA + sB) = 0 \tag{1}$$

where *each of the* $(m + 1)$ *coefficients* u_i *is a row vector of* n *constant components*. Since this is identically true for all r, s the coefficients of powers of r vanish; so that the following *minimal chain* of vector equations is obtained:

$$0 = u_0 B,\ u_0 A = u_1 B,\ u_1 A = u_2 B,\ \ldots,\ u_{m-1} A = u_m B,\ u_m A = 0. \qquad (2)$$

THEOREM 3. *If m is the least minimal index of row dependence the vectors $u_0, u_1, \ldots, u_m$ determined by this chain of equations are linearly independent: and so also are the m vectors $u_1 B, u_2 B, \ldots, u_m B$.*

Proof. The theorem is obvious if $m = 0$. If $m > 0$, the rank $(n - \nu)$ of B must be less than n, so that n components for the vector u_0 can be found (in ν distinct ways) satisfying the n scalar equations implied by

$$u_0 B = 0, \qquad u_0 \neq 0.$$

Next the relation $u_1 A = u_0 B$ is virtually a set of n nonhomogeneous linear equations to determine the n components of u_1 in terms of those of u_0 and the elements of A and B. (This step is possible if and only if the rank $n - \nu_1$ of A is the same as that of the augmented matrix $\{A, u_0 B\}$ which has $(n + 1)$ *rows*. Since the whole chain is already known to exist, at least one of the ν values of u_0 will satisfy this and provide ν_1 possible values of u_1).

Let this process be continued for constructing $u_0, u_1, \ldots, u_p$ until u_{p+1} is the first such vector to be linearly related to its predecessors. Then scalar constants a_i (zero or otherwise) exist in $\mathcal{F}$ such that

$$0 = u_{p+1} + a_1 u_p + a_2 u_{p-1} + \ldots + a_{p+1} u_0. \qquad (3)$$

Let $(p + 1)$ new vectors v_i be formed,

$$\begin{aligned} v_0 &= u_0, \\ v_1 &= u_1 + a_1 u_0, \\ v_2 &= u_2 + a_1 u_1 + a_2 u_0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ v_p &= u_p + a_1 u_{p-1} + \ldots + a_p u_0, \end{aligned} \qquad (4)$$

which are palpably linearly independent, since $u_0, \ldots, u_p$ are. Then

$$0 = v_0 B,\ v_0 A = v_1 B,\ \ldots,\ v_{p-1} A = v_p B,\ v_p A = 0 \qquad (5)$$

as is at once seen by substituting for each v in terms of the u_i. For example $v_0 A - v_1 B = u_0 A - (u_1 + a u_0) B = u_0 A - u_1 B = 0$. But this is a chain implying a relation $\theta\Lambda = 0$ with u replaced by $v\ (\neq 0)$ and m by p. Since m is minimal $m \leqq p$. Hence $u_0, u_1, \ldots, u_m$ are linearly independent since $u_0, \ldots, u_p$ are.

For the second part of the theorem let $m > 0$. If $u_q A$ is the first of the sequence $u_0 A$, $u_1 A$, ... to be linearly dependent upon its predecessors, let

$$u_q A + \beta_1 u_{q-1} A + \ldots + \beta_q u_0 A = 0 \qquad (6)$$

or $w_q A = 0$, where $w_q = u_q + \beta_1 u_{q-1} + \ldots + \beta_q u_0$. By constructing $w_0, w_1, \ldots, w_{q-1}$ analogously to the v_i in (4) it again follows that a chain $0 = w_0 B, \ldots, w_q A = 0$ exists, which in turn cannot be shorter than the chain (2). Hence $q \geqq m$: and this proves the theorem.

Reduction to Canonical Form.

§ 5. Consider the following matrix relation

$$\begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \\ \hdashline P_0 \end{bmatrix} (rA + sB) = \left[\begin{array}{cccc:c} r & & & & \\ s & r & & & \\ & s & & & \\ & & \ddots & & \\ & & & r & \\ & & & s & \\ \hdashline & & & rA_0+sB_0 & rA_1+sB_1 \end{array}\right] \begin{bmatrix} u_1 B \\ u_2 B \\ \cdot \\ \cdot \\ \cdot \\ u_m B \\ \hdashline C_0 \end{bmatrix}, \quad m > 0, \qquad (1)$$

or $P\Lambda = XQ^{-1}$, where $u_0, u_1, \ldots, u_m$ are the first $(m+1)$ rows of P. Since by Theorem 3 these are linearly independent, P itself may be made a nonsingular constant matrix by choosing a suitable submatrix P_0 for its remaining $(n - m - 1)$ rows. If $m = 0$ then $P\Lambda$ has a zero top row and we pass on to consider lower rows. If $m > 0$, then the m rows $u_i B$ are also linearly independent, so that a choice of a further submatrix C_0 is possible, the whole being nonsingular and written Q^{-1}. Owing to the chain of relations $u_i A = u_{i+1} B$ the first $(m+1)$ rows of the product $P\Lambda$ agree with the corresponding rows of XQ^{-1}. For example the i^{th} row gives

$$ru_{i-1} A + su_{i-1} B = su_{i-1} B + ru_i B.$$

Hence (1) is identically true provided that the remaining $(n - m - 1)$ rows of X are identical with those of $P\Lambda Q$. A canonical minimal submatrix L_{m_1} of X has now been semi-isolated, such that

$$X = \begin{bmatrix} L_{m_1} & \cdot \\ rA_0 + sB_0, & rA_1 + sB_1 \end{bmatrix}, \quad m = m_1, \qquad (2)$$

where, for example,

$$L_0 = 0,\ L_1 = \begin{bmatrix} r \\ s \end{bmatrix},\ L_2 = \begin{bmatrix} r & \cdot \\ s & r \\ \cdot & s \end{bmatrix}, \ldots, \qquad (3)$$

https://doi.org/10.1017/S0013091500008002 Published online by Cambridge University Press

m_1 being the lowest index of row dependence. If X contains a second such index m_2, then m_2 will in fact be lowest row-index in the sub-matrix $rA_1 + sB_1$; but it will emerge more directly by selecting a new solution v_0 of the equation $uB = 0$, and forming a new chain (since $\mu > 1$)

$$v_0 B = 0,\ v_1 B = v_0 A,\ \ldots,\ v_{m_2} A = 0, \qquad m_2 \geqq m_1. \tag{4}$$

THEOREM 4. *The* $(m_2 + 1)$ *vectors* v *are linearly independent of themselves and of the* u *vectors. Also all the vectors* $u_i B$, $v_i B$ $(i > 0)$ *are linearly independent.*

Proof. Let v_p be the first such vector which is linearly dependent upon its predecessors u or v. (i) If no vector u with suffix higher than p enters, let the relation be

$$0 = \sum_{r=0}^{p} a_r v_{p-r} + \sum_{r=0}^{p} \beta_r u_{p-r}, \qquad a_0 = 1.$$

Construct

$$\begin{aligned} w_0 &= a_0 v_0 + \beta_0 u_0, \\ w_1 &= a_0 v_1 + a_1 v_0 + \beta_0 u_1 + \beta_1 u_0, \text{ etc.} \end{aligned} \tag{5}$$

exactly as in Theorem 3. Then the w vectors will form a chain, independent of the u vectors, such that

$$w_0 \neq 0,\ w_0 B = 0,\ w_1 B = w_0 A,\ \ldots,\ w_h A = 0, \tag{6}$$

where $h = p - 1 < m_2$. This contradicts the assumption. The proof that the $u_i B$, $v_i B$ are unrelated is analogous to that in Theorem 3. (ii) If however terms u_q $(q > p)$ enter the relation, write it as

$$\sum_{r=0}^{p} a_r v_{p-r} + \sum_{r=0}^{p} \beta_r u_{p-r} = \gamma_0 u_q + \gamma_1 u_{q+1} + \ldots + \gamma_{m_1 - q} u_{m_1} \tag{7}$$

where $a_0 = 1$, $\gamma_0 \neq 0$, $q > p$. Let w_p denote either side of this equality and let h be defined by

$$p \leqq h = p + m_1 - q < m_1. \tag{8}$$

From $w_p = \gamma_0 u_q + \ldots + \gamma_{m_1 - 1} u_{m_1}$ further vectors $w_{p+1}, w_{p+2}, \ldots, w_h$ may be derived by successively adding unity to each suffix of w and u, and deleting terms of suffix exceeding m_1. The concluding vector is then $w_h = \gamma_0 u_{m_1}$. With those defined by (5) the whole set $w_0, w_1, \ldots, w_h$ is then a chain of index less than m_1, which again involves a contradiction. The proof that the $u_i B$, $v_i B$ are unrelated is analogous, starting with an identity such as (7) but with A appearing as final factor of each term. Again a chain $w_0, \ldots, w_h$ would exist, where $w_0, \ldots, w_p$ are defined by (5), and $w_{p+1}, \ldots, w_h$ by the rule just given. This proves the theorem.

This theorem allows us to take the v_i to be the first $(m_2 + 1)$ rows in P_0, and the $v_{i+1} B$ the first m_2 rows in C_0. The result is

$$\begin{bmatrix} u_i \\ v_i \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} (rA + sB) = \begin{bmatrix} L_{m_1} & \cdot & \cdot \\ \cdot & L_{m_2} & \cdot \\ \cdots & \cdots & \cdots \\ & \cdots\cdots & \end{bmatrix} \begin{bmatrix} u_{i+1} B \\ v_{i+1} B \\ \cdots \\ \cdots \end{bmatrix}$$

where two canonical minimal submatrices have now been semi-isolated. No new feature arises in further steps until all μ submatrices L_m have been semi-isolated. Among themselves they are completely isolated in the form

$$L = \operatorname{diag}(L_{m_1}, L_{m_2}, \ldots, L_{m_\mu}).$$

This exhausts all possible row dependence. Column dependence is then sought in μ' possible ways, but owing to the isolation of each L_{m_i} in its own row, such column dependence is independent of columns occupied by L. The result is a submatrix

$$L' = \operatorname{diag}(L'_{m_1'}, \ldots, L'_{m'_{\mu'}}),$$

and any further submatrix X_0 not lying in the rows or columns of L and L' must be nonsingular. This can be reduced to rational or classical canonical form S say, and finally all remaining nonzero elements other than those of L, L' and S can be removed by the methods earlier explained.[1]

§ 6. Also, for directly obtaining the rational form of the nonsingular portions of the pencil, vector chains of the same general type $u_{i+1} A = u_i B$ may be formed but for which $u_0 \neq 0$, $u_0 A \neq 0$, $|A| \neq 0$. They must then be examined in descending order of their length, as in the rational case[2] for the collineatory group. The method is sufficiently illustrated by the following example:

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} (rA + sB) = \begin{bmatrix} r, & s, & \\ & r, & s \\ a_0 s, & a_1 s, & r + a_2 s \end{bmatrix} \begin{bmatrix} u_0 A \\ u_1 A \\ u_2 A \end{bmatrix}.$$

In this example $u_3 = a_0 u_0 + a_1 u_1 + a_2 u_2$ is the first of such a chain to be related to its predecessors.

[1] *Canonical Matrices* (1932), 127-8.

[2] *Canonical Matrices*, 49.

https://doi.org/10.1017/S0013091500008002 Published online by Cambridge University Press

It is to be noted that in the example of § 3 (9) the chain appearing in the first four rows of Λ is not a true minimal. The failure is due to the presence of the lower element s in the first column. Every vector satisfying $u_0 B = 0$ must be of the form

$$[\alpha, \beta, 0, 0, -\beta, 0],$$

where α, β are arbitrary constants. Taking $\alpha = 1, \beta = 0$, the shortest chain is obtained as

$$u_0 = [1, 0, 0, 0, 0, 0], \;\; u_1 = [0, 0, 0, 0, 1, 0], \;\; u_2 = [0, 0, 0, 0, 0, 1]$$

where $u_0 B = u_0 A - u_1 B = u_1 A - u_2 B = u_2 A = 0$.

It may also be noted that the same method will furnish every submatrix of type

$$R_e = \begin{bmatrix} r & & & \\ s & r & & \\ & & \ddots & \\ & & s & r \end{bmatrix}, \quad |R_e| = r^e$$

due to a zero latent root, and belonging to the nonsingular core. All such are found according to ascending value of e by use of every vector u_0 for which $u_0 B = 0$ but which does not lead to a minimal chain. A modified chain now appears, following the same law except that it terminates abruptly with u_{e-1} at a point where it is impossible to satisfy the equation $u_e B = u_{e-1} A$ by any vector u_e.

https://doi.org/10.1017/S0013091500008002 Published online by Cambridge University Press