A functorial version of a construction of Hochschild and Mostow for representations of Lie algebras

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Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a complemented ideal of $\mathfrak{g}$, and $W$ an $\mathfrak{h}$-module. Hochschild and Mostow have described the construction of a $\mathfrak{g}$-module "induced" from $W$, which is finite-dimensional provided $W$ is finite-dimensional and satisfies a nilpotent action condition. This note describes a modification of their construction which is functorial and a weak adjoint to the restriction functor from $\mathfrak{g}$-modules to $\mathfrak{h}$-modules.

Throughout this paper we shall suppose that $\mathfrak{g}$ is a Lie algebra over a field $k$, that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, and that there is a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. $U_\mathfrak{h}, U_\mathfrak{g}$ will denote the universal enveloping algebras of $\mathfrak{h}$ and $\mathfrak{g}$. Clearly, every $g \in \mathfrak{g}$ can be written uniquely as $g = h + s$ with $h \in \mathfrak{h}$ and $s \in \mathfrak{s}$. This allows us to define, (with Hochschild and Mostow [1]), a composition $*$ by

\[(1) \quad g * u = hu + (su-us) \quad \text{for} \quad g \in \mathfrak{g}, \quad u \in U_\mathfrak{h}.
\]

It can be shown that $su-us \in U_\mathfrak{h}$, hence (1) determines a $\mathfrak{g}$-module structure on $U_\mathfrak{h}$.

We shall use $\text{mod-}\mathfrak{h}, \text{mod-}\mathfrak{g}$ to denote the categories of right $\mathfrak{h}$- and $\mathfrak{g}$-modules, and $F : \text{mod-}\mathfrak{g} \to \text{mod-}\mathfrak{h}$ to denote the restriction functor. Now let $W \in \text{mod-}\mathfrak{h}$. Then $\text{hom}_k(U_\mathfrak{h}, W)$ has a $\mathfrak{g}$-module structure given by

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\[ f^g(u) = f(g \ast u) \] for \( f \in \text{hom}_K(U^n, W) \), \( g \in \mathbb{G} \), and \( u \in U^n \).

**Construction of the functor \( I \)**

Define a map \( \hat{j}_W : W \to \text{hom}_K(U^n, W) \) by setting \( \hat{j}_W(w)(u) = w \cdot u \) for \( w \in W \) and \( u \in U^n \). It is easy to check that \( \hat{j}_W \) is a \( U^n \)-monomorphism.

We define a \( \mathbb{G} \)-submodule \( IW \) of \( \text{hom}_K(U^n, W) \) by setting \( IW = \{ \text{im} \hat{j}_W \} \cdot U^n \).

Now let \( j_W \) be \( \hat{j}_W \) with codomain restricted to be \( FIW \). Let \( W, W' \in \text{mod-h} \) and \( \psi \in \text{hom}_{U^n}(W, W') \). We define \( I\psi : IW \to IW' \) by \([I\psi](f)(u) = (\psi \circ f)(u)\) for \( u \in U^n \) and \( f \in IW \). We must show \((I\psi)(f) \in IW'\). Since \( f \in IW \), \( f \) may be written as

\[ f = \sum_{i=1}^n j_W(w_i) \cdot x_i \]

for suitable \( w_i \in W \), \( x_i \in U^n \). Then, for \( u \in U^n \),

\[ ([I\psi](f))(u) = \left[ \psi \circ \sum_{i=1}^n j_W(w_i) \cdot x_i \right](u) \]

\[ = \psi \left[ \sum_{i=1}^n x_i (w_i \ast u) \right] \]

\[ = \sum_{i=1}^n \psi(w_i) \cdot (x_i \ast u) \]

\[ = \left[ \sum_{i=1}^n j_W', \psi(w_i) \right] \cdot x_i(u), \]

so \((I\psi)(f) = \sum_{i=1}^n j_W', \psi(w_i) \cdot x_i \in IW'\). If \( W'' \) is another \( h \)-module and \( \psi' \in \text{hom}_{U^n}(W', W'') \), then \((\psi' \circ \psi) \circ f = \psi' \circ (\psi \circ f)\); it follows that \( I \) has the multiplicative property of a functor.

**Lemma 1.** \( j_W \) is natural in \( W \).

**Proof.** We must show that if \( \psi \in \text{hom}_{U^n}(W, W') \) then \( F I \psi \circ j_W = j_W' \circ \psi \). Suppose \( w \in W \) and \( u \in U^n \). Then

\[ f^g(u) = f(g \ast u) \] for \( f \in \text{hom}_K(U^n, W) \), \( g \in \mathbb{G} \), and \( u \in U^n \).
\[(\Psi \circ j_\mathfrak{h})_W(u)(u) = (\Psi \circ j_\mathfrak{h}(\omega))(u) = \psi(\omega, u) = \psi(\omega) \cdot u = j_\mathfrak{h}, (\psi)(u) = (j_\mathfrak{h}, \circ \psi)(\omega)(u)\]
as required.

**Lemma 2.** \(I\) is a faithful functor.

**Proof.** If \(I\psi = 0\), then for all \(\omega \in \mathcal{W}\), \(0 = (I\psi)(j_\mathfrak{h}(\omega))\); so
\[0 = [(I\psi)(j_\mathfrak{h}(\omega))](I_{U^\mathfrak{h}}) = \psi(\omega)\]. That is, \(\psi = 0\). \(\Box\)

**Theorem 3.** The functor \(I : \text{mod-}\mathfrak{h} \rightarrow \text{mod-}g\), described above, is an injective weak left adjoint to \(F\). That is, for \(W \in \text{mod-}\mathfrak{h}\) and \(V \in \text{mod-}g\), there is an injection
\[\theta_{WV} : \text{hom}_{U^\mathfrak{h}}(IW, V) \rightarrow \text{hom}_{U^g}(W, FV)\]
which is natural in \(W\) and \(V\).

**Proof.** For \(\phi \in \text{hom}_{U^\mathfrak{h}}(IW, V)\), we define \(\theta_{WV}(\phi) = F\phi \circ j_\mathfrak{h}\). The naturality of \(\theta_{WV}\) follows from that of \(j_\mathfrak{h}\) and the definition of \(\theta_{WV}\).
We must show that \(\theta_{WV}\) is injective. Suppose that \(\phi_1, \phi_2 \in \text{hom}_{U^\mathfrak{h}}(IW, V)\), and that \(F\phi_1 \circ j_\mathfrak{h} = F\phi_2 \circ j_\mathfrak{h}\). Then \(\phi_1\) and \(\phi_2\) coincide on \(\text{im} j_\mathfrak{h}\).
Since \(\phi_1, \phi_2\) are \(U^g\)-homomorphisms, it follows that they must coincide on \((\text{im} j_\mathfrak{h})U^g = IW\). \(\Box\)

**Theorem 4** (compare Hochschild and Mostow [1] and Zassenhaus [2]). Let \(\mathfrak{g}\) be a finite-dimensional Lie algebra over a field \(k\) of characteristic zero, and let \(\mathfrak{h}\) be an ideal of \(\mathfrak{g}\) with complementary subalgebra \(\mathfrak{g}\). Let \(W\) be a finite-dimensional \(\mathfrak{h}\)-module on which \([\mathfrak{h}, \mathfrak{h}]\) acts nilpotently. Then \(IW\), as defined above, is a finite-dimensional \(\mathfrak{g}\)-module.

**Proof.** If \(\{0\} = \mathcal{W}_0 < \mathcal{W}_1 < ... < \mathcal{W}_n = W\) is a composition series for \(W\), then set \(S(W) = (\mathcal{W}_n/\mathcal{W}_{n-1}) \oplus ... \oplus (\mathcal{W}_2/\mathcal{W}_1) \oplus (\mathcal{W}_1/\mathcal{W}_0)\). By the Jordan-Hölder theorem, \(S(W)\) is determined up to isomorphism. Clearly, a subalgebra of \(\mathfrak{h}\) acts nilpotently on \(W\) if and only if it annihilates \(S(W)\). Let us write \(d = \dim_k W\), and let \(\text{ann}_{U^\mathfrak{h}}(M)\) denote the annihilator in \(U^\mathfrak{h}\) of an \(\mathfrak{h}\)-module \(M\). Obviously,
Since, by hypothesis, $[\mathfrak{h}, \mathfrak{g}] \subseteq \text{ann}_{U_{\mathfrak{h}}}(S(W))$, it follows that \( \text{ann}_{U_{\mathfrak{h}}}(S(W)) \) is a \( \mathfrak{g} \)-submodule of \( W \). Hence \( (\text{ann}_{U_{\mathfrak{h}}}(S(W)))^d \) is a \( \mathfrak{g} \)-submodule of \( U_{\mathfrak{h}} \).

If \( f \in \text{hom}_K(U_{\mathfrak{h}}, W) \) and \( f(\text{ann}_{U_{\mathfrak{h}}}(W)) = \{0\} \), then for all \( x \in U_{\mathfrak{g}} \),

\[
\tilde{f} \left[ (\text{ann}_{U_{\mathfrak{h}}}(S(W)))^d \right] \subseteq f \left[ x \cdot (\text{ann}_{U_{\mathfrak{h}}}(S(W)))^d \right] \subseteq f(\text{ann}_{U_{\mathfrak{h}}}(W)) \quad \text{by (2)},
\]

\[
= \{0\}.
\]

Now \( \text{im} j_W \) annihilates \( \text{ann}_{U_{\mathfrak{h}}}(W) \); so \( IW = (\text{im} j_W) \cdot U_{\mathfrak{g}} \) annihilates \( (\text{ann}_{U_{\mathfrak{h}}}(S(W)))^d \). Let us write \( J = (\text{ann}_{U_{\mathfrak{h}}}(S(W)))^d \). Then it is easy to see that \( IW \) is embedded in \( \text{hom}_K(U_{\mathfrak{h}}/J, W) \). Since \( W \) is finite-dimensional, \( \text{ann}_{U_{\mathfrak{h}}}(W) \) is of finite codimension in \( U_{\mathfrak{h}} \). Hence, by (2), \( \text{ann}_{U_{\mathfrak{h}}}(S(W)) \) is of finite codimension in \( U_{\mathfrak{h}} \). Now we appeal to a result of Zassenhaus [2, page 263], which states that if \( X, Y \) are ideals of \( U_{\mathfrak{h}} \) of finite codimension, then so is \( XY \). We deduce from this that \( J \) is of finite codimension in \( U_{\mathfrak{h}} \), so that \( \dim_K \text{hom}_K(U_{\mathfrak{h}}/J, W) < \infty \), and so \( \dim_K IW < \infty \). \( \square \)

References


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