

ITERATING THE COFINALITY- ω CONSTRUCTIBLE MODEL

UR YA'AR 

Abstract. We investigate iterating the construction of C^* , the L -like inner model constructed using first order logic augmented with the “cofinality ω ” quantifier. We first show that $(C^*)^{C^*} = C^* \neq L$ is equiconsistent with ZFC, as well as having finite strictly decreasing sequences of iterated C^* s. We then show that in models of the form $L[U]$ we get infinite decreasing sequences of length ω , and that an inner model with a measurable cardinal is required for that.

§1. Introduction. The model C^* , introduced by Kennedy, Magidor, and Väänänen in [5], is the model of sets constructible using the logic $\mathcal{L}(Q_\omega^{\text{cf}})$ —first order logic augmented with the “cofinality ω ” quantifier. To be precise, for a regular cardinal κ we define a logic $\mathcal{L}(Q_\kappa^{\text{cf}})$ where we add to the syntax of first order logic a quantifier Q_κ^{cf} whose semantics are given by

$$\mathcal{M} \models Q_\kappa^{\text{cf}} xy\varphi(x, y, \bar{b}) \iff \left\{ (c, d) \in M^2 \mid \mathcal{M} \models \varphi(c, d, \bar{b}) \right\}$$

is a linear order of cofinality κ .

This logic, first introduced by Shelah [10], is a proper extension of first order logic, which is fully compact. Using this logic we construct inner models of set theory, akin to Gödel’s constructible universe L :

DEFINITION 1.1. C_κ^* , the class of $\mathcal{L}(Q_\kappa^{\text{cf}})$ -constructible sets, is defined by induction:

$$\begin{aligned} L'_0 &= \emptyset, \\ L'_{\alpha+1} &= \text{Def}_{\mathcal{L}(Q_\kappa^{\text{cf}})}(L'_\alpha), \\ L'_\beta &= \bigcup_{\alpha < \beta} L'_\alpha \text{ for limit } \beta, \\ C_\kappa^* &= \bigcup_{\alpha \in \mathbf{Ord}} L'_\alpha, \end{aligned}$$

where for any logic \mathcal{L} extending first-order logic and a set M ,

$$\text{Def}_{\mathcal{L}}(M) := \left\{ \left\{ a \in M \mid (M, \in) \models_{\mathcal{L}} \varphi(a, \bar{b}) \right\} \mid \varphi \in \mathcal{L}; \bar{b} \in M^{<\omega} \right\}.$$

We focus on the case $\kappa = \omega$ and denote $C^* := C_\omega^*$. As shown in [5], C^* is a model of ZFC, and in fact it is the same as $L[\mathbf{Ord}_\omega]$ where $\mathbf{Ord}_\omega = \{\alpha \in \mathbf{Ord} \mid \text{cf}(\alpha) = \omega\}$.

Received September 14, 2021.

2020 *Mathematics Subject Classification*. Primary 03E45, Secondary 03E47, 03E55.

Key words and phrases. inner models, cofinality logic, measurable cardinal, Namba forcing.

© The Author(s), 2023. Published by Cambridge University Press on behalf of The Association for Symbolic Logic. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

0022-4812/23/8804-0017

DOI:10.1017/jsl.2022.93



Like in the case of L , one can phrase the formula “ $V = C^*$ ”, i.e., $\forall x \exists \alpha (x \in L'_\alpha)$ where L'_α is the α th level in the construction of C^* . Unlike L , however, it is not always true that $C^* \models V = C^*$, which is equivalent to the question whether $(C^*)^{C^*} = C^*$. This is clearly the case if $V = L$, so the interesting question is whether this can hold with $C^* \neq L$. In Section 2 we show that this is consistent relative to the consistency of ZFC. Next we investigate the possibilities of $C^* \not\models V = C^*$. In such a case, it makes sense to define recursively the iterated C^* s:

$$\begin{aligned} C^{*0} &= V, \\ C^{*(\alpha+1)} &= (C^*)^{C^{*\alpha}} \text{ for any } \alpha, \\ C^{*\alpha} &= \bigcap_{\beta < \alpha} C^{*\beta} \text{ for limit } \alpha. \end{aligned}$$

This type of construction was first investigated by McAloon [8] regarding HOD, the class of hereditarily ordinal definable sets, where he showed that it is equiconsistent with ZFC that there is a strictly decreasing sequence of iterated HOD^s of length ω , and the intersection of the sequence can be either a model of ZFC or of ZF + \neg AC. Harrington also showed (in unpublished notes; cf. [14]) that the intersection might not even be a model of ZF. Jech [4] used forcing with Suslin trees to show that it is possible to have a strictly decreasing sequence of iterated HOD^s of any arbitrary ordinal length, and later Zadrożny [13] improved this to an **Ord** length sequence. In [14] Zadrożny generalized McAloon’s method and gave a more flexible framework for coding sets by forcing, which he used to give another proof of this result. HOD can be described also as the model constructed using second order logic (as shown by Myhill and Scott [9]), so it is natural to ask which of the results for HOD apply to other such models, specifically to C^* . In Section 3 we show that unlike the case of HOD, without large cardinals we can only have finite decreasing sequences of iterated C^{*s} , and that the existence of an inner model with a measurable cardinal is equivalent to the existence of an inner model with a strictly decreasing C^* sequence of length ω .

§2. Relative consistency of “ $V = C^* \neq L$ ”. In this section we follow the method of Zadrożny [14] to obtain the following result:

THEOREM 2.1. *If ZFC is consistent then so is ZFC + $V = C^* \neq L + 2^{\aleph_0} = \aleph_1$.*

The idea¹ of Zadrożny’s proofs, which are based on results of McAloon’s [7, 8], is to add a generic object (to make $V \neq L$), code it using some other generic object, then code the coding, and so on, iterating until we catch our tail. Our coding tool will be the modified Namba forcing of [5, Section 6], which adds a countable cofinal sequence to any element of some countable sequence of regular cardinals $> \aleph_1$ (and only to them).

DEFINITION 2.2. Suppose $S = \langle \lambda_n \mid n < \omega \rangle$ is a sequence of regular cardinals $> \aleph_1$ such that every λ_n occurs infinitely many times in the sequence. Let $\langle B_n \mid n < \omega \rangle$

¹I’d like to thank Kameryn Williams for his exposition of this and related results in his blog—<http://kamerynjw.net/2019/12/04/omegath-hod.html>.

be a partition of ω into infinite sets. The *modified Namba forcing with respect to S* is defined as follows. Conditions are trees T with ω levels, consisting of finite sequences of ordinals, such that if $(\alpha_0, \dots, \alpha_i) \in T$ and $i \in B_n$ then:

1. $\alpha_i < \lambda_n$.
2. $|\{\beta \mid (\alpha_0, \dots, \alpha_{i-1}, \beta) \in T\}| \in \{1, \lambda_n\}$.
3. For every m there are $\alpha_{i+1}, \dots, \alpha_{k-1}$ such that $k \in B_m$ and $|\{\beta \mid (\alpha_0, \dots, \alpha_{k-1}, \beta) \in T\}| = \lambda_m$.

A condition T' extends another condition T , $T' \leq T$, if $T' \subseteq T$.

If $\langle T_n \mid n < \omega \rangle$ is a generic sequence of conditions, then the stems of the trees T_n form a sequence $\langle \alpha_n \mid n < \omega \rangle$ such that $\langle \alpha_i \mid i \in B_n \rangle$ is cofinal in λ_n . Thus, in the generic extension $\text{cf}(\lambda_n) = \omega$ for all $n < \omega$, and KMV show that these are the only regular cardinals that change their cofinality to ω . Furthermore, it is shown that revised countable support iterations of this forcing preserves ω_1 .

In [5, Theorem 6.7], these tools are used to produce a model of $\text{ZFC} + V = C^* \neq L + 2^{\aleph_0} = \aleph_2$, but this requires an inaccessible cardinal (as proven there as well). Here we show that we can get a model of $\text{ZFC} + V = C^* \neq L$ from ZFC alone, and this model will satisfy $2^{\aleph_0} = \aleph_1$. These two results covers all possibilities, since in [5, Corollary to Theorem 5.20], it is shown that the statement $V = C^*$ implies that $2^{\aleph_0} \in \{\aleph_1, \aleph_2\}$, and for any $\kappa > \aleph_0$ $2^\kappa = \kappa^+$.

We say that a set X is *nice* if it is a countable set of ordinals which does not contain any of its limit points, and such that $L[X]$ agrees with L on ω_1 and on regular cardinals starting from $\aleph_{\text{sup } X+2}$. Note that if $S \in L$ is a countable sequence of regular L cardinals $> \aleph_1$, and X is generic for the modified Namba forcing corresponding to S , then X is nice as the size of the forcing is $\leq \aleph_{\text{sup } X+1}$ (we assume GCH, at least starting from this point). Assume $V = L[A_0]$ where A_0 is nice. Set $\mathbb{P}_0 = \{1\}$, and inductively we assume that \mathbb{P}_n forces the existence of a *nice* set of ordinals A_n , denote $\xi_n = \text{sup } A_n$ and we set $\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_{n+1}$ where $\dot{\mathbb{Q}}_{n+1}$ is the modified Namba forcing to add a Namba sequence E_α to each $\aleph_{\xi_n+\alpha+2}^L$ such that $\alpha \in A_n$ (note that by niceness, $\aleph_{\xi_n+\alpha+2}^L$ is still a regular uncountable cardinal in $V^{\mathbb{P}_n}$). We can require that $E_\alpha \subseteq (\aleph_{\xi_n+\alpha+1}^L, \aleph_{\xi_n+\alpha+2}^L)$, so that $A_{n+1} := \bigcup \{E_\alpha \mid \alpha \in A_n\}$ does not contain any of its limit points, and as we noted, it is nice. \mathbb{P}_ω is the full support (which is in our case also the revised countable support) iteration. Let $G \subseteq \mathbb{P}_\omega$ generic, and denote $A = \bigcup_{n < \omega} A_n$. By the properties of the modified Namba forcing, for any γ , $V[G] \models \text{cf}(\aleph_{\gamma+2}^L) = \omega$ iff $\gamma = \xi_n + \alpha$ for $\alpha \in A_n$.

- REMARK 2.3.
1. $\xi_{n+1} = \aleph_{\xi_n, 2}^L$, so inductively depends only A_0 and not on the generics. $\xi_\omega := \text{sup } A$ satisfies $\xi_\omega = \aleph_{\xi_\omega}^L$ and is of cofinality ω .
 2. From $\text{otp}(A_0)$ and A we can inductively reconstruct each $A_n - A_0$ is the first $\text{otp}(A_0)$ elements of A , and if we know A_n , then A_{n+1} are the first $\text{otp}(A_n) \cdot \omega$ elements of A above $\text{sup } A_n$.
 3. Hence for each n , $\text{otp}(A_n) = \text{otp}(A_0) \cdot \omega^n$.
 4. If $\alpha \in A_n$, then $E_\alpha = A \cap (\aleph_{\xi_n+\alpha+1}^L, \aleph_{\xi_n+\alpha+2}^L)$.

PROPOSITION 2.4. $(C^*)^{V[G]} = L[A] = V[G]$.

PROOF. By the properties of the modified Namba forcing, at each stage of the iteration the only cardinals of $L[A_0]$ receiving cofinality ω are the ones in A_n . The whole iteration will also add new ω sequences to $\text{sup } A$, but this already had cofinality ω as we noted earlier. So $V[G] \models \text{cf}(\gamma) = \omega$ iff either $V \models \text{cf}(\gamma) = \omega$ or there is n s.t. $V \models \text{cf}(\gamma) = \aleph_{\zeta_n + \alpha + 2}^L$ for $\alpha \in A_n$. And on the other hand, if $\alpha \in A$ then $V[G] \models \text{cf}(\aleph_{\zeta_n + \alpha + 2}^L) = \omega$. So

$$A = \bigcup_{n < \omega} \{ \alpha \in [\zeta_{n-1}, \zeta_n) \mid V[G] \models \text{cf}(\aleph_{\zeta_n + \alpha + 2}^L) = \omega \} \in (C^*)^{V[G]}$$

(where $\zeta_{-1} = 0$); hence $L[A] \subseteq (C^*)^{V[G]}$.

As we noted, for every $\alpha \in A$, E_α can be reconstructed from A and α , so $\langle E_\alpha \mid \alpha \in A \rangle$ is in $L[A]$. G can be reconstructed from this sequence; hence $V[G] \subseteq L[A] \subseteq (C^*)^{V[G]}$, so the equality follows. \dashv

To finish the proof of Theorem 2.1, we start with a model of $V = L +$ “there is no inaccessible cardinal,” and take, e.g., $A_0 = \omega$. Then $L[A]$ will satisfy $V = C^* \neq L$, and $2^{\aleph_0} = \aleph_1$ will still hold since by [5, Theorem 6.7], violating CH in a model of $V = C^*$ requires an inaccessible cardinal.²

§3. Iterating C^* .

THEOREM 3.1. *If ZFC is consistent then so is the existence of a model with a decreasing C^* -sequence of any given finite length.*

PROOF. Going back to the proof of Theorem 2.1, we note that for any n ,

$$(C^*)^{L[\bigcup_{k=0}^{n+1} A_k]} = L \left[\bigcup_{k=0}^n A_k \right]:$$

A_n can be computed from A_{n+1} using the cofinality- ω quantifier, which gives \supseteq , and on the other hand, from $\bigcup_{k=0}^n A_k$ we know exactly which ordinals will have cofinality ω in $L[\bigcup_{k=0}^{n+1} A_k]$, which gives \subseteq . So by starting, e.g., from $A_0 = \omega$, $L[\bigcup_{k=0}^n A_k]$ has the decreasing C^* chain

$$L \left[\bigcup_{k=0}^n A_k \right] = C^{*0} \supseteq C^{*1} \supseteq \dots \supseteq C^{*n} = L. \quad \dashv$$

We now show that without large cardinals this is best possible.

LEMMA 3.2. *Let $E = \{ \alpha < \omega_2^V \mid \text{cf}(\alpha) = \omega \}$.*

1. *If 0^\sharp does not exist, then $C^* = L[E]$.*
2. *If there is no inner model with a measurable cardinal, then $C^* = K[E]$ where K is the Dodd–Jensen core model.*

PROOF. 1. Clearly $E \in C^*$ so $L[E] \subseteq C^*$. Let $\alpha \in \mathbf{Ord}$. If $\text{cf}(\alpha) \geq \omega_2^V$ then also $L \models \text{cf}(\alpha) \geq \omega_2^V$ so in particular $L \models \text{cf}(\alpha) > \omega$. If $\text{cf}(\alpha) < \omega_2^V$, let $A \subseteq \alpha$

²It can in fact be proved that our forcing does not add new reals in general, but as we are looking for a consistency result this is not required.

be cofinal, so $|A| \leq \aleph_1$. By the covering theorem, there is $B \in L$, $A \subseteq B \subseteq \alpha$ s.t. $|B| = \aleph_1 + |A| = \aleph_1$. Let $\bar{\alpha} = \text{otp}(B)$, so $\bar{\alpha} < \omega_2^V$, and $\text{cf}(\alpha) = \text{cf}(\bar{\alpha})$, so $\text{cf}(\alpha) = \omega$ iff $\bar{\alpha} \in E$.

To summarize, we get that for every α , in $L[E]$ we can determine whether $\text{cf}(\alpha) = \omega$ or not, so $C^* \subseteq L[E]$.

2. The proof is exactly the same, noting that $K \subseteq C^*$ by [5, Theorem 5.5], and that our assumption implies the covering theorem holds for K . \dashv

THEOREM 3.3. *If there is no inner model with a measurable cardinal, then there is $k < \omega$ such that $C^{*k} = C^{*(k+1)}$.*

PROOF. By applying Lemma 3.2 clause 2 inside each C^{*n} , for every n we have $C^{*(n+1)} = K[E_n]$ where

$$E_n = \left\{ \alpha < \omega_2^{C^{*n}} \mid C^{*n} \models \text{cf}(\alpha) = \omega \right\}.$$

The sequence $\left\langle \left(\omega_1^{C^{*n}}, \omega_2^{C^{*n}} \right) \mid n < \omega \right\rangle$ is non-increasing in both coordinates; hence it stabilizes. Let k be such that $(\omega_1^{C^{*k}}, \omega_2^{C^{*k}}) = (\omega_1^{C^{*(k+1)}}, \omega_2^{C^{*(k+1)}})$, and we claim that $C^{*(k+1)} = C^{*(k+2)}$. To simplify notation we assume w.l.o.g. $k = 0$, i.e., $\omega_i^{C^*} = \omega_i^V$ for $i = 1, 2$ (so we can omit the superscript) and we want to show that $(C^*)^{C^*} = C^*$. We have

$$\begin{aligned} C^* &= K[\{\alpha < \omega_2 \mid V \models \text{cf}(\alpha) = \omega\}] = K[E_0], \\ (C^*)^{C^*} &= K[\{\alpha < \omega_2 \mid C^* \models \text{cf}(\alpha) = \omega\}] = K[E_1]. \end{aligned}$$

Clearly $E_1 \subseteq E_0$. On the other hand, if $\alpha \in \omega_2 \setminus E_1$, this means that $C^* \models \text{cf}(\alpha) = \omega_1$, and since $\omega_1^{C^*} = \omega_1$, we get that also $V \models \text{cf}(\alpha) = \omega_1$, so $\alpha \in \omega_2 \setminus E_0$, thus $E_1 = E_0$, and our claim is proved. \dashv

Our next goal is to show that this is precisely the consistency strength of a decreasing sequence:

THEOREM 3.4. *If there is an inner model with a measurable cardinal, then there is an inner model in which the sequence $\langle C^{*n} \mid n < \omega \rangle$ is strictly decreasing.*

We work in $L[U]$ where U is a normal ultrafilter on κ , and denote by M_α the α th iterate of $L[U]$ by U , $j_{\alpha,\beta} : M_\alpha \rightarrow M_\beta$ the elementary embedding, $\kappa_\alpha = j_{0,\alpha}(\kappa)$ and $U^{(\alpha)} = j_{0,\alpha}(U)$.

In [5, Theorem 5.16] the authors show that if $V = L[U]$ then $C^* = M_{\omega_2}[E]$ where $E = \{\kappa_{\omega \cdot n} \mid n < \omega\}$. We improve this by showing that C^* is unchanged after adding a Prikry sequence to κ , and then investigate the C^* -chain of $L[U]$. First we prove two useful lemmas.

LEMMA 3.5. *$E = \{\kappa_{\omega \cdot n} \mid n < \omega\}$ is generic over M_{ω_2} for the Prikry forcing on κ_{ω_2} defined from the ultrafilter $U^{(\omega^2)} \in M_{\omega_2}$.*

PROOF. We use Mathias's characterization of Prikry forcing:

FACT 3.6 (Mathias, cf. [6]). *Let M be a transitive model of ZFC, U a normal ultrafilter on κ , then $S \subseteq \kappa$ of order type ω is generic over M for the Prikry forcing defined from U iff for any $X \in U$, $S \setminus X$ is finite.*

So we need to show that for any $X \in U^{(\omega^2)}$, $E \setminus X$ is finite. The ultrafilter $U^{(\omega^2)}$ is defined by $X \in U^{(\omega^2)}$ iff $\exists \alpha < \omega^2$ $\{\kappa_\beta \mid \alpha \leq \beta < \omega^2\} \subseteq X$. For $X \in U^{(\omega^2)}$, choose some $\alpha < \omega^2$ such that $\{\kappa_\beta \mid \alpha \leq \beta < \omega^2\} \subseteq X$, then $E \setminus \alpha = \{\kappa_{\omega \cdot n} \mid \omega \cdot n < \alpha\}$, and since $\kappa_{\omega^2} = \sup\{\kappa_{\omega \cdot n} \mid n < \omega\}$, this set is finite. Hence $E = \{\kappa_{\omega \cdot n} \mid n < \omega\}$ satisfies the characterization. \dashv

LEMMA 3.7. For any $\beta < \kappa$ and any α , $M_\beta \models \text{cf}(\alpha) = \kappa$ iff $V \models \text{cf}(\alpha) = \kappa$.

PROOF. κ is regular in V ; thus it is regular in every M_β which is an inner model of V . If $M_\beta \models \text{cf}(\alpha) = \kappa$, then there is a cofinal κ -sequence in α (in both M_β and V), and since κ is regular we get $V \models \text{cf}(\alpha) = \kappa$. If $V \models \text{cf}(\alpha) = \kappa$, then the same argument rules out $M_\beta \models \text{cf}(\alpha) < \kappa$. So the only case left to rule out is $V \models \text{cf}(\alpha) = \kappa \wedge M_\beta \models \text{cf}(\alpha) > \kappa$. If $\beta = \gamma + 1$, then M_β is contained in M_γ and closed under κ -sequences in it, so they agree on cofinality κ , and by induction we get that they agree with V as well. So assume β is limit and let $\langle \alpha_\eta \mid \eta < \kappa \rangle$ be a cofinal sequence in α . By definition of the limit ultrapower, each α_η is of the form $j_{\bar{\beta}, \beta}(\bar{\alpha}_\eta)$ for some $\bar{\beta} < \beta$. We can also assume that each such $\bar{\beta}$ is large enough so that $\alpha \in \text{Range}(j_{\bar{\beta}, \beta})$. Since $\beta < \kappa$, there is some $\bar{\beta}$ fitting κ many α_η s, so without loss of generality we can assume $\bar{\beta}$ fits all of them. We can assume $\bar{\beta} > 0$ so κ is a fixed point of $j_{\bar{\beta}, \beta}$. If $\bar{\alpha} = \sup\{\bar{\alpha}_\eta \mid \eta < \kappa\}$ ³ then, since $\alpha = \sup\{j_{\bar{\beta}, \beta}(\bar{\alpha}_\eta) \mid \eta < \kappa\}$ and $\alpha \in \text{Range}(j_{\bar{\beta}, \beta})$, we must have that $\alpha = j_{\bar{\beta}, \beta}(\bar{\alpha})$. $\bar{\alpha}$ is of cofinality κ in V , so by the induction hypothesis also in $M_{\bar{\beta}}$, hence by elementarity $M_{\bar{\beta}} \models \text{cf}(\alpha) = \kappa$. \dashv

PROPOSITION 3.8. If $V = L[U]$ where U is a measure on κ , G is generic for Prikry forcing on κ , then $(C^*)^{V[G]} = (C^*)^V$.

PROOF. After forcing with Prikry forcing, the only change of cofinalities is that κ becomes of cofinality ω . So $V[G] \models \text{cf}(\alpha) = \omega$ iff $V \models \text{cf}(\alpha) \in \{\omega, \kappa\}$. We now follow the proof of [5, Theorem 5.16].

Consider M_{ω^2} , the ω^2 iterate of V , and let $E = \{\kappa_{\omega \cdot n} \mid n < \omega\}$. As we noted above, by [5, Theorem 5.16] we know that $(C^*)^V = M_{\omega^2}[E]$, so it is enough to show that also $(C^*)^{V[G]} = M_{\omega^2}[E]$.

Fix an ordinal α . As in the proof of [5, Theorem 5.16], $V \models \text{cf}(\alpha) = \omega$ iff $M_{\omega^2}[E] \models \text{cf}(\alpha) \in \{\omega\} \cup E \cup \{\sup E\}$. Regarding cofinality κ —by Lemma 3.7, $V \models \text{cf}(\alpha) = \kappa$ iff $M_{\omega^2} \models \text{cf}(\alpha) = \kappa$. As we noted, $M_{\omega^2} = L[u']$ where u' is a measure on κ_{ω^2} and by Lemma 3.5, E is Prikry generic over it; hence, since cofinality κ is unaffected by Prikry forcing on κ_{ω^2} , we get $V \models \text{cf}(\alpha) = \kappa$ iff $M_{\omega^2}[E] \models \text{cf}(\alpha) = \kappa$. Putting these facts together, in $M_{\omega^2}[E]$ we can detect whether $V \models \text{cf}(\alpha) \in \{\omega, \kappa\}$, so we know whether $V[G] \models \text{cf}(\alpha) = \omega$; hence we can construct $(C^*)^{V[G]}$ inside $M_{\omega^2}[E]$.

The other direction of the proof is almost the same as in [5, Theorem 5.16]: E is the set of ordinals in the interval $(\kappa, \kappa_{\omega^2})$ which have cofinality ω in $V[G]$ and are regular in the core model,⁴ which is contained in any C^* , so $E \in (C^*)^{V[G]}$, and from E one can define M_{ω^2} , so $M_{\omega^2}[E] \subseteq (C^*)^{V[G]}$. \dashv

³Note that we can't assume this sequence is in $M_{\bar{\beta}}$.

⁴Note that here we had to avoid κ which also satisfies this.

Now we can analyze the C^* -chain of $V = L[U]$. We stick to the notation $C^{*\alpha}$, starting from $C^{*0} = L[U] = M_0$, with M_α being the α th iterate of $L[U]$ and κ_α the α th image of the measurable cardinal. So by [5, Theorem 5.16] we have $C^{*1} = M_{\omega^2}[E_1]$ for $E_1 = \{\kappa_{\omega \cdot n} \mid n < \omega\}$. As we noted earlier M_{ω^2} is also of the form $L[U']$ for the measurable κ_{ω^2} , and by Lemma 3.5, E_1 is Prikry generic over it, so by Proposition 3.8,

$$C^{*2} = (C^*)^{M_{\omega^2}[E_1]} = (C^*)^{M_{\omega^2}},$$

which is again the ω^2 th iterate of M_{ω^2} , i.e., $M_{\omega^2+\omega^2}$, plus the corresponding sequence $E_2 = \{\kappa_{\omega^2+\omega \cdot n} \mid n < \omega\}$. So $C^{*2} = M_{\omega^2 \cdot 2}[E_2]$. We can continue inductively, and get the following:

THEOREM 3.9. *If $V = L[U]$ then for every $1 \leq m < \omega$ $C^{*m} = M_{\omega^2 \cdot m}[E_m]$ where M_α is the α th iterate of V , κ_α the α th image of the measurable cardinal, and $E_m = \{\kappa_{\omega^2 \cdot (m-1) + \omega \cdot n} \mid n < \omega\}$.*

To see that this gives us a *strictly* decreasing sequence of models, consider the following. For every m , $\kappa_{\omega^2 \cdot m}$ is the single measurable in $M_{\omega^2 \cdot m}$, by which we iterate, so it remains regular in all subsequent models, in particular in $M_{\omega^2 \cdot (m+1)}$. Since E_{m+1} is Prikry generic for $\kappa_{\omega^2 \cdot (m+1)}$ which is larger than $\kappa_{\omega^2 \cdot m}$, the latter remains regular in $M_{\omega^2 \cdot (m+1)}[E_{m+1}]$, while in $M_{\omega^2 \cdot m}[E_m]$ it is singular, so indeed

$$C^{*m} = M_{\omega^2 \cdot m}[E_m] \neq M_{\omega^2 \cdot (m+1)}[E_{m+1}] = C^{*(m+1)}.$$

This concludes the proof of Theorem 3.4.

To analyze $(C^{*\omega})^{L[U]}$, we will use the following theorem, due to Bukovský [1, 2] and Dehornoy [3]:

FACT 3.10. *If κ is measurable, M_α is the α th iterate of V by a normal ultrafilter on κ , and κ_α is the α th image of κ , then for any limit ordinal λ exactly one of the following holds:*

1. *If $\exists \alpha < \lambda$ s.t. $M_\alpha \models \text{cf}(\lambda) > \omega$ then $\bigcap_{\alpha < \lambda} M_\alpha = M_\lambda$.*
2. *If $\lambda = \alpha + \omega$ for some α , then $\langle \kappa_{\alpha+n} \mid n < \omega \rangle$ is Prikry generic over M_λ and $\bigcap_{\alpha < \lambda} M_\alpha = M_\lambda[\langle \kappa_{\alpha+n} \mid n < \omega \rangle]$.*
3. *Otherwise, $\bigcap_{\alpha < \lambda} M_\alpha$ is a quasi-generic extension of M_λ , hence satisfies ZF, but doesn't satisfy AC.*

COROLLARY 3.11. *If $V = L[U]$ then $C^{*\omega} = \bigcap_{\alpha < \omega^3} M_\alpha$ and it satisfies ZF but not AC.*

PROOF. By definition and our previous calculation,

$$C^{*\omega} = \bigcap_{m < \omega} C^{*m} = \bigcap_{1 \leq m < \omega} M_{\omega^2 \cdot m}[E_m]$$

and for each $m \geq 1$, $E_m \notin M_{\omega^2 \cdot (m+1)}[E_{m+1}]$ so

$$\bigcap_{1 \leq m < \omega} M_{\omega^2 \cdot m}[E_m] = \bigcap_{m < \omega} M_{\omega^2 \cdot m} = \bigcap_{\alpha < \omega^3} M_\alpha.$$

Since ω^3 is of cofinality ω but not of the form $\alpha + \omega$, the conclusion follows from Fact 3.10 clause 3. ⊣

§4. Conclusion and open questions. We summarize what is now known in terms of equiconsistency:

1. ZFC is equiconsistent with $V = C^* \neq L + 2^{\aleph_0} = \aleph_1$.
2. Existence of an inaccessible cardinal is equiconsistent with $V = C^* + 2^{\aleph_0} = \aleph_2$.
3. Existence of a measurable cardinal is equiconsistent with $\forall n < \omega (C^{*n} \supseteq C^{*(n+1)})$ and $C^{*\omega} \models \text{ZF} + \neg\text{AC}$.

Compared to the results regarding HOD, the following questions remain open:

- QUESTION 4.1. 1. *Is it possible, under any large cardinal hypothesis, that $\forall n < \omega$ $C^{*n} \supseteq C^{*(n+1)}$ and $C^{*\omega} \models \text{ZFC}$? More generally, for which ordinals α can we get a decreasing C^* sequence of length α ?*
2. *Is it possible, under any large cardinal hypothesis, that $\forall n < \omega$ $C^{*n} \supsetneq C^{*(n+1)}$ and $C^{*\omega} \not\models \text{ZF}$?*

A natural first attempt towards answering the first question would be to try and work in a model with more measurable cardinals. However, it seems that it would require at least *measurably many* measurables: in a forthcoming paper [12], we generalize [5, Theorem 5.16] and our Proposition 3.8 and show the following:

THEOREM 4.2. *Assume $V = L[\mathcal{U}]$ where $\mathcal{U} = \langle U_\gamma \mid \gamma < \chi \rangle$ is a sequence of measures on the increasing measurables $\langle \kappa^\gamma \mid \gamma < \chi \rangle$ where $\chi < \kappa^0$. Iterate V according to \mathcal{U} where each measurable is iterated ω^2 many times, to obtain $\langle M_\alpha^\gamma \mid \gamma < \chi, \alpha \leq \omega^2 \rangle$, with iteration points $\langle \kappa_{\omega \cdot n}^\gamma \mid \gamma < \chi, \alpha \leq \omega^2 \rangle$, and set M^χ as the directed limit of this iteration. Let G be generic over V for the forcing adding a Prikry sequence to every κ^γ . Set for every $\gamma < \chi$ $E^\gamma = \langle \kappa_{\omega \cdot n}^\gamma \mid 1 \leq n < \omega \rangle$ and $\hat{E}^\gamma = \langle \kappa_{\omega \cdot n}^\gamma \mid 0 \leq n < \omega \rangle$ then*

$$(C^*)^V = M^\chi[\langle E^\gamma \mid \gamma < \chi \rangle],$$

$$(C^*)^{V[G]} = M^\chi[\langle \hat{E}^\gamma \mid \gamma < \chi \rangle].$$

So, if $V = L[\mathcal{U}]$ as above, C^* is of the form $L[\mathcal{U}_1][G_1]$ for some sequence of measures and a sequence of Prikry sequences on its measures, and so C^{*2} is again of that form, where we iterated the measures in \mathcal{U} $\omega^2 \cdot 2$ many times and add Prikry sequences. So again, as we've done here, we'll get that $C^{*\omega}$ is the intersection of the models $M_{\omega^2 \cdot n}^\chi$ where we iterated each measure $\omega^2 \cdot n$ times. This is due to the facts that changing the order of iteration between the measures doesn't change the final result, and that the Prikry sequences "fall out" during the intersection. Now, we don't have a complete analysis of intersections of iterations by more than one measure, but Dehornoy proves the following more general fact:

FACT 4.3 [3, Section 5.3, Proposition 3]. *For every α let N_α be the α th iteration of V by some measure. Assume λ is such that for every $\alpha < \lambda$, $N_\alpha \models \text{cf}(\lambda) = \omega$ but there is no p such that $\lambda = p + \omega$. Then if M is a transitive inner model of ZFC containing $\bigcap_{\alpha < \lambda} N_\alpha$, then there is some $\alpha < \lambda$ such that $N_\alpha \subseteq M$.*

So, if we take N_α to be the iteration of V by the first measure in \mathcal{U} , we get that $C^{*\omega}$ contains $\bigcap_{\alpha < \omega^3} N_\alpha$, but doesn't contain any N_α for $\alpha < \omega^3$, so $C^{*\omega}$ cannot satisfy AC.

Furthermore, a recent result by Welch [11] shows that under the assumption that O^k (O-kukri) doesn't exist, i.e., there is no elementary embedding of an inner

model with a proper class of measurables to itself (and in particular no measurable limit of measurables), the characterization $(C^*)^V = M^\chi [\langle E^\gamma \mid \gamma < \chi \rangle]$ holds for any $\chi \leq \mathbf{Ord}$, and it seems likely that the analysis of the iterated C^* will be the same as well. Hence a different approach, or larger cardinals, would be required to answer this question.

A different line of inquiry stems from the following fact:

FACT 4.4 [5]. *If there is a proper class of Woodin cardinals then the theory of C^* is unchanged by forcing.*

So the question whether $C^* \models V = C^*$ cannot be changed under forcing in the presence of class many Woodin cardinals. If the sequence of $C^{*\alpha}$ is definable (perhaps up to some ordinal) then this will also be in the theory of C^* (note that on the face of it even the sequence up to ω may not be definable).

QUESTION 4.5. *What can be deduced on the sequence of iterated C^* from a proper class of Woodin cardinals?*

Acknowledgments. I would like to thank my advisor, Prof. Menachem Magidor, for his guidance and support without which this work would not have been possible. I would also like to thank the anonymous referee for their comments and suggestions.

REFERENCES

- [1] L. BUKOVSKÝ, *Changing cofinality of a measurable cardinal (an alternative proof)*. *Commentationes Mathematicae Universitatis Carolinae*, vol. 14 (1973), no. 4, pp. 689–698.
- [2] ———, *Iterated ultrapower and Prikry's forcing*. *Commentationes Mathematicae Universitatis Carolinae*, vol. 18 (1977), no. 1, pp. 77–85.
- [3] P. DEHORNOY, *Iterated ultrapowers and Prikry forcing*. *Annals of Mathematical Logic*, vol. 15 (1978), no. 2, pp. 109–160.
- [4] T. J. JECH, *Forcing with trees and ordinal definability*. *Annals of Mathematical Logic*, vol. 7 (1975), no. 4, pp. 387–409.
- [5] J. KENNEDY, M. MAGIDOR, and J. VÄÄNÄNEN, *Inner models from extended logics: Part 1*. *Journal of Mathematical Logic*, vol. 21 (2021), no. 2, p. 2150012.
- [6] A. R. D. MATHIAS, *On sequences generic in the sense of Prikry*. *Journal of the Australian Mathematical Society*, vol. 15 (1973), no. 4, pp. 409–414.
- [7] K. MCALOON, *Consistency results about ordinal definability*. *Annals of Mathematical Logic*, vol. 2 (1971), no. 4, pp. 449–467.
- [8] ———, *On the sequence of models HOD_n* . *Fundamenta Mathematicae*, vol. 82 (1974), no. 1, pp. 85–93.
- [9] J. MYHILL and D. SCOTT, *Ordinal definability*, *Axiomatic Set Theory* (D. S. Scott, editor), Proceedings of Symposia in Pure Mathematics, vol. XIII, American Mathematical Society, Providence, 1971, pp. 271–278.
- [10] S. SHELAH, *Generalized quantifiers and compact logic*. *Transactions of the American Mathematical Society*, vol. 204 (1975), pp. 342–364.
- [11] P. D. WELCH, *C^* in $L[E]$ -models below O^k* . Personal communication, 2022.
- [12] U. YA'AR, *Models for short sequences of measures in the cofinality- ω constructible model*, preprint, 2021, [arxiv:2109.04523](https://arxiv.org/abs/2109.04523).
- [13] W. ZADROŹNY, *Transfinite descending sequences of models HOD^α* . *Annals of Mathematical Logic*, vol. 20 (1981), no. 2, pp. 201–229.
- [14] ———, *Iterating ordinal definability*. *Annals of Pure and Applied Logic*, vol. 24 (1983), no. 3, pp. 263–310.

EINSTEIN INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY OF JERUSALEM
EDMOND J. SAFRA CAMPUS
GIVAT RAM JERUSALEM 91904, ISRAEL
E-mail: ur.yaar@mail.huji.ac.il