The Absolute Summability (A) of Fourier Series

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§ 1. In a recent paper\(^1\) Dr J. M. Whittaker has shown that the Fourier series

\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)
\]

of a function \(f(\theta)\) which has a Lebesgue integral in \((-\pi, \pi)\), is\(^2\) absolutely summable (A) to sum \(l\), if

\[
(a) \quad \int_0^\infty |\phi(t)| t^{-1} dt
\]

exists, where

\[
2\phi(t) = f(\theta + 2t) + f(\theta - 2t) - 2l.
\]

In this paper two other forms of criterion for absolute summability (A) of a Fourier series are obtained. In §2, it is shown that the series is absolutely summable (A), if

\[
(\beta) \quad \phi(t) \text{ is absolutely continuous in } (0, \delta).
\]

In §3, another criterion is found, viz.

\[
(\gamma) \quad \text{the existence of the integral } \int_0^\infty |\Phi(t)| t^{-2} dt,
\]

where

\[
\Phi(t) = \int_0^t \phi(u) du.
\]

In §4, the mutual relations of these three criteria are discussed, where it is shown that, while (\(\beta\)) is independent of (\(a\)) and (\(\gamma\)), (\(\gamma\)) includes (\(a\)).


\(^2\) A series

\[
\sum_{n=0}^{\infty} a_n,
\]

has been defined to be absolutely summable (A), if

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

is convergent in \((0 \leq x < 1)\) and if \(f(x)\) is of bounded variation in \((0, 1)\).
In the last article it is proved that a Fourier series may be absolutely summable \((A)\) at a point, without being convergent in the ordinary sense at that point.

§ 2. From the Poisson’s series\(^1\) (convergent for \(0 < x < 1\))

\[ P(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} x^n (a_n \cos n\theta + b_n \sin n\theta), \]

we get

\[ \frac{1}{2\pi} Q(x) = \frac{1}{2\pi} \{P(x) - l\} \]

\[ = \int_{0}^{\pi/2} \phi(t) \frac{1 - x^2}{1 - 2x \cos 2t + x^2} dt, \]

\[ = \int_{0}^{\delta} + \int_{\delta}^{\pi/2} \phi(t) \frac{1 - x^2}{1 - 2x \cos 2t + x^2} dt, \]

\[ = Q_1(x) + Q_2(x), \text{ say,} \]

where \(\delta\) is a constant such that \(0 < \delta < \frac{\pi}{2}\).

It is easy to prove that

\[ \left| \int_{0}^{x_1} |Q'_2(x)| dx, \right. \]

where \(0 < x_1 < 1\), is less than a constant and hence \(Q_2(x)\) is of bounded variation in \((0, 1)\).

Now suppose \(\phi(t)\) is absolutely continuous in \((0, \delta)\). Hence, integrating by parts

\[ Q_1(x) = \phi(\delta) \tan^{-1} \left( \frac{1 + x}{1 - x} \tan \delta \right) - \int_{0}^{x} \tan^{-1} \left( \frac{1 + x}{1 - x} \tan t \right) \left( \frac{d\phi(t)}{dt} \right) dt \]

\[ = J(x) - K(x). \]

Here \(J(x)\) is obviously a function of bounded variation in \((0, 1)\), whilst

\[ \int_{0}^{x_1} \left| K'(x) \right| dx = \int_{0}^{x_1} \left| \int_{0}^{x} \frac{d\phi(t)}{dt} \cdot \left( \tan^{-1} \left( \frac{1 + x}{1 - x} \tan t \right) \right) dt \right| dx \]

\[ \leq \int_{0}^{x_1} dx \int_{0}^{x} \left| \frac{d\phi(t)}{dt} \right| \left| \frac{\sin 2t}{1 - 2x \cos 2t + x^2} \right| dt \]

\[ = \int_{0}^{x_1} \left| \frac{d\phi(t)}{dt} \right| U(x_1, t) dt \]

inverting the order of integration\(^2\); here

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\(^2\) Ibid., 1 (1927), 630.
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\[
U(x, t) = \int_0^{x_i} \left| \frac{\sin 2t}{1 - 2x \cos 2t + x^2} \right| dx
\]

\[
= \tan^{-1} \left( \frac{1 + x_i \tan t}{1 - x_i \tan t} \right) - t \leq \frac{\pi}{2}.
\]

Therefore \(\int_0^{x_i} |K'(x)| \, dx \leq \frac{\pi}{2} \int_0^{x_i} \left| \frac{d\phi(t)}{dt} \right| \, dt < c,
\]

where \(c\) is a constant.\(^1\) Hence \(K(x)\) is a function of bounded variation in \((0, 1)\). Therefore \(Q_1(x)\) and consequently \(Q(x)\) from (2) is a function of bounded variation in \((0, 1)\), so that the series (1) is absolutely summable \((A)\) at \(\theta\), if it converges in virtue of \(\phi(t)\) being absolutely continuous in \((0, \delta)\).

§3. Let

\[
W(x, t) = \frac{1 - x^2}{1 - 2x \cos 2t + x^2},
\]

and

\[
\Phi(t) = \int_0^t \Phi(u) \, du.
\]

Taking \(\delta = \frac{\pi}{4}\), we have

\[
\int_0^{x_i} |Q'_1(x)| \, dx = \int_0^{x_i} \int_0^\pi \Phi(t) \frac{\partial W(x, t)}{\partial x} \, dt \, dx
\]

\[
= \int_0^{x_i} \left[ \Phi(t) \frac{\partial W(x, t)}{\partial x} \right]_0^\pi \Phi(t) \frac{\partial^2 W(x, t)}{\partial t \partial x} \, dt \, dx
\]

\[
\leq A + \int_0^{x_i} \int_0^\pi \Phi(t) \left. \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| \, dt \, dx,
\]

where \(A\) is a constant. Inverting the order of integration, we have

\[
\int_0^{x_i} |Q'_1(x)| \, dx \leq A + \int_0^\pi |\Phi(t)| \left( \int_0^{x_i} \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| \, dx \right) \, dt.
\]

Now

\[
\int_0^{x_i} \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| \, dx = \int_0^{x_i} \left| 4 \sin 2t \frac{1 + x^4 - 6x^2 + 2x (1 + x^2) \cos 2t}{(1 - 2x \cos 2t + x^2)^3} \right| \, dx
\]

\[
= \int_0^{x_i} |V(x, t)| \, dx,
\]

where

\[
V(x, t) = 4 \sin 2t \frac{1 + x^4 - 6x^2 + 2x (1 + x^2) \cos 2t}{(1 - 2x \cos 2t + x^2)^3}.
\]

\(^1\) Ibid., 593.
Let us write
\[ t_1 = \sin^{-1} \frac{1 - x_1}{\sqrt{2} \sqrt{(1 + x_1^2)}}, \quad (0 < t_1 < \pi/4). \]

Then, if \( 0 < t \leq t_1 \), we have
\[
\int_{0}^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| dx = \int_{0}^{x_i} V(x, t) \, dx
\]
\[ = \frac{4 \sin 2t \cdot x_1 (1 - x_1^2)}{(1 - 2x_1 \cos 2t + x_1^2)^2}. \]

Now since
\[
(1 - 2x_1 \cos 2t + x_1^2)^2 = (1 - x_1)^4 + 8x_1(1 + x_1^2) \sin^2 t - 16x_1^2 \sin t \cos^2 t
\]
\[ \geq 8x_1(1 - x_1)^2 \sin^2 t, \]
it follows that, for \( 0 < t \leq t_1 \),
\[
\int_{0}^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| dx \leq \frac{(1 + x_1) \cos t}{(1 - x_1) \sin t} < \frac{\pi^2}{2t^2}.
\]

If \( t_1 \leq t \leq \frac{1}{4} \pi \), and \( x' \), such that \( (\sqrt{2} - 1) \leq x' < 1 \), be given by
\[
\cos 2t = \frac{6x''^2 - 1 - x''^4}{2x'(1 + x''^2)},
\]
we see that
\[
\int_{0}^{x_1} \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| dx \leq \int_{0}^{1} \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| dx
\]
\[ = \int_{0}^{\sqrt{2} - 1} V(x, t) \, dx + \int_{\sqrt{2} - 1}^{x'} V(x, t) \, dx - \int_{x'}^{1} V(x, t) \, dx
\]
\[ = \frac{8 \sin 2t \cdot x'(1 - x''^2)}{(1 - 2x' \cos 2t + x''^2)^3}
\]
\[ < D \cos t \cosec^2 t < \frac{1}{4} \pi^2 D t^{-2}, \]
where \( D \) is a constant.

Thus we have
\[
\int_{0}^{x_1} |Q'(x)| \, dx \leq A + \left( \int_{0}^{t_1} \int_{t_1}^{\pi/4} |\Phi(t)| \left| \frac{\partial^2 W(x, t)}{\partial t \partial x} \right| dx \right) dt
\]
\[ < A + \frac{1}{2} \pi^2 \int_{0}^{t_1} |\Phi(t)| \, t^{-2} \, dt + \frac{1}{2} \pi^2 D \int_{t_1}^{\pi/4} |\Phi(t)| \, t^{-2} \, dt
\]
\[ < A + B \int_{0}^{\pi/4} |\Phi(t)| \, t^{-2} \, dt,
\]
where \( B \) is a constant.
Hence $Q_1(x)$ will be of bounded variation in $(0, 1)$ and consequently the Fourier series will be absolutely summable $(A)$, provided that

$$\int_0^\infty |\Phi(t)| t^{-\alpha} \, dt$$

exists.

§ 4. The criterion $(\gamma)$ includes $(a)$.

The proof of this is quite straightforward and is therefore omitted.

$(\gamma)$ is not included in $(a)$.

Take

$$\phi(t) = \rho t^{\rho-1} \sin \frac{1}{t} - \frac{1}{t^\rho} \cos \frac{1}{t}, \quad (1 < \rho < 2),$$

so that

$$\Phi(t) = t^\rho \sin \frac{1}{t}.$$ 

Then $(\gamma)$ exists, but $(a)$ does not exist.

$(\beta)$ is included neither in $(a)$ nor in $(\gamma)$.

Thus

$$\phi(t) = \left(\log \frac{1}{t}\right)^{-1}$$

satisfies $(\beta)$, but neither $(a)$ nor $(\gamma)$.

Again $(\beta)$ being an especial case of Jordan’s test, cannot include $(a)$ or $(\gamma)$.

§ 5. The existence of the integral

$$\int_0^\infty |\Phi(t)| t^{-\alpha} \, dt$$

is not a sufficient condition for the convergence of the corresponding Fourier series.

For if we take

$$\phi(t) = rt^{\alpha-1} \sin \frac{1}{t} - \frac{1}{t^\alpha} \cos \frac{1}{t}, \quad (1 < r < \frac{3}{2}),$$

so that

$$\Phi(t) = t^\alpha \sin \frac{1}{t}.$$ 

Then we have

$$I = \frac{1}{\pi} \int_0^\infty \left(rt^{\alpha-1} \sin \frac{1}{t} - \frac{1}{t^\alpha} \cos \frac{1}{t}\right) \cdot \frac{\sin (2n+1)t}{t} \, dt$$

$$= \frac{1}{\pi} \int_0^\infty r \sin \frac{1}{t} \cdot \frac{\sin(2n+1)t}{t^{\alpha-\alpha}} \, dt - \frac{1}{\pi} \int_0^\infty \frac{1}{t^{\alpha-\alpha}} \cos \frac{1}{t} \cdot \frac{\sin(2n+1)t}{t} \, dt$$

$$= I_1 - I_2.$$ 

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\[1\] See G. H. Hardy, Messenger of Math., 49 (1919–20), 150.
Now $\lim_{n \to \infty} I_1$ is zero, but by means of results due to Du Bois-Reymond,\(^1\) it can be proved that $I_2$ does not tend to any definite limit, as $n \to \infty$. Hence the corresponding Fourier series will not converge at $\theta$, although

$$\int_0^\infty |\Phi(t)| t^{-2} dt$$

exists.

Thus it has been shown that a Fourier series may be non-convergent at a point, but nevertheless absolutely summable $(A)$ at that point.

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See also G. H. Hardy, Quarterly Journal, 44 (1913), 242-263.