§ 3. Incentre.

*The internal angular bisectors of a triangle are concurrent.**

The following demonstration † is different from the usual one.

Figure 23.

Let AL be the internal bisector of ∠ A, and let the internal bisector of ∠ B cut it at I.

Then $$AI : LI = BA : BL$$
$$= CA : CL;$$

therefore the internal bisector of ∠ C passes through I.

The point of concurrency, which will be denoted by I, is the centre of the circle inscribed in ABC. This circle is often called the *incircle*,‡ and the centre of it the *incentre.*‡

The radius of the incircle is usually denoted by r.

The following method § of inscribing a circle in a given triangle will be better understood after a perusal of § 4 (5). As regards practical execution it is the simplest yet obtained.

Figure 24.

Along CA take AP equal to AB, and CQ equal to CB.

With A as centre and PQ as radius describe a circle cutting AC, AB in the points S, T.

With S as centre and PQ as radius describe a circle cutting the first circle in two points; the straight line joining these two points passes through the incentre.

With T as centre and PQ as radius describe a circle cutting the first circle in two points; the straight line joining these two points passes through the incentre.

Hence the incentre is determined as well as the radius of the incircle.

The proof will be evident from the following considerations.

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* Euclid's *Elements*, IV. 4.

† Todhunter's *Elements of Euclid*, p. 312 (1864).

‡ See the note on p. 32.

§ Mr E. Lemoine in the Report (second part) of the 21st session of the *Association Française pour l'avancement des sciences*, p. 49 (1892).

The dotted straight lines bisect AS, AT perpendicularly.

Now $$AS = AT = PQ = CP - CQ = b + c - a;$$

therefore if E, F be the mid points of AS, AT

$$AE = AF = \tfrac{1}{2}(b + c - a) = s_1$$

and E, F are points of contact of the incircle.

(1) The area of a triangle is equal to the rectangle * under the semiperimeter † of the triangle and the radius of the incircle.

This is expressed, $\triangle = sr,$

where $s = \tfrac{1}{2}(a + b + c).$

(2) *If P be any point inside ABC, and PA, PB, PC be denoted by* α, β, γ, *and the radii of the incircles of PBC, PCA, PAB by* ρ_1, ρ_2, ρ_3, *then*

$$(\rho_2 + \rho_3)\alpha + (\rho_3 + \rho_1)\beta + (\rho_1 + \rho_2)\gamma = (r - \rho_1)a + (r - \rho_2)b + (r - \rho_3)c.$$

For

$$\begin{aligned} 2ABC &= r(a + b + c), \\ 2PBC &= \rho_1(a + \beta + \gamma), \\ 2PCA &= \rho_2(b + \gamma + \alpha), \\ 2PAB &= \rho_3(c + \alpha + \beta). \end{aligned}$$

But $$ABC = PBC + PCA + PAB;$$

therefore $$\rho_1(a + \beta + \gamma) + \rho_2(b + \gamma + \alpha) + \rho_3(c + \alpha + \beta) = r(a + b + c);$$

whence the result follows.

If P be outside ABC,

$$(\rho_2 + \rho_3)\alpha + (\rho_3 - \rho_1)\beta + (-\rho_1 + \rho_2)\gamma = (r + \rho_1)a + (r - \rho_2)b + (r - \rho_3)c.$$

(3) *If I be the incentre of ABC, and AI, BI, CI be produced to meet the circumcircle in U, V, W, then the sides of UVW are perpendicular to AI, BI, CI.*

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* This is established in the course of the proof of Heron's theorem regarding the area of a triangle. See § 8.

† The semiperimeter of a triangle is usually denoted, in this country and North America, by s; on the continent of Europe it is generally denoted by p. Euler, who was one of the first if not the first to introduce the notation a, b, c for the sides of ABC, denotes the semiperimeter $\tfrac{1}{2}(AB + BC + CA)$ by S. See an article by him entitled *Variæ demonstrationes geometriæ* printed in *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ* for the years 1747-8, I. 53 (1750).

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FIGURE 25.

Join U, V, W with A, B, C.

The arc BU = CU, CV = AV, AW = BW ;

therefore arc UBW = arc CU + arc AW ;

therefore $\angle$ UCI = $\angle$ UIC ;

therefore UI = UC = UB.

Similarly VI = VC = VA,

and WI = WA = WB.

Hence AWIV, BUIW, CVIU are kites ;

therefore VW, WU, UV are perpendicular to AI, BI, CI.

(4) *The angles of UVW are respectively equal to*

$$\tfrac{1}{2}(B+C),\ \tfrac{1}{2}(C+A),\ \tfrac{1}{2}(A+B).$$

For

$$\begin{aligned}\angle WUV &= \angle AUV + \angle AUW\\ &= \angle ABV + \angle ACW\\ &= \tfrac{1}{2}(B+C).\end{aligned}$$

Hence whatever be the size of the angles A, B, C, triangle UVW is always acute-angled.

(5) The angles of ABC expressed in terms of the angles of UVW are

$$\begin{aligned}A &= -U+V+W = 180^\circ - 2U\\ B &= U-V+W = 180^\circ - 2V\\ C &= U+V-W = 180^\circ - 2W.\end{aligned}$$

Compare § 5, (8).

(6) $UVW : ABC = R : 2r.$

Join the circumcentre O with A, B, C.

Then

$$\begin{aligned}2UVW &= \text{hexagon } AWBUCV\\ &= OBUC + OCVA + OAWB\\ &= \tfrac{1}{2}(OU \cdot BC + OV \cdot CA + OW \cdot AB)\\ &= \tfrac{1}{2}R \cdot 2s = Rs;\end{aligned}$$

and $2ABC = 2rs.$

(7) *If ABC, $A_1B_1C_1$, $A_2B_2C_2$, ... $A_nB_nC_n$ be a series of triangles all inscribed in the same circle and each of which is derived from the preceding in the same manner as UVW was derived from ABC in (3); then when the whole number m increases indefinitely, the*

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*triangle $A_{2m}B_{2m}C_{2m}$ tends towards a limiting position $\alpha\beta\gamma$, the triangle $A_{2m+1}B_{2m+1}C_{2m+1}$ tends also towards a limiting position $\alpha'\beta'\gamma'$, the two limiting triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are equilateral, and symmetrically placed with reference to the centre of the circle.**

FIGURE 26.

In triangle $A_1B_1C_1$, the perpendicular from A_1 to the opposite side is A_1A, the diameter of the circumcircle is A_1OD;

therefore the bisector of $\angle B_1A_1C_1$ is also † the bisector of $\angle AA_1D$;

therefore the vertex A_2 is at the middle of the arc AD intercepted by $\angle OA_1A$.

Hence in general, to obtain the vertex A_{n+1} draw the diameter OA_n and the straight line $A_{n-1}A_n$; the mid point of the arc intercepted by the inscribed angle thus formed is the vertex sought.

In this manner, step by step, the vertices A_2, A_3, A_4, ... are determined, and each time the inscribed angle diminishes by half. This angle therefore tends to become zero, and the two lines $A_{n+1}A_n$ and $A_{n-1}A_n$ end by coalescing with the diameter OA_n. Now since the first of these two lines is an angular bisector and the second is the corresponding perpendicular of the triangle $A_nB_nC_n$, this triangle tends to become isosceles, that is, in the limit, $\angle B_n = \angle C_n$.

Similarly $\angle C_n = \angle A_n$ and $\angle A_n = \angle B_n$;

hence the triangle $A_nB_nC_n$ tends to become equilateral.

The inscribed angles which give the vertices of even order are quartered each time. Hence the halves AA_2, A_2A_4, ... of the arcs intercepted by these angles form the terms of a geometrical series whose ratio is $\frac{1}{4}$. Since none of the arcs AA_2, A_2A_4, ... $A_{2m-1}A_{2m}$ encroaches on the preceding, their sum represents the distance AA_{2m}. This sum is, in the limit,

$$A\alpha = \tfrac{4}{3}AA_2.$$

Thus the position of the limiting equilateral triangle $\alpha\beta\gamma$ is known.

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* Both (7) and (8) were proposed at a competitive examination in France in 1881. For the proofs see Vuibert's *Journal*, VII. 121-3 (1883).

† See § 5, (34).

In the same way the triangles of odd order tend towards the equilateral triangle $\alpha'\beta'\gamma'$ which is such that

$$A_1\alpha' = \tfrac{1}{3}A_1A_3.$$

To prove that the equilateral triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are symmetrical with respect to the centre O, it is sufficient to prove that α and α' are diametrically opposite, or that $D\alpha = A_1\alpha'$.

Now
$$D\alpha = 2A\,A_2 - A\alpha = \tfrac{2}{3}AA_2,$$
$$A_1\alpha' = \tfrac{1}{3}A_1A_3 \qquad = \tfrac{2}{3}AA_2.$$

(8) *If the radius R be taken as unity, the product of the numbers which measure the diameters of the circles inscribed in the triangles ABC, $A_1B_1C_1 \ldots A_nB_nC_n$ tends towards a limit when n increases indefinitely.*

Let $\triangle$, $\triangle_1$ ... $\triangle_n$ denote the areas of the triangles ABC, $A_1B_1C_1 \ldots A_nB_nC_n$; and d, d_1, ... d_n the numbers which measure the diameters of their incircles.

Then
$$\triangle_1 : \triangle = R : 2r = 1 : d;$$

therefore
$$d = \frac{\triangle}{\triangle_1}$$

Similarly
$$d_1 = \frac{\triangle_1}{\triangle_2} \quad d_2 = \frac{\triangle_2}{\triangle_3}, \ldots d_n = \frac{\triangle_n}{\triangle_{n+1}};$$

therefore
$$dd_1d_2 \ldots d_n = \frac{\triangle}{\triangle_{n+1}}.$$

Now when n increases indefinitely, $\triangle_{n+1}$ approaches the area of the equilateral triangle inscribed in a circle of radius 1, that is $3\sqrt{3}/4$; hence $dd_1d_2 \ldots d_n$ approaches the limit $4\triangle/3\sqrt{3}$.

(9) It has been seen that the series of triangles $A_1B_1C_1$, $A_2B_2C_2$, etc., deduced successively from ABC and from each other can be continued indefinitely far. Can this series be extended backwards indefinitely far, and if not, when will it stop? To answer the question a solution must be found for the problem:

Given a triangle ABC inscribed in a circle, construct another inscribed triangle RST such that A, B, C shall be the mid points of the arcs ST, TR, RS.

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From § 3, (4) it appears that whatever be the size of the angles R, S, T, the triangle ABC must be acute-angled. This being granted, draw the perpendiculars AX, BY, CZ of ABC, and produce them to meet the circumcircle in R, S, T. These are the vertices of the triangle sought.

The demonstration follows from the fact that H is the incentre of triangle RST. See § 5 (15).

By operating in a similar manner on RST, etc., the series may be continued backwards. It is plain, however, that as soon as a triangle is reached which is not acute-angled, the process comes to an end.

It may happen that a triangle is reached which has one angle right. Let RST be this triangle, R the right angle.

Draw RR_1 perpendicular to ST. Then the triangle antecedent to RST is the straight line R_1R, which may be considered as a triangle R_1RR. The side RR of this triangle is infinitely small and its direction is the tangent at A.

(10) Each triangle of the series considered in (7) has its angles equal to half the sum of the angles taken two and two of the preceding triangle. Consider a series of triangles such that each has its sides equal to half the sum of the sides taken two and two of the preceding triangle.

Starting with triangle ABC whose sides are a, b, c, the triangle $A_1B_1C_1$ is to be constructed whose sides are $\frac{1}{2}(b+c)$, $\frac{1}{2}(c+a)$, $\frac{1}{2}(a+b)$.

This second triangle is always possible even when a, b, c are taken at random, provided they be positive. For

$$\frac{c+a}{2} + \frac{a+b}{2} > \frac{b+c}{2}$$

$$\frac{a+b}{2} + \frac{b+c}{2} > \frac{c+a}{2}$$

$$\frac{b+c}{2} + \frac{c+a}{2} > \frac{a+b}{2}$$

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FIGURE 1.

Bisect the sides of ABC at A′, B′, C′. The angular contours A′CB′, B′AC′, C′BA′ straightened out will be the sides of the second triangle $A_1B_1C_1$.

Suppose triangle ABC to be formed by an endless thread which marks out the perimeter. Take the mid points of BC, CA, AB, and stretch the thread between these points, and the second triangle is obtained.

The same process may be repeated on triangle $A_1B_1C_1$ and so on indefinitely. The limiting triangle which is thus obtained may be proved to be the equilateral triangle whose side is $\frac{1}{3}(a+b+c)$.

Can this process be extended backwards indefinitely far? To answer the question a solution must be found for the problem :

Given a triangle whose sides are a, b, c, construct the triangle whose sides are

$$-a+b+c, \quad a-b+c, \quad a+b-c.$$

FIGURE 24.

In triangle ABC inscribe the circle DEF ;

then
$$AE = AF = \frac{-a+b+c}{2}$$

$$BF = BD = \frac{a-b+c}{2}$$

$$CD = CE = \frac{a+b-c}{2}.$$

Hence the triangle whose sides are equal to

$$AE+AF, \quad BF+BD, \quad CD+CE$$

will be the triangle sought.

Take, as before, the endless thread which marks out the perimeter of ABC at the points D, E, F and stretch it between these points.

Now this triangle is not always possible. For, in order that it may be possible, there must exist the inequality

$$a-b+c+a+b-c > -a+b+c, \text{ or } 3a > b+c.$$

Similarly $$3b > c + a, \text{ and } 3c > a + b.$$

By the addition of a, b, c these three inequalities may be transformed into $$2a > s, \ 2b > s, \ 2c > s.$$

But in every triangle the semiperimeter is greater than any one side. Hence the necessary and sufficient condition that the triangle antecedent to ABC may be possible is that each side of ABC must be greater than a quarter and less than a half of the perimeter.

The whole of (10) and a small part of (9) have been taken from a paper by Mr Édouard Collignon read at the Oran meeting (1888) of the *Association Française pour l'avancement des sciences*. See the Report of this meeting, Second Part, pp. 4-24. Mr Collignon's paper begins with a discussion of certain numerical series, and the results obtained are applied to the triangle, the quadrilateral, and to polygons of any number of sides.