CENTRALIZERS OF \( p \)-SUBGROUPS IN SIMPLE LOCALLY FINITE GROUPS

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Abstract. In Ersoy et al. [J. Algebra 481 (2017), 1–11], we have proved that if \( G \) is a
locally finite group with an elementary abelian \( p \)-subgroup \( A \) of order strictly greater than
\( p^2 \) such that \( CG(A) \) is Chernikov and for every non-identity \( \alpha \in A \) the centralizer \( CG(\alpha) \)
does not involve an infinite simple group, then \( G \) is almost locally soluble. This result is a
consequence of another result proved in Ersoy et al. [J. Algebra 481 (2017), 1–11], namely:
if \( G \) is a simple locally finite group with an elementary abelian group \( A \) of automorphisms
acting on it such that the order of \( A \) is greater than \( p^2 \), the centralizer \( CG(A) \) is Chernikov
and for every non-identity \( \alpha \in A \), the set of fixed points \( CG(\alpha) \) does not involve an infinite
simple groups then \( G \) is finite. In this paper, we improve this result about simple locally
finite groups: Indeed, suppose that \( G \) is a simple locally finite group, consider a finite non-
abelian subgroup \( P \) of automorphisms of exponent \( p \) such that the centralizer \( CG(P) \) is
Chernikov and for every non-identity \( \alpha \in P \) the set of fixed points \( CG(\alpha) \) does not involve
an infinite simple group. We prove that if \( Aut(G) \) has such a subgroup, then \( G \cong PSL_p(k) \)
where \( \text{char } k \neq p \) and \( P \) has a subgroup \( Q \) of order \( p^2 \) such that \( CG(P) = Q \).

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1. Introduction. In [2], we have proved the following result:

THEOREM 1.1. [2, Theorem 1.1]. Let \( p \) be a prime and \( G \) a locally finite group containing
an elementary abelian \( p \)-subgroup \( A \) of rank at least 3 such that \( CG(A) \) is Chernikov and
\( CG(\alpha) \) involves no infinite simple groups for any \( \alpha \in A^\# \). Then \( G \) is almost locally soluble.

To prove Theorem 1.1, we gave the following characterization of \( PSL_p(k) \) where \( \text{char } k \neq p \).

THEOREM 1.2. [2, Theorem 1.2]. An infinite simple locally finite group \( G \) admits an
elementary abelian \( p \)-group of automorphisms \( A \) such that \( CG(A) \) is Chernikov and \( CG(\alpha) \)
involves no infinite simple groups for any \( \alpha \in A^\# \) if and only if \( G \) is isomorphic to \( PSL_p(k) \)
for some locally finite field \( k \) of characteristic different from \( p \) and \( A \) has order \( p^2 \).

In this paper, we will improve Theorem 1.2. Indeed, we will prove a similar result
without assuming \( A \) is an elementary abelian, but instead, we prove for any subgroup of
exponent \( p \).

THEOREM 1.3. Let \( G \) be an infinite simple locally finite group, \( P \) a subgroup of
automorphisms of exponent \( p \) such that

(1) \( CG(P) \) is Chernikov;
(2) For every $\alpha \in P \setminus \{1\}$, the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group.

Then $G \cong \text{PSL}_p(k)$ where $k$ is an infinite locally finite field of characteristic $p$ and $P$ has a subgroup $Q$ of order $p^2$ such that $C_G(P) = C_G(Q) = Q$.

2. Preliminaries. Let us recall some definitions of the concepts mentioned in the theorems. First, consider $C_{p^n} = \{x \in \mathbb{C} : x^{p^n} = 1\}$. Here, $(C_{p^n}, .)$ defines a group isomorphic to a cyclic group of order $p^n$. Observe that if $m|n$ then $C_{p^m} \leq C_{p^n}$, and with the inclusion maps, these sets form a direct system, where the direct limit

$$\lim_{n \in \mathbb{N}} C_{p^n}$$

is denoted by $C_{p^\infty}$, which consists of all complex $p^n$-th roots of unity, and forms a group under complex multiplication. This group is called the quasi-cyclic $p$-group.

**Definition 2.1.** A group is called a Chernikov group if it is a finite extension of a direct product of finitely many copies of some quasi-cyclic $p_i$-groups, for possibly distinct primes $p_i$.

**Definition 2.2.** Let $\chi$ be a group-theoretical property. If a group $G$ has a normal subgroup of finite index satisfying $\chi$, then $G$ is called almost $\chi$.

**Definition 2.3.** Let $G$ and $H$ be two groups. If $G$ has a normal subgroup $K$ such that $G/K$ has a subgroup isomorphic to $H$, then $G$ is said to involve a subgroup isomorphic to $H$.

**Definition 2.4.** A group satisfies the minimal condition, namely $\text{min}$, if any non-empty set of subgroups has a minimal subgroup. A group satisfies $\text{min}$-$p$ if any non-empty set of $p$-subgroups has a minimal subgroup.

Kegel–Wehrfritz and Sunkov proved independently that a locally finite group satisfying minimal condition is a Chernikov group (see [5, 9]). For detailed discussion of groups satisfying $\text{min}$ and $\text{min}$-$p$, see [6].

3. Main results. First, we need the following proposition:

**Proposition 3.1.** Let $\mathcal{G}$ be a simple linear algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$, let $g \in \mathcal{G}$ be an element of prime order $p \neq q$ such that $C_{\mathcal{G}}(g)$ is a non-abelian group which does not involve any infinite simple groups. Then

(i) The identity component $C_{\mathcal{G}}(g)^0$ of the centralizer of $g$ in $\mathcal{G}$ is a maximal torus of $\mathcal{G}$,

(ii) $\mathcal{G} \cong \text{PGL}_p(\mathbb{F}_q)$.

**Proof.** Since $\mathcal{G}$ is a simple linear algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$ and $g \in G$ a semisimple element, $g$ is contained in a maximal torus $T$ of $\mathcal{G}$. By [7, Propositions 14.1 and 14.2], $C_{\mathcal{G}}(g)^0$ is connected reductive, containing a maximal torus $T$, and involving no infinite simple groups. Hence, $C_{\mathcal{G}}(g)^0 = T$. By [7, Proposition 14.20], the exponent of $C_{\mathcal{G}}(g)/C_{\mathcal{G}}(g)^0$ divides $p$, hence either $C_{\mathcal{G}}(g)$ is connected, and hence a torus, or $C_{\mathcal{G}}(g)/C_{\mathcal{G}}(g)^0$ is a finite group of exponent $p$.

Since $C_{\mathcal{G}}(g)$ is not abelian, one has $C_{\mathcal{G}}(g)$ a finite extension of an abelian group $T$, so it has finite rank. Recall that an infinite group $G$ is said to have finite rank $r$ if every finitely
generated subgroup is \( r \)-generated. In [1, Theorem 1.8], we have shown that when a simple linear algebraic group \( \overline{G} \) over the algebraic closure of \( \mathbb{F}_q \) has an element \( g \) of order \( p \) with \( C_{\sigma}(g) \) has finite rank, then one of the following cases occur:

1. \( \overline{G} \) is of type \( A_l \) and \( p > l \),
2. \( \overline{G} \) is of type \( B_l, C_l \) and \( p > 2l - 1 \),
3. \( \overline{G} \) has type \( D_l \) and \( p = 2l - 3 \),
4. \( \overline{G} \) is isomorphic to one of \( E_6, E_7, E_8, F_4 \), or \( G_2 \) and \( p > 11, 17, 29, 17 \), or \( 5 \), respectively.

On the other hand, since \( C_{\overline{G}}(g)/C_{\overline{G}}(g)^0 \) has exponent \( p \), by [10, Corollary 4.4] and [7, Proposition 14.20], we get \( p \) is a torsion prime. The list of torsion primes of linear algebraic groups is defined as follows: for type \( A_l \), these are the primes that divide \( l + 1 \). For types \( B_l, C_l, D_l, G_2 \), the prime is 2. For types \( E_6, E_7, F_4 \), the primes are 2 and 3, and for type \( E_8 \), the primes are 2, 3, 5 (see [8]).

Hence, one deduce that the only possible case that may occur is \( \overline{G} \) has type \( A_{p-1} \), indeed \( \overline{G} \cong PGL_p(\overline{\mathbb{F}}_q) \).

**Theorem 3.2.** Let \( G \) be an infinite simple locally finite group with a finite non-abelian \( p \)-group of automorphisms \( P \) such that

1. \( C_G(P) \) is Chernikov,
2. For every \( \alpha \in P \setminus \{1\} \), the set of fixed points \( C_G(\alpha) \) does not involve an infinite simple group

Then, \( G \) is isomorphic to \( PSL_p(k) \) where \( k \) is a locally finite field of characteristic \( q \neq p \) and \( P \) is metabelian.

**Proof.** Since \( P \) is a finite \( p \)-group and \( C_G(P) \) satisfies \( min-p \), by [2, Lemma 2.1], \( G \) satisfies \( min-p \). Then, by [4, Theorem B], \( G \) is a simple group of Lie type over a locally finite field \( k \) of characteristic \( q \). Now assume that \( q = p \). Clearly \( G \) contains a root subgroup, which is an infinite elementary abelian \( p \)-subgroup. Hence, \( G \) can not satisfy \( min-p \). Hence, \( q \neq p \), that is, \( G \) is isomorphic to a simple group of Lie type over an infinite locally finite field of characteristic \( q \neq p \).

Now, by [3, Lemma 4.3], there exists a simple linear algebraic group \( \overline{G} \) of adjoint type, a Frobenius map \( \sigma \) on \( \overline{G} \) and a sequence of natural numbers \( n_i \) such that

\[
G = \bigcup_{i \in \mathbb{N}} O^i(\overline{G}_{\sigma^{n_i}}).
\]

By assumption, the centralizer of any non-identity element does not involve an infinite simple group, so [2, Lemma 2.3] implies that \( P \) consists of inner-diagonal automorphisms of \( G \). Hence, \( P \leq \bigcup_{i \in \mathbb{N}} \overline{G}_{\sigma^{n_i}} \). Therefore, \( P \leq \overline{G}_{\sigma^j} \) for some \( j \in \mathbb{N} \).

Choose \( 1 \neq z \in \mathbb{Z}(P) \). Clearly, \( P \leq C_{\overline{G}}(z) \). Now, \( C_{\overline{G}}(z) = \bigcup_{i \in \mathbb{N}} O^i(C_{\overline{G}}(z)_{\sigma^n}) \).

By assumption, \( C_G(z) \) does not involve an infinite simple group. Now, suppose that \( C_{\overline{G}}(z) \) involves a simple linear algebraic group \( H \). Consider the union of fixed points of \( \sigma^j \) on \( H \), denoted by \( H_j = H_{\sigma^j} \). Clearly, \( H_j \leq H_{j+1} \) and infinitely many of \( H \) involves finite simple groups such that their union form an infinite locally finite simple group. Hence, we get a contradiction and we deduce \( C_{\overline{G}}(z) \) does not involve a simple linear algebraic group.

By [2, Lemma 2.4], \( C_{\overline{G}}(z) \) is metabelian. Hence, \( P \) is metabelian. On the other hand, since \( P \) is not abelian, \( C_{\overline{G}}(z) \) is not abelian.

By Proposition 3.1, \( \overline{G} \) is isomorphic to \( PGL_p(\overline{\mathbb{F}}_q) \). Hence, \( G \) is isomorphic to either \( PSL_p(k) \) or \( PSU_p(k) \). Following the argument in the proof of Theorem 1.2 in [2], since
the Weyl group of $PSU_p(k)$ has no elements of order $p$, and $PT/T$ embeds in the Weyl group, $PSU_p(k)$ has no such non-abelian subgroup $P$. Therefore, $G \cong PSL_p(k)$ where $k$ is an infinite locally finite field of characteristic $q \neq p$.

Then, we prove the main result of the paper:

**Proof of Theorem 1.3.** Assume first that $P$ is abelian. Then by Theorem 1.2, the result follows with $|P| = p^2$.

Now, assume $P$ is non-abelian. By Theorem 3.2, $G \cong PSL_p(k)$ where $k$ is a locally finite field of characteristic $q \neq p$. Let $z \in Z(P)$, observe that $P \leq CG(z) \leq CG(z)$ where $G$ is the corresponding simple linear algebraic group and $\sigma$ is the Frobenius map such that $G = \bigcup_{i \in \mathbb{N}} O(i)(G_{\sigma^i})$, which exist by [3, Lemma 4.3]. Denote the maximal torus of $G$ containing $z$ by $T$. By Proposition 3.1(i), $CG(z)^0 = T$. Indeed, by [10, Corollary 1.7], $T$ is the unique maximal torus containing $z$. Since $P$ is not abelian, $CG(z)/CG(z)^0$ can not be 1, hence by [7, Proposition 14.20], it has exponent $p$. Let $y$ be any element of $CG(z)\backslash CG(z)^0$. Then, $Q = \langle y, z \rangle$ has order $p^2$. Indeed, $CG(z)^0 = T$, and $y \in N_G(T)$. Hence, $y$ induces an element $w$ of order $p$ in the Weyl group. Now, $z \in C_T(w)$. The computation in the proof of Theorem 1.2 in [2] shows that indeed $C_T(w)$ has order $p$, hence $CG(Q) = Q$. This $Q$ is the required subgroup.

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**References**


