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Nearly Perfect Complexes and Galois Module Structure

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Abstract. We define a generalization of the Euler characteristic of a perfect complex of modules for the group ring of a finite group. This is combined with work of Lichtenbaum and Saito to define an equivariant Euler characteristic for $\mathbb{G}_m$ on regular projective surfaces over $\mathbf{Z}$ having a free action of a finite group. In positive characteristic we relate the Euler characteristic of $\mathbb{G}_m$ to the leading terms of the expansions of L-functions at $s = 1$.

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1. Introduction

A perfect complex of modules for a ring R is a bounded complex of finitely generated projective R-modules. The Euler characteristic of a perfect complex lies in the Grothendieck group $K_0(R)$ of all finitely generated projective R-modules. Perfect complexes and their Euler characteristics have been a basic tool in homological algebra, topology, algebraic geometry and, more recently, the theory of Galois module structure.

Suppose G is a finite group and that $R = \mathbf{Z}G$ is the integral group ring of G. The main object of this paper is to define invariants in $K_0(\mathbf{Z}G)$ associated with certain complexes of $\mathbf{Z}G$-modules which are of obvious arithmetic interest, but which are not perfect. Such complexes arise naturally from the cohomology of class field theory. In Section 2 we define the concept of a 'nearly perfect complex' of $\mathbf{Z}G$-

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modules, and we show in Theorem 2.3 that each such complex has a generalized Euler characteristic in $K_0(\mathbf{Z}G)$.

The motivation for this paper is the work of Lichtenbaum [L1] and Saito [Sa] on the cohomology of $\mathbb{G}_m$ on surfaces X which are proper over Spec($\mathbf{Z}$). In particular, Lichtenbaum discovered that for surfaces over a finite field, one could use duality theorems to assign a numerical Euler characteristic to $\mathbb{G}_m$, despite the fact that some of the cohomology groups of $\mathbb{G}_m$ may not be finitely generated. The cohomology groups of a perfect complex of $\mathbf{Z}G$-modules must be finitely generated, so that the cohomology of $\mathbb{G}_m$ on X cannot in general be computed by a perfect complex. The duality theorems used by Lichtenbaum identify the divisible subgroups of the cohomology groups of $\mathbb{G}_m$ on X with the Pontryagin duals of finitely generated groups. Theorem 2.3 arose from the idea that in defining Euler characteristics, one can compensate for the existence of divisible subgroups in the cohomology of a complex provided one has a fixed isomorphism between these divisible groups and the Pontryagin duals of other finitely generated groups.

In Section 3 we describe an arithmetic application to surfaces X having a free action of a finite group G. We assume that X is regular, geometrically connected and proper over Spec($\mathbf{Z}$). We will also assume that Brauer group of X is finite, which is conjectured to always be the case. Let $K(X)$ be the function field of X. We show how the above work of Lichtenbaum and Saito leads via Theorem 2.3 to an Euler characteristic $\chi^G(X, \mathbb{G}_m)$ for $\mathbb{G}_m$ on X which lies in $K_0(\mathbf{Z}G)$ if $K(X)$ has characteristic $p > 0$, and in $K_0(\mathbf{Z}[1/2][G])$ if $K(X)$ has characteristic 0. When $K(X)$ has characteristic p, we show how Lichtenbaum's work on the leading terms of zeta functions at $s = 1$ leads to a formula of the form

$$f(\chi^G(X, \mathbb{G}_m)) = L_{X,1}. \tag{1.1}$$

Here $f\colon K_0(\mathbf{Z}G) \to G_0(\mathbf{Z}G)$ is the forgetful homomorphism to the Grothendieck group of all finitely generated $\mathbf{Z}G$-modules. The class $L_{X,1}$ in $G_0(\mathbf{Z}G)$ has order 1 or 2, and is defined by the signs of all the conjugates of the totally real algebraic numbers which are the leading terms at $s = 1$ of the L-functions of symplectic representations of the group G of the cover $X \to X/G$. The class $L_{X,1}$ actually pertains to the difference between an Euler characteristic of $\mathbb{G}_a$ and $\mathbb{G}_m$ (c.f. Proposition 3.10). One arrives at (1.1) from the fact that $L_{X,1} = -L_{X,1}$ and by using work of Nakajima [N] to show that the class associated to $\mathbb{G}_a$ in $G_0(\mathbf{Z}G)$ is trivial because G acts freely on X. In a later paper we will prove a generalization of (1.1) to the case in which G acts tamely on X in which both multiplicative and additive Euler characteristics occur.

2. Euler Characteristics of Nearly Perfect Complexes

Let G be a finite group. If A is an Abelian group, let A_{div} be the subgroup of all divisible elements of A, and let $A_{\text{codiv}} = A/A_{\text{div}}$.

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DEFINITION 2.1. A nearly perfect complex of G-modules is a triple $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ of the following kind.

(a) $C^\bullet$ is a bounded complex

$$C^\bullet\colon \cdots \to C^m \to C^{m+1} \to \cdots \to C^n \to \cdots \tag{2.1}$$

of cohomologically trivial $\mathbf{Z}G$-modules. If some of the C^i are non-zero, we let m (resp. n) be the smallest (resp. largest) integer for which $C^i \neq 0$. If all of the $C^i = 0$, let $n = -1$ and $m = 0$. Define length $(C^\bullet) = n - m + 1$.

(b) For each integer i, L_i is a torsion-free finitely generated $\mathbf{Z}G$-module. Let $H^i(C^\bullet)$ be the i-th cohomology group of $C^\bullet$. We require τ_i to be a G-isomorphism $\tau_i\colon \mathrm{Hom}_{\mathbf{Z}}(L^i, \mathbf{Q}/\mathbf{Z}) \to H^i(C^\bullet)_{\mathrm{div}}$.

(c) For all i, the group $H^i(C^\bullet)_{\mathrm{codiv}}$ is finitely generated over $\mathbf{Z}$.

DEFINITION 2.2. Two nearly perfect complexes $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ and $(C'^\bullet, \{L'_i\}_i, \{\tau'_i\}_i)$ are quasi-isomorphic if the following is true:

(a) There is an isomorphism between $C^\bullet$ and $C'^\bullet$ in the derived category of the homotopy category of $\mathbf{Z}G$-modules.

(b) There is a $\mathbf{Z}G$-module isomorphism $L'_i \to L_i$ for each i.

(c) There is a commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}}(L^i, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\tau_i} & H^i(C^\bullet)_{\mathrm{div}} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{Z}}(L'^i, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\tau'_i} & H^i(C'^\bullet)_{\mathrm{div}} \end{array}$$

in which the left vertical isomorphism is induced by (b) and the right one by (a).

THEOREM 2.3. *Suppose* $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *is a nearly perfect complex.*

(a) *One can define an Euler characteristic* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) \in K_0(\mathbf{Z}G)$ *which depends only on the quasi-isomorphism class of* $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$.

(b) *The image of* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *in* $G_0(\mathbf{Z}G)$ *is equal to*

$$\sum_i (-1)^i \cdot ([H^i(C^\bullet)_{\mathrm{codiv}}] - [\mathrm{Hom}_{\mathbf{Z}}(L^i, \mathbf{Z})]).$$

(c) *Suppose that the cohomology groups* $H^i(C^\bullet)$ *are finitely generated as* $\mathbf{Z}G$*-modules, so that the* L_i *and* τ_i *are trivial. In this case the usual construction* (cf. [H, Lemma III.12.3], [M, p. 263]) *produces a perfect complex* $P^\bullet$ *of* $\mathbf{Z}G$*-modules which is quasi-isomorphic to* $C^\bullet$*, and* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *is the Euler characteristic in* $K_0(\mathbf{Z}G)$ *of* $P^\bullet$.

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Remark. Suppose R is a Dedekind ring, and that the fraction field F of R is a number field. One can then replace $\mathbf{Z}$ by R and $\mathbf{Q}/\mathbf{Z}$ by F/R in Definitions 2.1 and 2.2 and in Theorem 2.3. This leads to Euler characteristics in $K_0(RG)$ for nearly perfect complexes over RG.

We now outline the construction of $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\})$.

From $(C^\bullet, \{L_i\}_i, \{\tau_i\})$ we will construct a nearly perfect complex $(D^\bullet, \{L_i'\}_i, \{\tau_i'\})$ with the following properties (cf. Corollaries 2.10 and 2.8). If length $(C^\bullet) = n - m + 1 = 0$, then both $C^\bullet$ and $D^\bullet$ are the zero complex. If length $(C^\bullet) = 1$, then C^n is the only non-zero term of $C^\bullet$, but C^n may not be projective. In this case, D^{n-1} will be the only non-zero term of $D^\bullet$, and D^{n-1} will be a finitely generated projective $\mathbf{Z}G$-module. Finally, if length $(C^\bullet) > 1$, then length $(D^\bullet) <$ length$(C^\bullet)$.

If length $(C^\bullet) \leqslant 1$ we will define $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\})$ via the class of the projective module D^{n-1} in $K_0(\mathbf{Z}G)$, as in Definition 2.11. If length $(C^\bullet) > 1$, then length $(D^\bullet) <$ length$(C^\bullet)$, and we define $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\})$ in terms of $\chi(D^\bullet, \{L_i'\}_i, \{\tau_i'\})$ in Definition 2.11. This leads to an inductive definition of $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\})$, which one must show is independent of the choices involved in constructing $(D^\bullet, \{L_i'\}_i, \{\tau_i'\})$ from $(C^\bullet, \{L_i\}_i, \{\tau_i\})$.

Define $\delta_i: C^i \to C^{i+1}$ to be the ith boundary map in $C^\bullet$, and let $Z^i = \ker(\delta_i)$ be the ith cycle group. Thus $Z^n = C^n$, since $C^\bullet$ has no terms above degree n. The exact sequence

$$0 \to Z^{n-1} \to C^{n-1} \to C^n \to H^n(C^\bullet) \to 0 \tag{2.2}$$

defines an extension class $\alpha = \alpha_n \in \mathrm{Ext}^2_{\mathbf{Z}G}(H^n(C^\bullet), Z^{n-1})$. The main idea in the construction of $(D^\bullet, \{L_i'\}_i, \{\tau_i'\})$ is to replace the final terms $\cdots \to C^{n-2} \to C^{n-1} \to C^n$ of $C^\bullet$ by $\cdots \to C^{n-2} \to D^{n-1}$, where D^{n-1} is a cohomologically trivial $\mathbf{Z}G$-module constructed using the extension class α and the given isomorphism $\tau_n: \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}/\mathbf{Z}) \to H^n(C^\bullet)_{\mathrm{div}}$. The first step in this is to use α to construct a class in $\mathrm{Ext}^1_{\mathbf{Z}G}(M^n, Z^{n-1})$ for a suitable module M^n we now define.

We have an exact diagram of the following kind:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Z}) & \to & M^n & \to & N^n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) & \to & \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n & \to & F^n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}/\mathbf{Z}) & \to & H^n(C^\bullet) & \to & H^n(C^\bullet)_{\mathrm{codiv}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.3}$$

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To explain this diagram, note first that $\mathrm{Ext}^1_{\mathbf{Z}}(L^n, \mathbf{Z}) = 0$ since in Definition 2.1(b) L^n is a free finitely generated $\mathbf{Z}$-module. Hence, the left column is exact. In the bottom row we have identified $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}/\mathbf{Z})$ with $H^n(C^\bullet)_{\mathrm{div}}$ using the isomorphism τ_n of Definition 2.1(b). Because $H^n(C^\bullet)_{\mathrm{codiv}}$ is finitely generated, we can find a finitely generated projective module F^n along with a homomorphism from F^n to $H^n(C^\bullet)$ as in the middle column of diagram (2.3) which induces a surjection from F^n to $H^n(C^\bullet)_{\mathrm{codiv}}$. We will further require that $F^n = \{0\}$ if $H^n(C^\bullet)_{\mathrm{codiv}} = 0$. The homomorphism $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \to H^n(C^\bullet)$ in the middle column of diagram (2.3) is the one compatible with the left column and the bottom row. The modules M^n and N^n are defined to be the kernels of the vertical homomorphisms, and (2.3) is exact by the snake lemma.

Applying the functor $\mathrm{Hom}_{\mathbf{Z}G}(*, Z^{n-1})$ to the middle column of (2.3) gives a long exact sequence

$$\begin{aligned}\cdots &\to \mathrm{Ext}^1_{\mathbf{Z}G}(\mathrm{Hom}(L^n, \mathbf{Q}) \bigoplus F^n, Z^{n-1}) \to \mathrm{Ext}^1_{\mathbf{Z}G}(M^n, Z^{n-1})\\ &\xrightarrow{\phi_1} \mathrm{Ext}^2_{\mathbf{Z}G}(H^n(C^\bullet), Z^{n-1}) \to \mathrm{Ext}^2_{\mathbf{Z}G}(\mathrm{Hom}(L^n, \mathbf{Q}) \bigoplus F^n, Z^{n-1}) \to \cdots\end{aligned} \tag{2.4}$$

LEMMA 2.4. *The module M^n in* (2.3) *is a finitely generated torsion-free* $\mathbf{Z}G$-*module. The boundary map*

$$\mathrm{Ext}^1_{\mathbf{Z}G}(M^n, Z^{n-1}) \xrightarrow{\phi_1} \mathrm{Ext}^2_{\mathbf{Z}G}(H^n(C^\bullet), Z^{n-1}) \tag{2.5}$$

in (2.4) *is an isomorphism.*

Proof. Since N^n and $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Z})$ are finitely generated over $\mathbf{Z}$, diagram (2.3) shows M^n is as well. Because $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n$ is torsion-free, so is M^n.

To analyze (2.4) we use the spectral sequence

$$H^p(G, \mathrm{Ext}^q_{\mathbf{Z}}(D_1, D_2)) => \mathrm{Ext}^{p+q}_{\mathbf{Z}G}(D_1, D_2) \tag{2.6}$$

for G-modules D_1 and D_2. The projective dimension of $\mathbf{Z}$ is 1 (cf. [K, p. 171 and 191]). Therefore

$$\mathrm{Ext}^q_{\mathbf{Z}}(D_1, D_2) = 0 \quad \text{if} \quad q \geqslant 2. \tag{2.7}$$

If D_1 is finitely generated and torsion free, then $\mathrm{Ext}^1_{\mathbf{Z}}(D_1, D_2) = 0$. If D_1 is a finitely generated projective $\mathbf{Z}G$-module, then $\mathrm{Hom}_{\mathbf{Z}}(D_1, D_2)$ is a summand of an induced G-module, so $\mathrm{Hom}_{\mathbf{Z}}(D_1, D_2)$ is G-cohomologically trivial. Any $\mathbf{Q}$-vector space is also G-cohomologically trivial.

One finds from (2.6), (2.7) and the above remarks that if $q > 1$ then

$$\begin{aligned}&\mathrm{Ext}^q_{\mathbf{Z}G}(\mathrm{Hom}(L^n, \mathbf{Q}) \bigoplus F^n, Z^{n-1})\\ &\quad = \mathrm{Ext}^q_{\mathbf{Z}G}(\mathrm{Hom}(L^n, \mathbf{Q}), Z^{n-1}) \bigoplus \mathrm{Ext}^q_{\mathbf{Z}G}(F^n, Z^{n-1}) = 0\end{aligned} \tag{2.8}$$

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and

$$\mathrm{Ext}^1_{\mathbf{Z}G}(M^n, Z^{n-1}) = H^1(G, \mathrm{Hom}_{\mathbf{Z}}(M^n, Z^{n-1})). \tag{2.9}$$

The homomorphism

$$\mathrm{Ext}^1_{\mathbf{Z}G}(\mathrm{Hom}(L^n, \mathbf{Q}) \bigoplus F^n, Z^{n-1}) \to \mathrm{Ext}^1_{\mathbf{Z}G}(M^n, Z^{n-1}) \tag{2.10}$$

the long exact sequence (2.4) is trivial, since its domain is a $\mathbf{Q}$-vector space and, by (2.9), its range is a group annihilated by $\#G$. Thus (2.4), (2.9) and (2.10) establish the isomorphism stated in Lemma 2.4.

COROLLARY 2.5. *There is a unique class* $\beta \in \mathrm{Ext}^1_{\mathbf{Z}G}(M^n, Z^{n-1})$ *which maps to the extension class* $\alpha \in \mathrm{Ext}^2_{\mathbf{Z}G}(H^n(C^\bullet), Z^{n-1})$ *of the sequence* (2.2) *under the boundary isomorphism* ϕ_1 *of Lemma* 2.4.

DEFINITION 2.6. Let

$$0 \to Z^{n-1} \to D^{n-1} \to M^n \to 0 \tag{2.11}$$

be an exact sequence representing the extension class β of Corollary 2.5. Let $D^i = C^i$ if $i < n-1$, and let $D^i = 0$ for $i > n-1$. Let

$$D^\bullet\colon \cdots \to D^m \to D^{m+1} \to \cdots \to D^{n-1} \to 0 \to \cdots$$

be the complex such that the boundary $\lambda_i\colon D^i \to D^{i+1}$ is as follows. If $i < n-2$ then λ_i is the boundary map $\delta_i\colon C^i \to C^{i+1}$ of $C^\bullet$. If $i \geqslant n-1$, then λ_i is the zero homomorphism. Finally, if $i = n-2$, then $\lambda_{n-2}\colon D^{n-2} = C^{n-2} \to D^{n-1}$ is the composition of the boundary map $\delta_{n-2}\colon C^{n-2} \to Z^{n-1}$ with the inclusion $Z^{n-1} \to D^{n-1}$ in sequence (2.11).

PROPOSITION 2.7. *The module* D^{n-1} *is cohomologically trivial for* G.

Proof. By Corollary 2.5, the cup product of β with the extension class of the middle column

$$0 \to M^n \to \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n \to H^n(C^\bullet) \to 0 \tag{2.12}$$

of (2.3) is ± 1 times the extension class α of sequence (2.2). Therefore splicing together (2.11) and (2.12) gives an exact sequence

$$0 \to Z^{n-1} \to D^{n-1} \to \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n \to H^n(C^\bullet) \to 0 \tag{2.13}$$

which has extension class $\pm\alpha$. Since C^{n-1} and C^n in (2.2) were assumed to be cohomologically trivial for G, cup product with $\pm\alpha$ induces isomorphisms in Tate cohomology

$$\hat{H}^j(\Gamma, H^n(C^\bullet)) \to \hat{H}^{j+2}(\Gamma, Z^{n-1}) \tag{2.14}$$

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for all integers j and all subgroups $\Gamma \subset G$. Since $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n$ in sequence (2.13) is cohomologically trivial for G, this implies D^{n-1} in (2.13) must also be cohomologically trivial for G.

COROLLARY 2.8. *Suppose $n = m$ in sequence* (2.1), *so that $C^i = 0$ if $i \neq n$ and $C^n \neq 0$. Then $D^i = 0$ unless $i = n - 1$. The module $D^{n-1} = M^n$ is a finitely generated projective $\mathbf{Z}G$-module, and*

$$\mathrm{rank}_{\mathbf{Z}G}(D^{n-1}) = \mathrm{rank}_{\mathbf{Z}G}(F^n) + \frac{1}{\#G}(\mathrm{rank}_{\mathbf{Z}}(L^n) - \mathrm{rank}_{\mathbf{Z}}(H^n(C^\bullet)_{\mathrm{codiv}})), \tag{2.15}$$

where $C^n = H^n(C^\bullet)$.

Proof. If $m = n$, then $Z^{n-1} = 0$. Hence, sequence (2.11) shows $D^{n-1} = M^n$. By Lemma 2.4, M^n is a finitely generated torsion-free $\mathbf{Z}G$-module, and D^{n-1} is cohomologically trivial by Proposition 2.7. Hence, $D^{n-1} = M^n$ must be a finitely generated projective $\mathbf{Z}G$-module. The equality (2.15) follows from $D^{n-1} = M^n$, the top row and right column of (2.3), $\mathrm{rank}_{\mathbf{Z}}(\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Z})) = \mathrm{rank}_{\mathbf{Z}}(L^n)$, and the fact that finitely generated projective $\mathbf{Z}G$-modules are locally free.

LEMMA 2.9. *If $i < n-1$, then $H^i(D^\bullet) = H^i(C^\bullet)$. If $i > n-1$, then $H^i(D^\bullet) = 0$. Finally, there is an exact sequence*

$$0 \to H^{n-1}(C^\bullet) \to H^{n-1}(D^\bullet) \to M^n \to 0. \tag{2.16}$$

Proof. In view of (2.11), the exact sequence (2.16) is

$$0 \to Z^{n-1}/B^{n-1} \to D^{n-1}/B^{n-1} \to D^{n-1}/Z^{n-1} \to 0, \tag{2.17}$$

where $B^{n-1} \subset Z^{n-1}$ is the group of $n-1$ boundaries of both $C^\bullet$ and $D^\bullet$. The rest of the Lemma is clear.

COROLLARY 2.10. *Suppose $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ is a nearly perfect complex, as in Definition* 2.1. *Let $L_i' = L_i$ if $i \leqslant n-1$, and let $L_i' = 0$ if $i > n-1$. Define $\tau_i' = \tau_i\colon \mathrm{Hom}_{\mathbf{Z}}\,(L_i, \mathbf{Q}/\mathbf{Z}) \to H^i(C^\bullet)_{\mathrm{div}} = H^i(D^\bullet)_{div}$ if $i \leqslant n-1$, and let $\tau_i' = 0$ if $i > n-1$. Then $(D^\bullet, \{L_i'\}_i, \{\tau_i'\})$ is a nearly perfect complex. If $n > m$, then*

$$D^\bullet\colon \cdots \to 0 \to D^m \to \cdots \to D^{n-1} \to 0 \to \cdots \tag{2.18}$$

has $\mathrm{length}(D^\bullet) \leqslant n - m < \mathrm{length}(C^\bullet) = n - m + 1$.

Proof. Since M^n is finitely generated by Lemma 2.4, sequence (2.16) gives an isomorphism $H^{n-1}(C^\bullet)_{\mathrm{div}} \to H^{n-1}(D^\bullet)_{\mathrm{div}}$. The Corollary is now clear from Lemma 2.9.

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DEFINITION 2.11. If $\mathrm{length}(C^\bullet) = n - m + 1 = 0$, then $C^\bullet$ and $D^\bullet$ are both the zero complex, and we let

$$\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = 0. \tag{2.19}$$

Suppose now that $\mathrm{length}(C^\bullet) = n - m + 1 = 1$. By Corollary 2.8, D^{n-1} has a class $[D^{n-1}]$ in $K_0(\mathbf{Z}G)$, so we can let

$$\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = (-1)^{n-1} \cdot [D^{n-1}] + (-1)^n \cdot [F^n]. \tag{2.20}$$

We now assume by induction that $n_0 \geqslant 1$ is an integer such that $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ has been defined whenever $\mathrm{length}(C^\bullet) = n - m + 1 \leqslant n_0$. If $\mathrm{length}(C^\bullet) = n_0 + 1$ then using Corollary 2.10 we can define

$$\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = \chi(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i) + (-1)^n \cdot [F_n]. \tag{2.21}$$

PROPOSITION 2.12. *The nearly perfect complex* $(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$ *depends on* $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$, *the projective module* F^n *and the homomorphism* $\lambda\colon F^n \to H^n(C^\bullet)$ *induced by the middle column of diagram* (2.3). *The class* $\chi(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i) + (-1)^n \cdot [F_n]$ *does not depend on the choice of* F^n *or* $\lambda\colon F^n \to H^n(C^\bullet)$. *Therefore in Definition* 2.11, $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *depends only on the nearly perfect complex* $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *and not on further choices.*

To begin the proof of this Proposition, we observe that the choice of F^n and $\lambda\colon F^n \to H^n(C^\bullet)$ determines the diagram (2.3). This fixes the long exact sequence (2.4), so from Corollary 2.5 and Definition 2.6 we see that $(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$ is also determined by these choices together with the original nearly perfect complex $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$.

If $C^\bullet$ has length 0, then $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = 0$ so Proposition 2.12 holds. We now assume $C^\bullet$ has positive length, and that by induction, Proposition 2.12 is true for all nearly perfect complexes of length less than that of $C^\bullet$.

In the course of the proof, we will show by induction the following

HYPOTHESIS 2.13. *Suppose* $(C_1^\bullet, \{L_{1,i}\}_i, \{\tau_{1,i}\}_i)$ *and* $(C_2^\bullet, \{L_{2,i}\}_i, \{\tau_{2,i}\}_i)$ *are nearly perfect complexes having the following properties.*

(a) *The length of each of these complexes is less than that of* $C^\bullet$.
(b) *There is a morphism of complexes* $C_1^\bullet \to C_2^\bullet$ *which is a term by term isomorphism.*
(c) *There are isomorphisms* $L_{1,i} \to L_{2,i}$ *which are compatible with the cohomology isomorphisms* $H^i(C_1^\bullet) \to H^i(C_2^\bullet)$ *induced by the morphism in (b), the* $\tau_{1,i}$ *and the* $\tau_{2,i}$.

Then

$$\chi(C_2^\bullet, \{L_{2,i}\}_i, \{\tau_{2,i}\}_i) = \chi(C_1^\bullet, \{L_{1,i}\}_i, \{\tau_{1,i}\}_i), \tag{2.22}$$

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where the two sides of this equality are well defined because by assumption, Proposition 2.12 *holds for complexes of length less than that of* $C^\bullet$.

This induction hypothesis is clearly true if $C^\bullet$ has length 1, since then $C_1^\bullet$ and $C_2^\bullet$ have length 0.

LEMMA 2.14. *Let* $\tilde{\lambda}: F^n \to H^n(C^\bullet)_{\mathrm{codiv}}$ *be the homomorphism induced by* $\lambda: F^n \to H^n(C^\bullet)$, *so that* $\tilde{\lambda}$ *appears in the right column of diagram* (2.3). *Suppose* $\lambda': F^n \to H^n(C^\bullet)$ *is another homomorphism such that* $\tilde{\lambda} = \tilde{\lambda}'$. *Then assuming Hypothesis* 2.13, λ *and* λ' *lead to the same value for* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$. *Therefore* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *depends only on the (surjective) homomorphism* $\tilde{\lambda}: F^n \to H^n(C^\bullet)_{\mathrm{codiv}}$.

Proof. From diagram (2.3), we see that there is a homomorphism $\mu: F^n \to \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}/\mathbf{Z})$ such that $\lambda = \lambda' + i \circ \mu$, where $i: \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}/\mathbf{Z}) \to H^n(C^\bullet)$ is the inclusion in the bottom row of (2.3). Since F^n is projective and the homomorphism $\pi: \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \to \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}/\mathbf{Z})$ in the left column of (2.3) is surjective, we can lift μ to a homomorphism $\mu': F^n \to \mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q})$ such that

$$\lambda = \lambda' + i \circ \pi \circ \mu'. \tag{2.23}$$

Let T be the automorphism of $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n$ defined by

$$T\left(a \bigoplus b\right) = (a + \mu'(b)) \bigoplus b. \tag{2.24}$$

Define (2.3)′ to be the diagram (2.3) when λ' is used instead of λ. Then we see that there is a unique isomorphism $\tilde{T}$ from diagram (2.3) to diagram (2.3)′ which fixes the left and right columns and which is the automorphism T on the module $\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Q}) \bigoplus F^n$ in the middle of the diagram. This $\tilde{T}$ induces an automorphism of $H^n(C^\bullet)$ and an isomorphism $M^n \to M'^n$ from M^n to the corresponding module M'^n in diagram (2.3)′. Following through the construction in Definition 2.6, we see that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n-1} & \longrightarrow & D^{n-1} & \longrightarrow & M^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^{n-1} & \longrightarrow & D'^{n-1} & \longrightarrow & M'^n & \longrightarrow & 0 \end{array} \tag{2.25}$$

in which the left vertical homomorphism is the identity map, the right vertical homomorphism is the above isomorphism $M^n \to M'^n$, and the middle homomorphism is an isomorphism.

Let $(D^\bullet, \{L_{1,i}\}_i, \{\tau_{1,i}\}_i)$ and $(D'^\bullet, \{L_{2,i}\}_i, \{\tau_{2,i}\}_i)$ be the nearly perfect complex structures resulting from Definition 2.6 when one uses λ and λ', respectively. Via (2.25) we get a morphism of complexes $D^\bullet \to D'^\bullet$ which is the identity on all

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terms except D^{n-1}, and which is the middle vertical isomorphism of (2.25) on D^{n-1}. This morphism induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}(C^\bullet) & \longrightarrow & H^{n-1}(D^\bullet) & \longrightarrow & M^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{n-1}(C^\bullet) & \longrightarrow & H^{n-1}(D'^\bullet) & \longrightarrow & M'^n & \longrightarrow & 0 \end{array} \tag{2.26}$$

in which the left and right isomorphisms are those induced by $\tilde{T}$. By construction $\tilde{T}$ induces the identity isomorphism of $H^{n-1}(C^\bullet)_{\mathrm{div}} = H^{n-1}(D^\bullet)_{\mathrm{div}} = H^{n-1}(D'^\bullet)_{\mathrm{div}}$. Hence the identity isomorphism $L_{1,i} \to L_{2,i}$ for each i is compatible with $D^\bullet \to D'^\bullet$, $\tau_{1,i}$ and $\tau_{2,i}$. It follows now from our induction hypothesis (2.22) that

$$\begin{aligned} &\chi(D^\bullet, \{L_{2,i}\}_i, \{\tau_{2,i}\}_i) + (-1)^n \cdot [F^n] \\ &\quad = \chi(D'^\bullet, \{L_{1,i}\}_i, \{\tau_{1,i}\}_i) + (-1)^n \cdot [F^n]. \end{aligned} \tag{2.27}$$

Therefore we get the same value for $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ whether we use λ or λ'. This proves Lemma 2.14.

Proof of Proposition 2.12 and Hypothesis 2.13. Let F_1^n be another choice of a projective $\mathbf{Z}G$-module and a surjective homomorphism $\lambda_1: F_1^n \to H^n(C^\bullet)$. Since both of the induced homomorphisms $\tilde{\lambda}: F^n \to H^n(C^\bullet)_{\mathrm{codiv}}$ and $\tilde{\lambda}_1: F_1^n \to H^n(C^\bullet)_{\mathrm{codiv}}$ are surjective, we can form a pull back square

$$\begin{array}{ccc} F^n \bigoplus F_1^n & \longrightarrow & F_1^n \\ \downarrow & & \downarrow{\scriptstyle \tilde{\lambda}_1} \\ F^n & \underset{\tilde{\lambda}}{\longrightarrow} & H^n(C^\bullet)_{\mathrm{codiv}} \end{array} \tag{2.28}$$

It follows from Lemma 2.14 that to show we get the same value for $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ whether we use $\tilde{\lambda}$ or $\tilde{\lambda}_1$, we can reduce to the case in which $\tilde{\lambda}_1$ is the composition of a surjection $h: F_1^n \to F^n$ and $\tilde{\lambda}$. Since the value of $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ does not depend on how $\tilde{\lambda}$ is lifted to λ, we can furthermore assume that $\lambda_1: F_1^n \to H^n(C^\bullet)$ is the composition of h and $\lambda: F^n \to H^n(C^\bullet)$.

Define $(2.3)_1$ to be the diagram (2.3) which results from using λ_1 instead of λ, and let M_1^n and N_1^n be the modules appearing in $(2.3)_1$ at the positions corresponding to M^n and N^n. We have a commutative exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_2 & \longrightarrow & N_1^n & \longrightarrow & N^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_2 & \longrightarrow & F_1^n & \longrightarrow & F^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{\lambda}_1} & & \downarrow{\scriptstyle \tilde{\lambda}} & & \\
0 & \longrightarrow & 0 & \longrightarrow & H^n(C^\bullet)_{\text{codiv}} & \longrightarrow & H^n(C^\bullet)_{\text{codiv}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}
\qquad (2.29)
$$

in which the right columns are the right columns of diagrams $(2.3)_1$ and (2.3). The middlerow of (2.29) splits because F^n is projective. Hence, the top row of (2.29) also splits, and F_2 is projective because F^n and F_1^n are. We thus have compatible isomorphisms $N_1^n = N^n \bigoplus F_2$ and $M_1^n = M^n \bigoplus F_2$. Using the fact that F_2 is projective, we see that the construction of Definition 2.6 leads to an isomorphism $D_1^{n-1} = D^{n-1} \bigoplus F_2$, where $D_1^\bullet$ is the complex which results from using λ_1 instead of λ. The homomorphism $D_1^{n-2} = D^{n-2} = C^{n-2} \to D_1^{n-1}$ is the composition of $D^{n-2} \to D^{n-1}$ and the inclusion $D^{n-1} \to D_1^{n-1} = D^{n-1} \bigoplus F_2$ which is the identity map onto the first summand of D^{n-1}. If $F_2 = 0$, then $\tilde{\lambda} = \tilde{\lambda}_1$ and there is nothing to prove because of Lemma 2.14. So we assume in what follows that F_2 is non-trivial.

We see from this description that $H^{n-1}(D_1^\bullet) = H^{n-1}(D^\bullet) \bigoplus F_2$, and that the nearly perfect complex structure $(D_1^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$ of $D_1^\bullet$ is the one which results from $(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$ by simply adding the projective module F_2 to D^{n-1}.

In view of Definition 2.11 and (2.29), the expressions for $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ which result from using $\tilde{\lambda}_1$ and $\tilde{\lambda}$, respectively, are

$$\chi(D_1^\bullet, \{L_i'\}_i, \{\tau_i'\}_i) + (-1)^n \cdot ([F^n] + [F_2]) \qquad (2.30)$$

and

$$\chi(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i) + (-1)^n \cdot ([F^n]). \qquad (2.31)$$

We will now show (2.30) and (2.31) are equal.

Choose a free module F_3 and a G-morphism $\lambda_3 \colon F_3 \to H^{n-1}(D^\bullet)$ which induces a surjection $F_3 \to H^{n-1}(D^\bullet)_{\text{codiv}}$. We further require $F_3 = \{0\}$ if $H^{n-1}(D^\bullet)_{\text{codiv}} = 0$. We compute $(D_1^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$ by choosing the projective module $F_3 \bigoplus F_2$ together with the homomorphism

$$\lambda_4 \colon F_3 \bigoplus F_2 \to H^{n-1}(D_1^\bullet) = H^{n-1}(D^\bullet) \bigoplus F_2$$

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which is the identity map on the second summand and λ_3 on the first. From the diagram of the form (2.3) which results and from Definitions 2.6 and 2.11, we see that there is a single nearly perfect complex $(D_2^\bullet, \{L_i''\}_i, \{\tau_i''\}_i)$ of length less than length$(C^\bullet)$ such that

$$\chi(D^\bullet, \{L_i'\}_i, \{\tau_i'\}_i) = \chi(D_2^\bullet, \{L_i''\}_i, \{\tau_i''\}_i) + (-1)^{n-1} \cdot [F_3'] \tag{2.32}$$

and

$$\chi(D_1^\bullet, \{L_i'\}_i, \{\tau_i'\}_i) = \chi(D_2^\bullet, \{L_i''\}_i, \{\tau_i''\}_i) + (-1)^{n-1} \cdot [F_3' \bigoplus F_2], \tag{2.33}$$

where $F_3' = D^{n-1}$ is projective if length$(C^\bullet) = 1$, and $F_3' = F_3$ if length$(C^\bullet) > 1$. Here we may use Hypothesis 2.13 to assert that $\chi(D_2^\bullet, \{L_i''\}_i, \{\tau_i''\}_i)$ has the same value on the right sides of (2.32) and (2.33). It follows easily from (2.32) and (2.33) that (2.30) and (2.31) are equal.

The proof of Proposition 2.12 will now be complete once we show that our induction hypothesis (2.22) holds for nearly perfect complexes $(C_1^\bullet, \{L_{1,i}\}_i, \{\tau_{1,i}\}_i)$ and $(C_2^\bullet, \{L_{2,i}\}_i, \{\tau_{2,i}\}_i)$ of the same length as $C^\bullet$. This may be proved by using the inductive definition of $\chi(C_j^\bullet, \{L_{j,i}\}_i, \{\tau_{j,i}\}_i)$ to reduce to the case of complexes of length less than that of $C^\bullet$; we will leave the details to the reader.

The proof of part (a) of Theorem 2.3 is now completed by

PROPOSITION 2.15. *Suppose* $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *and* $(C'^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$ *are quasi-isomorphic in the sense of Definition 2.2. Then* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = \chi(C'^\bullet, \{L_i'\}_i, \{\tau_i'\}_i)$.

Proof. By [HRD, Sect. I.3], we can reduce to the case in which there is a morphism of complexes $C^\bullet \to C'^\bullet$ which gives rise to the quasi-isomorphism in the derived category which is referred to in Definition 2.2(a).

A shift by 1 to either the left or the right of the terms of $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ multiplies $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ by -1. Thus we can reduce to the case in which at least one of $C^\bullet$ or $C'^\bullet$ is not the zero complex, $C^i = C'^i = 0$ if $i < 0$, and $C'^0 \neq 0$ or $C^0 \neq 0$. Let $n' \geqslant 0$ (resp. $n \geqslant 0$) be the largest non-negative integer for which $C'^n \neq 0$ (resp. $C^n \neq 0$). We wish to reduce to the case $n' = n$, by replacing $C^\bullet$ (resp. $C'^\bullet$) by a quasi-isomorphic longer complex if $n' > n$ (resp. if $n > n'$). We will treat only the case $n' > n$, since the case $n > n'$ is similar. If $n' > n$ we can add a non-zero finitely generated free module F to both C^n and $C^{n+1} = 0$ and use the identity map $F \to F$ to construct a perfect complex $(C_2^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ from $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ which has length one greater than $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$. Since $H^{n+1}(C_2^\bullet) = 0$, we can use the trivial free module surjecting onto $H^{n+1}(C_2^\bullet)$ to compute $\chi(C_2^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ via the counterpart of diagram (2.3). This leads to $\chi(C_2^\bullet, \{L_i\}_i, \{\tau_i\}_i) = \chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$. In this way we can increase the length of $C^\bullet$ to be able to assume $n' = n$.

We will prove Proposition 2.15 by induction on $n = n'$. Suppose $n = n' = 0$. Then C'^0 and C^0 are the only non-zero terms of $C'^\bullet$ and $C^\bullet$, respectively. The

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morphism $C^\bullet \to C'^\bullet$ must induce an isomorphism $C^0 = H^0(C^\bullet) \to H^0(C'^\bullet) = C'^0$. Hence, the case $n = n' = 0$ is treated by Hypothesis 2.13, which was proved in the course of the proof of Proposition 2.12.

We now suppose $n = n' > 0$ and that Proposition 2.15 is true for all pairs of complexes which have trivial terms in negative dimension and highest terms in dimension less than n. As in the definition of $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$, we choose a free module F^n together with a morphism $F^n \to H^n(C^\bullet)$ inducing a surjection $F^n \to H^n(C^\bullet)_{\mathrm{codiv}}$. We now use the isomorphisms $H^n(C^\bullet) \to H^n(C'^\bullet)$ and $L^n \to L'^n$ in Definition 2.2 to take the diagram (2.3) used to define $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ to the diagram (2.3)′ used to define $\chi(C'^\bullet, \{L'_i\}_i, \{\tau'_i\}_i)$. In particular, we have an induced isomorphism $M^n \to M'^n$. Via Lemma 2.4, Corollary 2.5 and Definition 2.6, this leads to a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n-1} & \longrightarrow & D^{n-1} & \longrightarrow & M^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z'^{n-1} & \longrightarrow & D'^{n-1} & \longrightarrow & M'^n & \longrightarrow & 0 \end{array} \tag{2.34}$$

in which the right vertical arrow is an isomorphism. The morphism of complexes $C^\bullet \to C'^\bullet$ together with (2.34) now gives a morphism of complexes $D^\bullet \to D'^\bullet$ which induces cohomology isomorphisms $H^i(D^\bullet) \to H^i(D'^\bullet)$ if $i \neq n-1$. When $i = n-1$, one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}(C^\bullet) & \longrightarrow & H^{n-1}(D^\bullet) & \longrightarrow & M^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{n-1}(C'^\bullet) & \longrightarrow & H^{n-1}(D'^\bullet) & \longrightarrow & M'^n & \longrightarrow & 0 \end{array} \tag{2.35}$$

in which the left and right vertical homomorphisms are isomorphisms, so that the middle vertical homomorphism is as well. This proves $D^\bullet \to D'^\bullet$ is a quasi-isomorphism. The diagram (2.35) also shows that the nearly perfect complex structures on $D^\bullet$ and $D'^\bullet$ which are constructed in Corollary 2.10 are quasi-isomorphic in the sense of Definition 2.2. Since $D^\bullet$ and $D'^\bullet$ have highest non-zero terms in degree less than $n = n'$, the proof of Proposition 2.15 now follows by induction.

Completion of the proof of Theorem 2.3. As noted before, Proposition 2.15 shows part (a) of Theorem 2.3. We now show parts (b) and (c) of Theorem 2.3 by induction on $\mathrm{length}(C^\bullet) = n - m + 1$.

If $\mathrm{length}(C^\bullet) < 0$, then $C^\bullet$ is the zero complex and $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = 0$ by (2.19), so we are done.

From diagram (2.3) we have

$$[M^n] = [\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Z})] + [F^n] - [H^n(C^\bullet)_{\mathrm{codiv}}] \quad \text{in } G_0(\mathbf{Z}G). \tag{2.36}$$

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Suppose length$(C^\bullet) = 1$. By Corollary 2.8, $C^i = 0$ if $i \neq n$, and $H^n(C^\bullet) = C^n \neq 0$. Furthermore, $D^{n-1} = M^n$ is a projective $\mathbf{Z}[G]$-module, and

$$\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = (-1)^{n-1} \cdot [M^n] + (-1)^n \cdot [F^n] \tag{2.37}$$

by (2.20). Combining (2.36) and (2.37) shows $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ has the image

$$(-1)^{n-1} \cdot [M^n] + (-1)^n \cdot [F^n] = (-1)^n \cdot ([H^n(C^\bullet)_{\text{codiv}}] - [\mathrm{Hom}_{\mathbf{Z}}(L^n, \mathbf{Z})])$$

in $G_0(\mathbf{Z}G)$ as asserted in Theorem 2.3(b). Suppose $H^n(C^\bullet) = C^n$ is finitely generated as a $\mathbf{Z}G$-module. Then $L^n = 0$, and diagram (2.3) shows $[F^n] - [M^n] = [H^n(C^\bullet)] = [C^n]$ in $K_0(\mathbf{Z}G)$ when we identify $K_0(\mathbf{Z}G)$ with the Grothendieck group of all finitely generated cohomologically trivial $\mathbf{Z}G$-modules (cf. [C, Prop. 4.1(b)]). Thus (2.37) gives $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = (-1)^n \cdot [C^n]$ in this case, which by Schanuel's Lemma is equivalent to the assertion of Theorem 2.3(c).

Finally, suppose length$(C^\bullet) > 1$, and let $(D^\bullet, \{L'_i\}_i, \{\tau'_i\}_i)$ be a nearly perfect complex of the kind constructed in Corollary 2.10. By Definition 2.11,

$$\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = \chi(D^\bullet, \{L'_i\}_i, \{\tau'_i\}_i) + (-1)^n \cdot [F_n]. \tag{2.38}$$

By induction, the image of $\chi(D^\bullet, \{L'_i\}_i, \{\tau'_i\}_i)$ in $G_0(\mathbf{Z}G)$ is

$$\sum_i (-1)^i \cdot ([H^i(D^\bullet)_{\text{codiv}}] - [\mathrm{Hom}_{\mathbf{Z}}(L'^i, \mathbf{Z})]).$$

In view of the construction of $(D^\bullet, \{L'_i\}_i, \{\tau'_i\}_i)$, this equals

$$\begin{aligned}&(-1)^{n-1} \cdot ([H^{n-1}(D^\bullet)_{\text{codiv}}] - [\mathrm{Hom}_{\mathbf{Z}}(L^{n-1}, \mathbf{Z})]) + \\ &\quad + \sum_{i<n-1} (-1)^i \cdot ([H^i(C^\bullet)_{\text{codiv}}] - [\mathrm{Hom}_{\mathbf{Z}}(L^i, \mathbf{Z})])\end{aligned} \tag{2.39}$$

By the exact sequence (2.16), we have

$$[H^{n-1}(D^\bullet)_{\text{codiv}}] = [H^{n-1}(C^\bullet)_{\text{codiv}}] - [M^n] \tag{2.40}$$

since M^n is finitely generated. Hence on combining (2.38), (2.39) and (2.40), we see that $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ has image in $G_0(\mathbf{Z}G)$ equal to

$$\sum_i (-1)^i \cdot ([H^i(C^\bullet)_{\text{codiv}}] - [\mathrm{Hom}_{\mathbf{Z}}(L^i, \mathbf{Z})])$$

as asserted in Theorem 2.3(b).

Suppose now that all the $H^i(C^\bullet)$ are finitely generated. Then the groups $H^i(D^\bullet)$ are as well by Lemmas 2.4 and 2.9, and $L^n = 0$. By composing the inclusion map

$Z^{n-1} \to D^{n-1}$ with multiplication by -1, if necessary, we conclude from (2.13) that there is an exact sequence

$$0 \to Z^{n-1} \to D^{n-1} \to F^n \to H^n(C^\bullet) \to 0 \tag{2.41}$$

which has the same extension class in $\mathrm{Ext}^2_{\mathbf{Z}G}(H^n(C^\bullet), Z^{n-1})$ as the exact sequence

$$0 \to Z^{n-1} \to C^{n-1} \to C^n \to H^n(C^\bullet) \to 0.$$

Let $D'^\bullet$ be the complex obtained from $D^\bullet$ by putting F^n in the nth position and using the morphism $D^{n-1} \to F^n$ of (2.41). It follows that $D'^\bullet$ is isomorphic in the derived category to $C^\bullet$.

Using [H, Lemma III.12.3] and [M, p. 263], we now construct perfect complexes $Q^\bullet$ and $P^\bullet$ together with quasi-isomorphisms $Q^\bullet \to D^\bullet$ and $P^\bullet \to C^\bullet$. We can furthermore assume that $Q^i = 0$ if $i > n-1$, since $D^i = 0$ for such i. Define $Q'^\bullet$ to be the complex obtained from $Q^\bullet$ by adjoining F^n in degree n, and by letting the morphism $Q^{n-1} \to F^n$ be the one induced by the isomorphism $Q^{n-1}/Z^{n-1}(Q^\bullet) = H^{n-1}(Q^\bullet) \to H^{n-1}(D^\bullet)$ followed by the map $H^{n-1}(D^\bullet) = D^{n-1}/Z^{n-1} \to F^n$ induced by the boundary map $D^{n-1} \to F^n$ of $D'^\bullet$. Then $Q'^\bullet$ is isomorphic to $D'^\bullet$ in the derived category and, hence, also to $C^\bullet$ and to $P^\bullet$. It follows that

$$\sum_i (-1)^i [Q'^i] = \sum_i (-1)^i [P^i] \tag{2.42}$$

in $K_0(\mathbf{Z}G)$. However, since $D^\bullet$ has smaller length than $C^\bullet$, we know by induction on the statement of Theorem 2.3(c) that

$$\chi(D^\bullet, \{L'_i\}_i, \{\tau'_i\}_i) = \sum_i (-1)^i [Q^i] = \sum_i (-1)^i [Q'^i] - (-1)^n [F^n]. \tag{2.43}$$

Now combining (2.38) with (2.43) and (2.42) shows

$$\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) = \sum_i (-1)^i [P^i]$$

and this is the assertion of Theorem 2.3(c).

3. Arithmetic Applications

In this section we will suppose that X is a regular two-dimensional scheme which is proper over Spec($\mathbf{Z}$), geometrically connected, and for which the Brauer group $Br(X)$ is finite. Let G be a finite group acting freely on X, which by definition means that the inertia group in G of each point of X is trivial. This implies that the quotient map $X \to X/G$ is étale. Let $K(X)$ be the function field of X. There are

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two cases; the geometric one, when $\mathrm{char}(K(X)) > 0$ and the arithmetic one, when $\mathrm{char}(K(X)) = 0$. Let $\mathbb{G}'_m = \mathbb{G}_m$ (resp. $\mathbb{G}'_m = \mathbb{G}_m \otimes_{\mathbf{Z}} \mathbf{Z}[1/2]$) in the geometric (resp. arithmetic) case. The cohomology groups of $\mathbb{G}'_m$ have been computed by Lichtenbaum [L1] in the geometric case, and by Saito [Sa] in the arithmetic case. In Section 3.1 and Section 3.2 we will show how their results lead naturally to a nearly perfect complex structure associated to the hypercohomology of $\mathbb{G}'_m$. One may thus use the Euler characteristic construction of the previous section to define an Euler characteristic $\chi(X, \mathbb{G}'_m)$ for $\mathbb{G}'_m$ on X which lies in $K_0(\mathbf{Z}G)$ (resp. $K_0(\mathbf{Z}[1/2][G])$) in the geometric (resp. arithmetic) case.

Let $f: K_0(\mathbf{Z}G) \to G_0(\mathbf{Z}G)$ be the forgetful map. In Section 3.3 we will use work of Lichtenbaum to prove:

THEOREM 3.1. *Suppose $K(X)$ has positive characteristic. Then $f(\chi(X, \mathbb{G}_m)) = L_{X,1}$ where $L_{X,1}$ is defined in Proposition 3.11. The class $L_{X,1}$ has order 1 or 2, and is determined by the signs of all the conjugates of the totally real algebraic numbers which are the leading terms at $s = 1$ of the Artin L-series associated to symplectic representations of the Galois group of the cover $X \to X/G$.*

As mentioned in Section 1, Theorem 3.1 is deduced from a result (Proposition 3.11) which involves Euler characteristics of both $\mathbb{G}_m$ and $\mathbb{G}_a$. In a later paper we will prove generalizations of these results to tame actions of G on X. In the arithmetic case, one could presumably develop a counterpart to Theorem 3.1 using a form of the conjecture of Birch and Swinnerton-Dyer.

3.1. THE COHOMOLOGICAL TRIVIALITY OF HYPERCOHOMOLOGY

PROPOSITION 3.2. *Let U be an integral scheme, and let G be a finite group acting freely on U. The quotient map $U \to V = V/G$ is étale. Suppose R is a ring and that $F^\bullet$ is a complex of sheaves of R-modules for the étale topology of V which is bounded below. Suppose that for each subgroup H of G, only finitely many of the cohomology groups $H^i(U/H, F^\bullet)$ are non-trivial. Then the hypercohomology $\mathbf{H}^*(U, F^\bullet)$ is isomorphic in the derived category of the homotopy category of R-modules to a bounded complex $C^\bullet$ of RG-modules which are cohomologically trivial for G.*

Proof. Let

$$I^\bullet: I_0 \to I_1 \to I_2 \to I_3 \to \cdots \tag{3.1}$$

be a complex of injective sheaves of R-modules for the étale topology on V for which there is a quasi-isomorphism $F^\bullet \to I^\bullet$. Then

$$\Gamma(U, I^\bullet): \Gamma(U, I_0) \to \Gamma(U, I_1) \to \Gamma(U, I_2) \to \cdots \tag{3.2}$$

is a complex of RG-modules which isomorphic to $\mathbf{H}^*(U, F^\bullet)$ in the derived category of the homotopy category of RG-modules.

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Let K be the function field of V, and let $j\colon \mathrm{Spec}(K) \to V$ be the natural morphism. Fix a separable closure K^s of K containing the function field N of U. Because G acts freely on U, N/K is a finite Galois extension, and we have a natural identification of G with $\mathrm{Gal}(N/K)$. For each RG-module M, let M^s be the inflation of M from G to $\mathrm{Gal}(K^s/K)$. Viewing M^s as a sheaf on the étale topology of $\mathrm{Spec}(K)$, we have an étale sheaf $j_*(M^s)$ on V. The functor which sends M to $j_*(M^s)$ is an exact fully faithful functor from the category of RG-modules to that of étale sheaves of R-modules on V. Furthermore, if J is any étale sheaf of R-modules on V, $\Gamma(U, J)$ is an RG-module, and $\mathrm{Hom}_R(j_*(M^s), J) = \mathrm{Hom}_{RG}(M, \Gamma(U, J))$. It follows that if J is injective, then $\Gamma(U, J)$ is an injective RG-module. In particular, the terms of (3.2) are injective RG-modules.

Let n be a positive integer such that $H^i(U/H, F^\bullet) = 0$ if $i > n$ and H is a subgroup of G. Define $M_n = \ker(\Gamma(U, I_n) \to \Gamma(U, I_{n+1}))$. Then

$$\Gamma(U, I_0) \to \cdots \Gamma(U, I_{n-1}) \to M_n \tag{3.3}$$

is isomorphic to $\mathbf{H}^*(U, F^\bullet)$ in the derived category, and

$$0 \to M_n \to \Gamma(U, I_n) \to \Gamma(U, I_{n+1}) \to \cdots \tag{3.4}$$

is exact. It will suffice to show that M_n is a cohomologically trivial G-module.

An injective G-module M is cohomologically trivial, since we can split the inclusion of such M into the induced G-module $M \otimes_{\mathbf{Z}} \mathbf{Z}G$, and induced G-modules are cohomologically trivial. Hence, (3.4) gives a resolution of M_n by cohomologically trivial G-modules. Therefore for each subgroup H of G, the cohomology groups of the complex

$$\Gamma(U, I_n)^H \to \Gamma(U, I_{n+1})^H \to \cdots \tag{3.5}$$

beginning in degree 0 give the H-cohomology groups of M_n. However, since $U \to V$ is an étale G-cover, $\Gamma(U, I_j)^H = \Gamma(U/H, I_j)$. Hence, if $i \geqslant n$, the $(i-n)$th cohomology group of (3.5) equals $H^i(U/H, F^\bullet)$, and this is 0 if $i > n$ by assumption. Thus (3.5) has trivial cohomology in dimensions greater than 0, and we conclude that M_n is a cohomologically trivial G-module.

3.2. THE COHOMOLOGY OF $\mathbb{G}_m$ ON SURFACES

The following result is shown in [L1, sect. 3.4 and sect. 4] in the geometric case and in [Sa] in the arithmetic case.

THEOREM 3.3 (Lichtenbaum–Saito). *Let X be a regular two-dimensional scheme which is proper over* $\mathrm{Spec}(\mathbf{Z})$*, geometrically connected, and for which the Brauer group $Br(X)$ is finite. Define $R = \mathbf{Z}$ in the geometric case, and let $R = \mathbf{Z}[1/2]$ in the arithmetic case. Let $\mathbb{G}'_m = R \otimes_{\mathbf{Z}} \mathbb{G}_m$. One has $H^i(X, \mathbb{G}'_m) = 0$ if $i > 4$.*

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The group $H^0(X, \mathbb{G}'_m)$ *if finite (resp. finitely generated over* R*) in the geometric (resp. arithmetic) case by the Dirichlet unit Theorem. The group* $H^2(X, \mathbb{G}'_m)$ *is finite by assumption. The group* $H^1(X, \mathbb{G}'_m)$ *is finitely generated over* R *by the Mordell–Weil Theorem. For* $i = 0, 1$*, there is a natural pairing* $H^i(X, \mathbb{G}'_m) \times H^{4-i}(X, \mathbb{G}'_m) \to \mathbf{Q}/R$ *which induces an isomorphism between* $H^{4-i}(X, \mathbb{G}'_m)$ *and* $\mathrm{Hom}_R(H^i(X, \mathbb{G}'_m), \mathbf{Q}/R)$.

We now apply Proposition 3.2 to $X = U$ and $F^\bullet$ the complex having $\mathbb{G}'_m$ in dimension 0 and the zero sheaf in other dimensions. This shows $\mathbf{H}^*(U, F^\bullet)$ is isomorphic to a complex of the kind in Definition 2.1(a). Thus Theorems 3.3 and 2.3 show:

COROLLARY 3.4. *Let* $L_i = H^{4-i}(X, \mathbb{G}'_m)/H^{4-i}(X, \mathbb{G}'_m)_{\mathrm{tor}}$ *for* $i \in \{3, 4\}$*. Let* $\tau_i\colon \mathrm{Hom}_R(L_i, \mathbf{Q}/R) \to H^i(X, \mathbb{G}'_m)_{\mathrm{div}}$ *be the isomorphism induced by Theorem 3.3. Define* $L_i = 0$ *and* $\tau_i = 0$ *if* $i \notin \{3, 4\}$*. Then* $(C^\bullet, \{L_i\}_i, \{\tau_i\}_i)$ *is a nearly perfect complex of* $R[G]$*-modules. By Theorem 2.3 of Section 2, and the Remark following it, we have an Euler characteristic* $\chi(C^\bullet, \{L_i\}_i, \{\tau_i\}_i) \in K_0(RG)$*; call this invariant* $\chi^G(X, \mathbb{G}'_m)$*. The image of* $\chi^G(X, \mathbb{G}'_m)$ *in* $G_0(RG)$ *is the class*

$$[H^0(X, \mathbb{G}'_m)] - [H^1(X, \mathbb{G}'_m)] + [H^2(X, \mathbb{G}'_m)] +$$
$$+ [\mathrm{Hom}_R(H^1(X, \mathbb{G}'_m), R)] - [\mathrm{Hom}_R(H^1(X, \mathbb{G}'_m)_{\mathrm{tor}}, \mathbf{Q}/R)] -$$
$$- [\mathrm{Hom}_R(H^0(X, \mathbb{G}'_m), R)] + [\mathrm{Hom}_R(H^0(X, \mathbb{G}'_m)_{\mathrm{tor}}, \mathbf{Q}/R)].$$

3.3. LEADING TERMS OF L-FUNCTIONS AT $s = 1$

In this section we will suppose that X is a smooth projective geometrically connected surface over a finite field, and that $Br(X)$ is finite. As before, G will be a finite group acting freely on X. Our goal is to prove Theorem 3.1 relating the image of $\chi(X, \mathbb{G}_m)$ in $G_0(\mathbf{Z}G)$ to the signs at infinity of the leading terms at $s = 1$ of the L-functions of symplectic representations of G.

The strategy of the proof is to work in the finer Grothendieck group $G_0T(\mathbf{Z}G)$ all finite $\mathbf{Z}G$-modules, in which one can define refined Euler characteristics $\chi_T^G(X, \mathbb{G}_m)$ and $\chi_T^G(X, \mathbb{G}_a)$ of $\mathbb{G}_m$ and the additive sheaf $\mathbb{G}_a$ (cf. Definition 3.5). We will use work of Lichtenbaum to show in Theorem 3.9 that the difference of $\chi_T^G(X, \mathbb{G}_a) - \chi_T^G(X, \mathbb{G}_m)$ is determined by the leading terms of L-functions at $s = 1$. To carry out this calculation, we use a ‘Hom-description’ of $G_0T(\mathbf{Z}G)$ due to Queyrut (cf. Proposition 3.7) which makes it possible to idenfity classes in $G_0T(\mathbf{Z}G)$ via suitable functions on the characters of G. The proof of Theorem 3.1, then follows from Theorem 3.9, Queyrut’s ‘Hom-description’ of the forgetful homomorphism $G_0T(\mathbf{Z}G) \to G_0(\mathbf{Z}G)$, and a result of Nakajima which shows $\chi_T^G(X, \mathbb{G}_a)$ has trivial image in $G_0(\mathbf{Z}G)$.

As in [L1], $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$ and the intersection pairing on divisors induces a pairing $H^1(X, \mathbb{G}_m) \times H^1(X, \mathbb{G}_m) \to \mathbf{Z}$ which is non-degenerate when

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tensored with $\mathbf{Q}$. Let $\lambda\colon H^1(X,\mathbb{G}_m)\to \mathrm{Hom}_{\mathbf{Z}}(H^1(X,\mathbb{G}_m),\mathbf{Z}) = H^1(X,\mathbb{G}_m)^D$ be the G-homomorphism induced by this pairing. Then λ has finite kernel and cokernel. Since $H^1(X,\mathbb{G}_m)^D$ is torsion-free, this implies $\mathrm{Ker}(\lambda) = H^1(X,\mathbb{G}_m)_{\mathrm{tor}}$. Recall that if A is an abelian group, then $A_{\mathrm{codiv}} = A/A_{\mathrm{div}}$, where A_{div} is the subgroup of divisible elements of A.

DEFINITION 3.5. Define classes $\chi_T^G(X,\mathbb{G}_m)$ and $\chi_T^G(X,\mathbb{G}_a)$ in $G_0T(\mathbf{Z}G)$ by

$$\begin{aligned}\chi_T^G(X,\mathbb{G}_m) &= [H^0(X,\mathbb{G}_m)] - [H^1(X,\mathbb{G}_m)_{\mathrm{tor}}] + [H^2(X,\mathbb{G}_m)] + \\ &\quad + [H^1(X,\mathbb{G}_m)^D/\lambda H^1(X,\mathbb{G}_m)] - \\ &\quad - [H^3(X,\mathbb{G}_m)_{\mathrm{codiv}}] + [H^4(X,\mathbb{G}_m)]\end{aligned}$$

and

$$\chi_T^G(X,\mathbb{G}_a) = [H^0(X,O_X)] - [H^1(X,O_X)] + [H^2(X,O_X)].$$

DEFINITION 3.6. Suppose v is a place of $\mathbf{Q}$. Define $\overline{\mathbf{Q}}_v$ to be an algebraic closure of $\mathbf{Q}_v$ containing an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. Let R_G (resp. $R_{G,v}$) be the character group of G over $\overline{\mathbf{Q}}$ (resp. $\overline{\mathbf{Q}}_v$). Define $\Omega_F = \mathrm{Gal}(\overline{F}/F)$ for $F = \mathbf{Q}$ and $F = \mathbf{Q}_v$. If v is a finite place of $\mathbf{Q}$, let $\overline{\mathbf{Z}}_v$ be the integral closure of the v-adic integers $\mathbf{Z}_v$ in $\overline{\mathbf{Q}}_v$. Let H_v be the group of character functions $f \in \mathrm{Hom}_{\Omega_{\mathbf{Q}_v}}(R_{G,v},\overline{\mathbf{Q}}_v^*)$ such that $f(\chi)$ is a unit if χ is the character of a projective $\overline{\mathbf{Z}}_vG$-module. If v is the infinite place of $\mathbf{Q}$, let H_v be the group of character functions $f \in \mathrm{Hom}_{\Omega_{\mathbf{Q}_v}}(R_{G,v},\overline{\mathbf{Q}}_v^*)$ such that $f(\chi)$ is real and positive if $\chi \in R_{G,v}$ is the character of a simple $\overline{\mathbf{Q}}_vG$-module of Schur index 2. For all v, define $(\overline{\mathbf{Q}})_v = \overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_v$. Choosing a place of $\overline{\mathbf{Q}}$ over v gives rise to an isomorphism $\mathrm{Hom}_{\Omega_{\mathbf{Q}_v}}(R_{G,v},\overline{\mathbf{Q}}_v^*) \to \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_G,(\overline{\mathbf{Q}})_v^*)$. Define $H_{(v)}$ to be the image of H_v under this isomorphism. Let $J(\overline{\mathbf{Q}})$ (resp. $J_f(\overline{\mathbf{Q}})$) be the group of ideles (resp. finite ideles) of $\mathbf{Q}$. Define $H_{\mathbf{A}} = \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}})) \cap (\prod_v H_{(v)})$ and $H_{\mathbf{A},f} = \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J_f(\overline{\mathbf{Q}})) \cap (\prod_{v\ \mathrm{finite}} H_{(v)})$.

PROPOSITION 3.7 (Queyrut [Q]). *Let v be a finite place of $\mathbf{Q}$ with residue field $k(v)$. There is a unique isomorphism*

$$\tau_v\colon \mathrm{Hom}_{\Omega_{\mathbf{Q}_v}}(R_{G,v},\overline{\mathbf{Q}}_v^*)/H_v \to G_0(k(v)G) \tag{3.6}$$

which sends $\mathrm{Det}_v(\alpha_v)$ to $[\overline{\mathbf{Z}}_vG/(\overline{\mathbf{Z}}_vG\cdot\alpha_v)]$ for $\alpha_v \in (\overline{\mathbf{Q}}_vG)^* \cap \overline{\mathbf{Z}}_vG$, where $\mathrm{Det}_v\colon (\overline{\mathbf{Q}}_vG)^* \to \mathrm{Hom}_{\Omega_{\mathbf{Q}_v}}(R_{G,v},\overline{\mathbf{Q}}_v^*)$ is the usual v-adic determinant map. The direct sum of these isomorphisms gives an isomorphism

$$\tau\colon \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J_f(\overline{\mathbf{Q}}))/H_{\mathbf{A},f} \to G_0T(\mathbf{Z}G) = \bigoplus_{v\ \mathrm{finite}} G_0(k(v)G). \tag{3.7}$$

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Let $\tilde{G}_0(\mathbf{Z}G)$ *be the kernel of the homomorphism* $G_0(\mathbf{Z}G) \to G_0(\mathbf{Q}G)$ *induced by tensoring* $\mathbf{Z}G$*-modules with* $\mathbf{Q}$ *over* $\mathbf{Z}$*. The natural map* $G_0T(\mathbf{Z}G) \to G_0(\mathbf{Z}G)$ *together with* τ *give an isomorphism*

$$\tau^{\text{stab}}: \frac{\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))}{\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*) \cdot H_{\mathbf{A}}} \to \tilde{G}_0(\mathbf{Z}G). \tag{3.8}$$

DEFINITION 3.8. Let V be a representation of G over $\overline{\mathbf{Q}}$. Define $L(t, V)$ to be the Artin L-function of V as a representation of the Galois group of the cover $X \to X/G$ of smooth surfaces, where $t = q^{-s}$ if q is the order of the field of constants of X and s is a complex variable. Let r_V be the order of zero of $L(t, V)$ at $t = q^{-1}$, corresponding to $s = 1$. (In fact, $r_V \leqslant 0$, since the poles of $L(t, V)$ arise from terms associated to $H^2(\overline{X}, \mathbf{Q}_l)$ in the l-adic formula for $L(t, V)$ when l is a prime not dividing q and $\overline{X} = \overline{\mathbf{F}}_q \otimes_{\mathbf{F}_q} X$.) Define

$$c_V = \lim_{t \to q^{-1}} (1 - qt)^{-r_V} L(t, V).$$

Let $c_{X,G} \in \text{Hom}(R_G, \overline{\mathbf{Q}}^*)$ be the function which sends the character χ_V of V to c_V. Define $i_f\colon \overline{\mathbf{Q}}^* \to J_f(\overline{\mathbf{Q}})$ to be the diagonal embedding into the finite ideles.

THEOREM 3.9. *The function* $c_{X,G}$ *lies in* $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*)$*. One has*

$$\tau(i_f(c_{X,G})) = \chi_T^G(X, \mathbb{G}_a) - \chi_T^G(X, \mathbb{G}_m) \tag{3.8}$$

in $G_0T(\mathbf{Z}G)$.

Proof. From the definition of Artin L-functions, one has $L(t, V^\alpha) = L(t, V)^\alpha$ as power series in t when $\alpha \in \text{Aut}(\mathbf{C}/\mathbf{Q})$. Since $L(t, V)$ is a rational function in t by the Weil conjectures, it follows that $c_{V^\alpha} = (c_V)^\alpha$. Hence $c_{X,G}$ lies in $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*)$.

Let v be a finite place of $\mathbf{Q}$ corresponding to the rational prime l, so $k(v) = \mathbf{Z}/l$. Fix an embedding $i_v\colon \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_v$. Let $\text{proj}_v\colon G_0T(\mathbf{Z}G) \to G_0(k(v)G)$ be the natural projection. The composition $i_v \circ c_{X,G}$ lies in $\text{Hom}_{\Omega_{\mathbf{Q}_v}}(R_{G,v}, \overline{\mathbf{Q}}_v^*)$. To prove Theorem 3.9, it will suffice to show for all v, l and i_v as above that

$$\tau_v(i_v \circ c_{X,G}) = \text{proj}_v(\chi_T^G(X, \mathbb{G}_a) - \chi_T^G(X, \mathbb{G}_m)), \tag{3.10}$$

where τ_v is the homomorphism defined in (3.6). By the theory of Brauer characters, two classes in $G_0(k(v)G)$ are equal if they have the same restrictions to every cyclic subgroup Γ of G of order prime to l. The restriction map $\text{res}_G^\Gamma\colon G_0(k(v)G) \to G_0(k(v)\Gamma)$ is induced by the induction map $\text{ind}_\Gamma^G\colon R_{\Gamma,v} \to R_{G,v}$ relative to the isomorphism (3.6). Since Artin L-functions respect induction, we are thus reduced to proving (3.10) when $G = \Gamma$ is a cyclic group of order prime to l, which we assume is the case for the rest of the proof.

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Define R to be the ring of integers of the maximal unramified extension of $\mathbf{Q}_l = \mathbf{Q}_v$ in $\overline{\mathbf{Q}}_v$. Let $\psi: G \to \overline{\mathbf{Q}}^*$ be a one-dimensional character of G, so that $\psi' = i_v \circ \psi$ is a character of G with values in R^*. The group ring $R[G]$ is semisimple; let $e_{\psi'}$ be the central idempotent associated to ψ'. If M is a finite $\mathbf{Z}G$-module, let $\text{length}_{\psi'}(M)$ be the length of a composition series for the finite R-module $e_{\psi'}(R \otimes_{\mathbf{Z}} M)$. There is a unique homomorphism $\text{ord}_{\psi'}: G_0T(\mathbf{Z}G) \to \mathbf{Q}^*$ which sends the class $[M]$ of a finite module M to $l^{\text{length}_{\psi'}(M)}$. By applying the exact functor $N \to e_{\psi'}(R \otimes_{\mathbf{Z}} N)$ to each of the modules appearing in Lichtenbaum's proof of [L1, Thm. 4.8], one finds an equality of non-zero fractional R-ideals

$$i(c_\psi)R = \frac{\text{ord}_{\psi'}(\chi_G^T(X, \mathbb{G}_a))}{\text{ord}_{\psi'}(\chi_G^T(X, \mathbb{G}_m))} \cdot R \tag{3.11}$$

In fact, Theorem 4.8 of [L1] for a projective surface is equivalent to (3.11) when ψ is the trivial character of the trivial group G. (Lichtenbaum's proof of [L1, Thm. 4.8] relies on [L2], which in turn depends on work of Tate and Milne on geometric counterparts of the Birch and Swinnerton–Dyer conjecture.) Unwinding the definitions in Proposition 3.7, we see (3.11) is equivalent to (3.10), so the proof is complete.

LEMMA 3.10. *There is a unique character function $h \in \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))$ with the following properties.*

(a) *For all χ, the finite components of $h(\chi)$ are equal to* 1, *and the infinite component is* 1 *if χ is not symplectic.*
(b) *Suppose χ is symplectic, and let ∞ be the infinite place of* $\mathbf{Q}$. *Let $\alpha_\infty \in (\overline{\mathbf{Q}})_\infty = \overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_\infty$ be the infinite component of an idele $\alpha \in J(\overline{\mathbf{Q}})$. Under each $\mathbf{Q}_\infty$-algebra map $(\overline{\mathbf{Q}})_\infty \to \overline{\mathbf{Q}_\infty} = \mathbf{C}$, the image of $h(\chi)_\infty$ is ± 1 and the image of $(c_{X,G}(\chi) \cdot h(\chi))_\infty$ is real and positive.*

Proof. Since symplectic characters are real-valued, $c_{X,G}(\chi)$ must be a totally real algebraic number if χ is symplectic. We then define $h(\chi)_\infty$ via the signs of all the conjugates of $c_{X,G}(\chi)$. One has $h \in \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))$ since $c_{X,G} \in \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*)$.

PROPOSITION 3.11. *Define $L_{X,1} \in G_0(\mathbf{Z}G)$ to be the image of the function h of Lemma* 3.10 *under the homomorphism $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}})) \to G_0(\mathbf{Z}G)$ resulting from* (3.8) *of Proposition* 3.7. *Then $L_{X,1}$ has order one or two, and is determined by the signs of the conjugates of the totally real algebraic numbers $c_{X,V}$ as V ranges over the symplectic representations of G. One has*

$$L_{X,1} = z(\chi_T^G(X, \mathbb{G}_a) - \chi_T^G(X, \mathbb{G}_m)), \tag{3.12}$$

where $z: G_0T(\mathbf{Z}G) \to G_0(\mathbf{Z}G)$ is the forgetful homomorphism.

Proof. View the finite ideles $J_f(\overline{\mathbf{Q}})$ as the subgroup of $J(\overline{\mathbf{Q}})$ having trivial infinite components. Then Theorem 3.9 shows that

$$\tau^{\text{stab}}(i_f(c_{X,G})) = z(\chi_T^G(X, \mathbb{G}_a) - \chi_T^G(X, \mathbb{G}_m)) \tag{3.13}$$

where τ^{stab} is the homomorphism resulting from (3.8). Let $i: \overline{\mathbf{Q}} \to J(\overline{\mathbf{Q}})$ be the diagonal embedding, and let $i_{(\infty)}: \overline{\mathbf{Q}} \to (\overline{\mathbf{Q}})_\infty$ be the natural embedding into ideles with trivial finite components. Then $i_f(c_{X,G}) = i(c_{X,G}) \cdot i_{(\infty)}(c_{X,G})^{-1}$. From Lemma 3.10 and Proposition 3.7, we see that the character function $i_{(\infty)}(c_{X,G})^{-1} \cdot h^{-1}$ lies in the subgroup $H_{\mathbf{A}}$ appearing in (3.8), while $i(c_{X,G})$ lies in $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*)$. Thus

$$\begin{aligned} \tau^{\text{stab}}(i_f(c_{X,G})) &= \tau^{\text{stab}}(i(c_{X,G})) \cdot \tau^{\text{stab}}(i_{(\infty)}(c_{X,G})^{-1} \cdot h^{-1}) \cdot \tau^{\text{stab}}(h) \\ &= 1 \cdot 1 \cdot L_{X,1}. \end{aligned}$$

Combining (3.13) and (3.14) shows Proposition 3.11.

Proof of Theorem 3.1. We wish to show that

$$f(\chi^G(X, \mathbb{G}_m)) = L_{X,1}, \tag{3.15}$$

where $\chi^G(X, \mathbb{G}_m) \in K_0(\mathbf{Z}G)$ is defined in Corollary 3.4, $f: K_0(\mathbf{Z}G) \to G_0(\mathbf{Z}G)$ is the forgetful homomorphism, and $L_{X,1}$ is defined in Proposition 3.11.

By Theorem 3.3,

$$H^3(X, \mathbb{G}_m)_{\text{codiv}} = \text{Hom}(H^1(X, \mathbb{G}_m)_{\text{tor}}, \mathbf{Q}/\mathbf{Z})$$

and

$$H^4(X, \mathbb{G}_m) = \text{Hom}(H^0(X, \mathbb{G}_m), \mathbf{Q}/\mathbf{Z}) = \text{Hom}(H^0(X, \mathbb{G}_m)_{\text{tor}}, \mathbf{Q}/\mathbf{Z}).$$

Substituting the right sides of these expressions into Corollary 3.4 when $R = \mathbf{Z}$ gives

$$\begin{aligned} &f(\chi^G(X, \mathbb{G}_m)) \\ &\quad = [H^0(X, \mathbb{G}_m)] - [H^1(X, \mathbb{G}_m)] + [H^2(X, \mathbb{G}_m)] + \\ &\qquad + [H^1(X, \mathbb{G}_m)^D] - [H^3(X, \mathbb{G}_m)_{\text{codiv}}] + [H^4(X, \mathbb{G}_m)], \end{aligned} \tag{3.16}$$

since $\text{Hom}_{\mathbf{Z}}(H^0(X, \mathbb{G}_m), \mathbf{Z}) = 0$ because $H^0(X, \mathbb{G}_m)$ is finite. As noted just prior to Definition 3.5, the kernel of the homomorphism $\lambda: H^1(X, \mathbb{G}_m) \to H^1(X, \mathbb{G}_m)^D$ is $H^1(X, \mathbb{G}_m)_{\text{tor}}$. Thus

$$[H^1(X, \mathbb{G}_m)] = [\lambda H^1(X, \mathbb{G}_m)] + [H^1(X, \mathbb{G}_m)_{\text{tor}}] \tag{3.17}$$

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in $G_0(\mathbf{Z}G)$. On substituting (3.17) into (3.16), we see from the definition of $\chi_T^G(X,\mathbb{G}_m)$ in Definition 3.5, that

$$f(\chi^G(X,\mathbb{G}_m)) = z(\chi_T^G(X,\mathbb{G}_m)). \tag{3.18}$$

By Proposition 3.11,

$$z(\chi_T^G(X,\mathbb{G}_m)) - z(\chi_T^G(X,\mathbb{G}_a)) = -L_{X,1} = L_{X,1} \tag{3.19}$$

since $L_{X,1}$ has order 1 or 2. Thus (3.18) and (3.19) show that to prove (3.15), it will suffice to show

$$z(\chi_T^G(X,\mathbb{G}_a)) = 0. \tag{3.20}$$

Since G acts freely on X, it is a Theorem of Nakajima [N] that the class $\chi_T^G(X,\mathbb{G}_a) = [H^0(X,\mathcal{O}_X)] - [H^1(X,\mathcal{O}_X)] + [H^2(X,\mathcal{O}_X)] \in G_0((\mathbf{Z}/p\mathbf{Z})G)$ is the class of a free $(\mathbf{Z}/p\mathbf{Z})G$-module, where p is the characteristic of the function field $K(X)$. Hence (3.20) holds, which completes the proof.

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