PATH PROPERTIES OF THE PRIMITIVES OF A BROWNIAN MOTION

ZHENGYAN LIN

(Received 16 June 1999; revised 28 June 2000)

Communicated by V. Stefanov

Abstract

Let \( \{W(t), t \geq 0\} \) be a standard Brownian motion. For a positive integer \( m \), define a Gaussian process

\[
X_m(t) = \frac{1}{m!} \int_0^t (t-s)^m \, dW(s).
\]

Watanabe and Lachal gave some asymptotic properties of the process \( X_m(\cdot) \), \( m \geq 1 \). In this paper, we study the bounds of its moduli of continuity and large increments by establishing large deviation results.

2000 Mathematics subject classification: primary 60F15, 60J65, 60G15; secondary 60G17.

Keywords and phrases: moduli of continuity, large increments, Brownian motion, primitive.

1. Introduction

Let \( \{W(t), t \geq 0\} \) be a standard Brownian motion. For a positive integer \( m \), define a Gaussian process

\[
X_m(t) = \frac{1}{m!} \int_0^t (t-s)^m \, dW(s),
\]

which was first mentioned by Shepp [4]. This class of processes arises in several domains of applied mathematics. For instance, the process \( X_1(\cdot) \), which has been studied at length, is the solution of Langevin’s equation under certain physical conditions. Wahba [5,6] used \( X_n(\cdot) \) to derive a correspondence between smoothing by splines and Bayesian estimation in certain stochastic models.

Watanabe [7] established a law of the iterated logarithm for \( X_1(\cdot) \) (in fact, his result concerns a larger class of Gaussian processes). Lachal [2,3] studied the law of the
iterated logarithm and regular points for $X_m(\cdot)$, $m \geq 1$. Moreover, Lachal [2] obtained some integral tests that precisely characterize the upper functions for $X_m$, which is an important result in the asymptotic study of $X_m$.

In this paper we study path behaviour of the process $X_m(\cdot)$. By establishing results on large deviations, we investigate the moduli of continuity and large increment properties for $X_m(\cdot)$, $m \geq 1$, and give their upper and lower bounds. Note that increments of $X_m(\cdot)$ are neither independent nor stationary, moreover $X_m(\cdot)$ is also not a stationary process. Usually, stationarity of increments is required for investigating the moduli of continuity and large increments of a process.

First of all, we give some moment results. We have

\begin{equation}
EX_m^2(t) = \frac{1}{(m!)^2} \int_0^t (t-s)^{2m} ds =: b_mt^{2m+1},
\end{equation}

where $b_m = (m!)^{-2}(2m+1)^{-1}$, and for any $h > 0$

\begin{equation}
E(X_m(t+h) - X_m(t))^2 = \frac{1}{(m!)^2} E \left( \int_0^{t+h} (t+h-s)^m dW(s) - \int_0^t (t-s)^m dW(s) \right)^2
\end{equation}

for some positive $b_{mj}, j = 2, \ldots, 2m+1$, where $b_{m2} = ((m-1)!)^{-2}(2m-1)^{-1}$. Equality (1.3) implies

\begin{equation}
E(X_m(t+h) - X_m(t))^2 = (1 + \delta(h/t))b_m h^2 t^{2m-1},
\end{equation}

where $0 < \delta(x) \to 0$ as $x \to 0$. Hence

\begin{equation}
E(X_m(t+h)X_m(t)) = \frac{1}{2} E \left( X_m^2(t+h) + X_m^2(t) - (X_m(t+h) - X_m(t))^2 \right)
\end{equation}

\begin{equation}
= \frac{1}{2} b_m \left( (t+h)^{2m+1} + t^{2m+1} \right) - \frac{1}{2} \sum_{j=2}^{2m+1} b_{mj} h^j t^{2m+1-j}.
\end{equation}

Put $Y_m(t) = X_m(t)/t^{m-1/2}$. By (1.2)

\begin{equation}
EY_m^2(t) = b_mt^2.
\end{equation}
Using (1.2), (1.3) and (1.5) we have

\[(1.7) \quad E(Y_m(t+h) - Y_m(t))^2 = E \left\{ \frac{X_m(t+h) - X_m(t)}{(t+h)^{m-1/2}} - \left( \frac{1}{t^{m-1/2}} - \frac{1}{(t+h)^{m-1/2}} \right) X_m(t) \right\}^2 \]

\[= \sum_{j=2}^{2m+1} \frac{b_{mj} h^j t^{2m+1-j}}{(t+h)^{2m-1}} + \frac{((t+h)^{m-1/2} - t^{m-1/2})^2}{t^{2m-1}(t+h)^{2m-1}} b_m t^{2m+1} \]

\[= -\frac{2((t+h)^{m-1/2} - t^{m-1/2})}{t^{m-1/2}(t+h)^{2m-1}} \left\{ \frac{1}{2} b_m ((t+h)^{2m+1} - t^{2m+1}) \right\} \]

\[= B_m h^2 + g_m(h, t), \]

where

\[B_m = b_{m2} + b_m \left\{ \left( m - \frac{1}{2} \right)^2 - \left( m - \frac{1}{2} \right)(2m + 1) \right\} \]

\[= b_{m2} - b_m \left( m - \frac{1}{2} \right) \left( m + \frac{3}{2} \right), \]

\[g_m(h, t) = O(h^3 t) \quad \text{as } h t \to 0, \]

which implies that

\[(1.8) \quad E(Y_m(t+h) - Y_m(t))^2 = (1 + o(1)) B_m h^2 \quad \text{as } h t \to 0. \]

2. Large deviations

First we quote a well-known lemma.

**Lemma 2.1 (Fernique).** Let \( G(t) \) be a Gaussian process on \([0, 1]\) with \( EG^2(t) \leq A^2 \) and \( E(G(t) - G(s))^2 \leq \sigma^2 |t - s| \), where \( \sigma(\cdot) \) is a continuous nondecreasing function satisfying

\[\int_1^\infty \sigma(e^{-x^2}) \, dx < \infty. \]

Then, for \( x \geq 2 \), we have

\[P \left\{ \sup_{0 \leq t \leq 1} |G(t)| \geq x \left( A + \int_1^\infty \sigma(e^{-y^2}) \, dy \right) \right\} \leq ce^{-x^2/2}, \]

where \( c \) is an absolute constant.
The following is a large deviation result for small time increments.

**Proposition 2.1.** For any \( \varepsilon > 0 \), there exist positive numbers \( h_0, x_0, c_1 \) and \( C_1 \) such that for any \( 0 < h \leq h_0 \) and \( x \geq x_0 \)

\[
P \left\{ \sup_{0 < t \leq 1 - h} \sup_{0 \leq s \leq h} \frac{|X_m(t + s) - X_m(t)|}{(t \vee h)^{m-1/2}} \geq (1 + \varepsilon) b_m^{1/2} h x \right\} \leq C_1 (e^{-c_1 x^2} + h^{-1} e^{-x^2/2}).
\]

**Proof.** For any \( t > 0 \) and integer \( r > 0 \), let \( t_r = [t/2^r/h]/(2^r/h) \), and write, for \( rh < 1 - h \),

\[
(2.1) \sup_{0 < t \leq 1 - h} \sup_{0 \leq s \leq h} \frac{|X_m(t + s) - X_m(t)|}{(t \vee h)^{m-1/2}} = \sup_{0 < t \leq rh} \sup_{0 \leq s \leq h} \frac{|X_m(t + s) - X_m(t)|}{(t \vee h)^{m-1/2}} \vee \sup_{rh \leq t \leq 1 - h} \sup_{0 \leq s \leq h} \frac{|X_m(t + s) - X_m(t)|}{t^{m-1/2}} =: I_1 \vee I_2.
\]

Noting \( t \vee h \geq (t + h)/2 \), we have

\[
I_1 \leq 2^{m-1/2} \sup_{0 < t \leq rh} \sup_{0 \leq s \leq h} \frac{|X_m(t + s) - X_m(t)|}{(t + h)^{m-1/2}} \leq 2^{m+1/2} \sup_{0 < t \leq (1 + r)h} |Y_m(t)|.
\]

Let \( Z_m(t) = Y_m(((1 + r)ht), 0 < t \leq 1 \). We will use Lemma 2.1 with \( A = b_m^{1/2} (1 + rh) \) and \( \sigma(s) = (2Bm)^{1/2} (1 + rh)s \). Put \( D = (1 + r)(b_m^{1/2} + (2Bm)^{1/2} \int_1^\infty e^{-y^2} dy) \). For any given \( \varepsilon > 0 \), take \( r = r(\varepsilon) \) to be specified later on. By Lemma 2.1, we have

\[
(2.2) P\{I_1 \geq b_m^{1/2} h x \} \leq P \left\{ \sup_{0 < t \leq 1} |Z_m(t)| \geq (b_m^{1/2} 2^{-(m+1/2)} D^{-1}) D h x \right\} \leq C e^{-c_1 x^2}
\]

for \( x \geq x_0 := b_m^{1/2} 2^{m+1/2} D \), where \( c_1 = b_m 2^{-2(m+1)} D^{-2}/2 \).

Consider \( I_2 \) now. We shall use a method similar to that in [1]. For \( rh < t \leq 1 - h \), \( 0 \leq s \leq h \), which implies that

\[
\frac{1}{t^{m-1/2}} \leq \left( 1 + \frac{1}{r} \right)^{m-1/2} \frac{1}{(t + s)^{m-1/2}} \leq \left( 1 + \frac{1}{r} \right)^{m-1/2} \frac{1}{(t + s)^{m-1/2}}
\]

for any \( j \geq 0 \), we have

\[
(2.3) \frac{|X_m(t + s) - X_m(t)|}{t^{m-1/2}} \leq \frac{|X_m((t + s)_j) - X_m(t_j)|}{t^{m-1/2}} + \frac{|X_m((t + s)_r) - X_m(t + s)|}{t^{m-1/2}}
\]
For the first term of the right hand side of (2.3), by (1.4) we have provided $r = r(\varepsilon)$ is large enough. Hence, noting that the number of points lying within the grid $[0, h] \times [rh, 1]$ with step $h/2^r$ is less than $2^{2r}/h$, we obtain

\[ E \left( \frac{X_m((t + s), r) - X_m(t, r)}{t_r^{m-1/2}} \right)^2 \leq \left( 1 + \frac{\varepsilon}{4} \right)^2 b_{m2} (1 + 2^{-r})^2 h^2 \leq \left( 1 + \frac{\varepsilon}{3} \right)^2 b_{m2} h^2, \]

provided $r = r(\varepsilon)$ is large enough. Hence, noting that the number of points lying within the grid $[0, h] \times [rh, 1]$ with step $h/2^r$ is less than $2^{2r}/h$, we obtain

\[ P \left\{ \sup_{r < t < 1 - h} \sup_{0 < s < h} \frac{|X_m((t + s), r) - X_m(t, r)|}{t_r^{m-1/2}} \geq \left( 1 + \frac{\varepsilon}{3} \right)^2 b_{m2} h \right\} \leq \frac{2^{2r}}{h} \sup_{r < t < 1 - h} \sup_{0 < s < h} P \left\{ \frac{|X_m((t + s), r) - X_m(t, r)|}{t_r^{m-1/2}} \geq \left( 1 + \frac{\varepsilon}{3} \right)^2 b_{m2} h \right\} \leq \frac{2^{2r}}{h} e^{-x^2/2} \]

by recalling the well-known inequality $1 - \Phi(x) \leq (1/\sqrt{2\pi}x) e^{-x^2/2}$. (Without loss of generality, assume that $x_0 \geq 1/\sqrt{2\pi}$.)

Consider the second term of the right hand side of (2.3). Note the following inequality:

\[ P \left\{ \sup_{i \in I} \sum_{j=0}^{\infty} X_{ij} \geq \sum_{j=0}^{\infty} x_j \right\} \leq \sum_{i \in I} P \left\{ \sum_{j=0}^{\infty} X_{ij} \geq \sum_{j=0}^{\infty} x_j \right\} \leq \#(I) \sup_{i \in I} P \{ \exists j \geq 0 : X_{ij} \geq x_j \} \leq \#(I) \sup_{i \in I} \sum_{j=0}^{\infty} P \{ X_{ij} \geq x_j \}, \]
where $X_{ij}, i \in I, j = 0, 1, \ldots,$ are random variables and $x_j, j = 0, 1, \ldots,$ are real numbers. Moreover, by (1.4) again, we have

$$E \left( \frac{X_m(t_{r+j+1}) - X_m(t_{r+j})}{r_{r+j+1}^{m-1/2}} \right)^2 \leq 2b_m^2h^2/2^{2(r+j+1)}$$

for any $0 < t \leq 1$, provided $r$ is large enough. Furthermore, we may demand

$$\sqrt{2} \sum_{j=0}^{\infty} 2^{-(r+j+1)/2} \leq \left( 1 + \frac{1}{r} \right)^{-m+1/2} \frac{c}{3}.$$

Then we have

$$(2.5) \
\P \left\{ \sup_{r < t \leq 1-h} \sup_{0 \leq s \leq h} \sum_{j=0}^{\infty} \left| X_m((t+s)_{r+j+1}) - X_m((t+s)_{r+j}) \right| \right\} \leq \left( 1 + \frac{1}{r} \right)^{-m+1/2} \frac{c}{3} b_m^{1/2} h x$$

for large $r$. Similarly, for the third term of the right hand side of (2.3) we have

$$(2.6) \quad \P \left\{ \sup_{r < t \leq 1-h} \sup_{0 \leq s \leq h} \sum_{j=0}^{\infty} \left| X_m(t_{r+j+1}) - X_m(t_{r+j}) \right| \right\} \leq \frac{2^r}{h} e^{-x^2/2}.$$

Combining (2.3)–(2.6) we obtain

$$(2.7) \quad P \left\{ I_2 \geq (1 + \epsilon) b_m^{1/2} h x \right\} \leq (2^r + 2^{r+1}) \frac{1}{h} e^{-x^2/2}.$$

(2.2) and (2.7) together imply the conclusion of Proposition 2.1. \qed

An analogue of Proposition 2.1 in the large increment case is the following.

**Proposition 2.2.** Let $a_T$ be a function of $T$ with $0 < a_T \leq T$ and $a_T / T \to 0$ as $T \to \infty$. Then for any $\epsilon > 0$, there exist positive numbers $T_0, x_1, c_2$ and $C_2$ such that for any $T \geq T_0$ and $x \geq x_1$,

$$P \left\{ \sup_{0 < t \leq T-a_T} \sup_{0 \leq s \leq a_T} \left| X_m(t+s) - X_m(t) \right| \right\} \leq (1 + \epsilon) b_m^{1/2} a_T x \leq C_2(e^{-\alpha x^2} + (a_T^{-1} e^{-x^2/2}).$$

The proof is similar to that of Proposition 2.1, and hence, is omitted.
We need another well-known lemma.

**Lemma 3.1 (Slepian).** Let $G(t)$ and $G^*(t)$ be Gaussian processes on $[0, T]$ for some $0 < T < \infty$, possessing continuous sample path functions with $E G(t) = E G^*(t) = 0$, $E G^2(t) = E G^{*2}(t) = 1$, and let $\rho(s, t)$ and $\rho^*(s, t)$ be their respective covariance functions. Suppose that we have $\rho(s, t) \geq \rho^*(s, t)$, $s, t \in [0, T]$. Then for any real $u$,

$$P \left\{ \sup_{0 \leq t \leq T} G(t) \leq u \right\} \geq P \left\{ \sup_{0 \leq t \leq T} G^*(t) \leq u \right\}.
$$

Put $\log x = \ln(e \vee x)$.

**Theorem 3.1.**

\[ \lim_{h \to 0} \sup_{0 < t < h} \sup_{0 < s < h} \frac{|X_m(t + s) - X_m(t)|}{b_{m2}^{1/2}(t \vee h)^{m-1/2} h (2 \log h^{-1})^{1/2}} \leq 1 \quad \text{almost surely,} \]

\[ \liminf_{h \to 0} \sup_{0 < t < h} \frac{|X_m(t + h) - X_m(t)|}{b_{m2}^{1/2}(t \vee h)^{m-1/2} h (2 \log h^{-1})^{1/2}} \geq 1 \quad \text{almost surely.} \]

**Remark 3.1.** It is interesting to find the exact factors such that equality signs in (3.1) and/or (3.2) hold. For Lévy's moduli of continuity of a Brownian motion $W(-)$, the ‘$(\log h^{-1})^{1/2}$’ makes the equality sign in (3.1) hold. For $X_m(-)$, there are certain difficulties because its increments are neither independent nor stationary.

**Proof.** First we prove (3.1). For any given $\varepsilon > 0$, by Proposition 2.1, there exist $c_1 = c_1(\varepsilon) > 0$ and $C_1 = C_1(\varepsilon) > 0$ such that

\[ P \left\{ \sup_{0 < t < h} \sup_{0 < s < h} \frac{|X_m(t + s) - X_m(t)|}{b_{m2}^{1/2}(t \vee h)^{m-1/2} h (2 \log h^{-1})^{1/2}} \geq (1 + \varepsilon)^2 \right\} \leq C_1 \left( \exp \left\{-2c_1(1 + \varepsilon)^2 \log h^{-1}\right\} + h^{-1} \exp \left\{- (1 + \varepsilon)^2 \log h^{-1}\right\} \right) \leq C_1(h^{2c_1} + h^{2\varepsilon}). \]

Taking $h_n = n^{-A}$ with $A > (2(\varepsilon \wedge c_1))^{-1}$, we obtain

\[ \sum_{n=1}^{\infty} P \left\{ \sup_{0 < t < h_n} \sup_{0 < s < h_n} \frac{|X_m(t + s) - X_m(t)|}{b_{m2}^{1/2}(t \vee h_n)^{m-1/2} h_n (2 \log h_n^{-1})^{1/2}} \geq (1 + \varepsilon)^2 \right\} < \infty, \]
which, in combination with the Borel-Cantelli lemma, implies

$$\limsup_{n \to \infty} \sup_{0 \leq t \leq \frac{1}{h_n}} \sup_{0 \leq s \leq h_n} \frac{|X_m(t + s) - X_m(t)|}{b_m^{1/2}(t \land h_n)^{m-1/2}h_n^{(2 \log h_n^{-1})^{1/2}}} \leq (1 + \varepsilon)^2 \quad \text{a.s.}$$

The procedure from (3.3) to (3.1) is routine, and hence, is omitted.

Next we show (3.2). Let $h_n = n^{-A_n}$ with $A_n = n^{(\log \log n)^{-1}} \uparrow \infty$ as $n \to \infty$. Define

$$Y(i) = \frac{X_m((i + 1)h_n) - X_m(ih_n)}{(ih_n)^{m-1/2}}, \quad 0 < i \leq n^{A_n} - 1.$$ 

By (1.4), $E(Y(i)^2) \geq b_m h_n^2$. We have that, for $i \leq j$,

$$E(Y(i)Y(j))$$

$$= \frac{1}{(m!)^2(ih_n)^{m-1/2}(jh_n)^{m-1/2}} \left\{ \int_0^{(i+1)h_n} (i+1)n - s)^m [(j+1)n - s)^m \right\}

= h_n^2 \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p} \binom{m}{q} \frac{1}{(m!)^2(2m - p - q + 1)(ij)^{m-1/2}}

\times \left\{ (i+1)^{2m-q+1}(j+1)^q - (i+1)^{2m-q+1}j^q - i^{2m-q+1}(j+1)^q + i^{2m-q+1}j^q \right\}

= h_n^2 \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p} \binom{m}{q} \frac{4(2m - q + 1)q(i/j)^{m-q+1/2}}{(m!)^2(2m - p - q + 1)(ij)^{m-1/2}}(1 + O(1/i)).$$

Let $n_1 = [A_n \log n]$, $Z(i) = Y(e^i)$, $i = 0, 1, \ldots, n_1$,

$$c_m = \sum_{p=0}^m \sum_{q=0}^m \binom{m}{p} \binom{m}{q} \frac{4(2m - q + 1)q}{(m!)^2(2m - p - q + 1)},$$

and $D_n = 3 \log \log n$. (3.4) implies that for $i \geq n_1 / 3$ and $j - i \geq D_n$,

$$E(Z(i)Z(j)) \leq h_n^2 c_m e^{-(j-i)/2}(1 + O(1/i)) \leq c_m (\log n)^{-1} h_n^2,$$

provided $n$ is large enough. Let $(\xi_i, i \geq 0)$ and $\zeta$ be independent normal random variables with means zero and $E\xi_i^2 = E\zeta^2 = c_m (\log n)^{-1} h_n^2$. Let $Z(i) = \zeta + \xi_i$ as $n \to \infty$ (recalling (1.4)).
Path properties of the primitives of a Brownian motion

\[EY_i^2 = EZ(i)^2 \text{ and } EZ(i)Z(j) \leq EY_i Y_j.\]

Let \(I = \{i : n_1/3 \leq i \leq n_1 - 1, i \mod D_n\}\), then \(\#(I) \geq n_1/(2D_n)\) for large \(n\). Hence by Slepian’s lemma and using the well-known inequality

\[1 - \Phi(x) \geq \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-x^2/2},\]

we obtain that for large \(n\)

\[(3.6) \quad P \left\{ \max_{n_1/3 \leq i \leq n_1 - 1, \mod D_n} Z(i) \leq (1 - \varepsilon)b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2} \right\}
\leq P \left\{ \max_{n_1/3 \leq i \leq n_1 - 1, \mod D_n} \gamma_i \leq (1 - \varepsilon)b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2} \right\}
\leq P \left\{ \max_{n_1/3 \leq i \leq n_1 - 1, \mod D_n} \xi_i \leq \left(1 - \frac{\varepsilon}{2}\right)b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2} \right\}
+ P \left\{ \xi \geq \frac{\varepsilon}{2}b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2} \right\}
\leq \left(1 - P \left\{ \xi_i > \left(1 - \frac{\varepsilon}{2}\right)b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2} \right\}\right)^{n_1/(2D_n)}
+ \exp \left\{ -\frac{\varepsilon^2 b_{m_2}}{4c_m (\log n)^{-1}} \log \log h_n^{-1} \right\}
\leq \left(1 - \frac{1}{(8\pi \log \log h_n^{-1})^{1/2}} \exp \left\{ -\left(1 - \frac{\varepsilon}{2}\right) \log \log h_n^{-1} \right\}\right)^{n_1/(2D_n)} + n^{-2}
= \left(1 - \frac{(\log h_n^{-1})^{(1-\varepsilon/2)}}{(8\pi \log \log h_n^{-1})^{1/2}}\right)^{n_1/(2D_n)} + n^{-2}
\leq \exp \left\{ -\frac{(\log h_n^{-1})^{(1-\varepsilon/2)n_1}}{2D_n(8\pi \log \log h_n^{-1})^{1/2}} \right\} + n^{-2} \leq 2n^{-2}.

Inequality (3.6) implies

\[\sum_{n=1}^{\infty} P \left\{ \max_{0 \leq i \leq n_1 - 1} Z(i) \leq (1 - \varepsilon)b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2} \right\} < \infty,
\]

and by the Borel-Cantelli lemma it follows that

\[(3.7) \quad \liminf_{n \to \infty} \max_{0 \leq i \leq n_1 - 1} \frac{Z(i)}{b_{m_2}^{1/2}h_n(2 \log \log h_n^{-1})^{1/2}} \geq 1 - \varepsilon \quad \text{a.s.}\]
And hence we conclude

\[
(3.8) \quad \liminf_{n \to \infty} \sup_{h_n \leq t \leq 1-h_n} \frac{X_m(t + h_n) - X_m(t)}{b_{m_2}^{1/2} t^{m-1/2} h_n (2 \log \log h_n^{-1})^{1/2}} \geq 1 - \varepsilon \quad \text{a.s.}
\]

Considering \( h_{n+1} < h \leq h_n \), we have

\[
(3.9) \quad \sup_{0 < t \leq 1-h} \frac{|X_m(t + h) - X_m(t)|}{b_{m_2}^{1/2} (t \vee h)^{m-1/2} h (2 \log \log h^{-1})^{1/2}} \geq \sup_{h_n \leq t \leq 1-h} \frac{|X_m(t + h_n) - X_m(t)|}{b_{m_2}^{1/2} t^{m-1/2} h_n (2 \log \log h_n^{-1})^{1/2}} \\
- 2 \sup_{h_n < t \leq 1 - (h_n - h_{n+1})} \sup_{0 \leq s \leq h_n - h_{n+1}} \frac{|X_m(t + h_{n+1} + s) - X_m(t + h_{n+1})|}{b_{m_2}^{1/2} (t + h_{n+1})^{m-1/2} (h_n - h_{n+1})} \times \frac{(t + h_{n+1})^{m-1/2} (h_n - h_{n+1}) (\log(h_n - h_{n+1})^{-1})^{1/2}}{(2 \log(h_n - h_{n+1})^{-1})^{1/2} t^{m-1/2} h_{n+1} (\log h_{n+1}^{-1})^{1/2}}.
\]

By the derivative calculus for the function \( f(x) = x^{-A} \), we have

\[
h_n - h_{n+1} = h_{n+1} A_n \frac{n \log n}{\log \log n} (1 + o(1)).
\]

Therefore,

\[
\lim_{n \to \infty} \sup_{h_n < t \leq 1 - (h_n - h_{n+1})} \frac{(t + h_{n+1})^{m-1/2} (h_n - h_{n+1}) (\log(h_n - h_{n+1})^{-1})^{1/2}}{t^{m-1/2} h_{n+1} (\log h_{n+1}^{-1})^{1/2}} = 0.
\]

Consequently we conclude (3.2) by (3.8), (3.9) and (3.1). This completes the proof of Theorem 3.1.

\[\square\]

4. Large increments

**THEOREM 4.1.** Let \( a_T \) be a continuous function of \( T \) with \( 0 < a_T \leq T \) and suppose that

\[
(4.1) \quad \lim_{n \to \infty} \sup_{n-1 < t \leq n} \frac{a_t}{\inf_{n-1 < t \leq n} a_t} = 1
\]

and

\[
(4.2) \quad \lim_{T \to \infty} \log(T/a_T)/\log \log T = \infty.
\]

Then

\[
(4.3) \quad \limsup_{T \to \infty} \sup_{0 < t \leq T - a_T} \sup_{0 < s \leq a_T} \frac{|X_m(t + s) - X_m(t)|}{b_{m_2}^{1/2} (t \vee a_T)^{m-1/2} a_T (2 \log(T/a_T))^{1/2}} \leq 1 \quad \text{a.s.}
\]
If, instead of (4.2), for any \( \varepsilon > 0 \) there exists \( T_0 > 0 \) such that for \( T > T_0 \)

\[
\left( \log \frac{T}{a_T} \right)^{(\log \log \log T)^{1/\varepsilon}} \geq \log T,
\]

(4.4)

\[
(\log a_T)^{2(1-\varepsilon) \log \log a_T} \geq \log T,
\]

(4.5)

then

\[
\liminf_{T \to \infty} \sup_{0 < t \leq T - a_T} \frac{|X_m(t + a_T) - X_m(t)|}{b_{m_2}^{1/2}(t \vee a_T)^{m-1/2} a_T (2 \log \log(T/a_T))^{1/2}} \geq 1 \quad \text{a.s.}
\]

(4.6)

**PROOF.** First we prove (4.3). Let \( \theta > 1 \) and for integers \( k \) and \( j \) let

\[
A_{kj} = \{ T : \theta^{k-1} < T \leq \theta^{k}, \ \theta^{j-1} < a_T \leq \theta^{j} \}.
\]

(4.7)

In the sequel, we always consider \( k \) and \( j \) such that \( A_{kj} \) is non-empty. For any \( A > 0 \), by condition (4.2), there exists \( k_0 \) such that for \( k \geq k_0 \)

\[
\log \theta^{k-j} / \log \log \theta^{k} \geq A,
\]

that is,

\[
j \leq k - [(A / \log \theta) \log k + \theta_1] =: k_1,
\]

(4.8)

where \( \theta_1 = A(\log \log \theta) / \log \theta \). Then, noting that \( b_{m_2}^{1/2}(t \vee a_T)^{m-1/2} a_T (2 \log \log(T/a_T))^{1/2} \) is an increasing function of both \( T \) and \( a_T \), we have

\[
\limsup_{T \to \infty} \sup_{0 < t \leq T - a_T} \frac{|X_m(t + s) - X_m(t)|}{b_{m_2}^{1/2}(t \vee a_T)^{m-1/2} a_T (2 \log \log(T/a_T))^{1/2}}
\]

\[
\leq \limsup_{k \to \infty} \sup_{-\infty < j \leq k_1} \sup_{0 < t \leq \theta^{k-j} - \theta_1} \sup_{0 \leq \theta^{j-1} \leq \theta} \frac{b_{m_2}^{1/2}(t \vee \theta)^{m-1/2} \theta^{j-1} (2 \log \theta^{k-j})^{1/2}}{\theta^{m+1/2} |X_m(t + s) - X_m(t)|}
\]

\[
\leq \limsup_{k \to \infty} \sup_{-\infty < j \leq k_1} \sup_{0 \leq \theta^{j-1} \leq \theta} \frac{b_{m_2}^{1/2}(t \vee \theta)^{m-1/2} \theta^{j-1} (2 \log \theta^{k-j})^{1/2}}{b_{m_2}^{1/2}(t \vee \theta)^{m-1/2} \theta^{j} (2 \log \theta^{k-j})^{1/2}}.
\]

(4.9)

Using Proposition 2.2 and (4.8) we have

\[
P \left\{ \sup_{-\infty < j \leq k_1} \sup_{0 \leq \theta^{j-1} \leq \theta} \sup_{0 \leq \theta^{j} \leq \theta} \frac{|X_m(t + s) - X_m(t)|}{b_{m_2}^{1/2}(t \vee \theta)^{m-1/2} \theta^{j} (2 \log \theta^{k-j})^{1/2}} \geq (1 + \varepsilon)^2 \right\}
\]

\[
\leq C_2 \sum_{j = -\infty}^{k_1} \left( \exp \left\{ -2c_2(1 + \varepsilon)^2 \log \theta^{k-j} \right\} + \theta^{k-j+1} \exp \left\{ -(1 + \varepsilon)^2 \log \theta^{k-j} \right\} \right)
\]

\[
\leq C_2 \sum_{j = -\infty}^{k_1} \left( \theta^{-2c_2(1+\varepsilon)^2} \theta^{k-j} + \theta^{-2c_2(1+\varepsilon)^2} \theta^{k-j} \theta^{1} \right)
\]

\[
\leq c \left( \theta^{-2c_2(1+\varepsilon)^2} (A / \log \theta) (k + \theta_1) + \theta^{-2c_2(1+\varepsilon)^2} (k + \theta_1) + \theta^{-2c_2(1+\varepsilon)^2} \theta^{k-j+1} \right) \leq c k^{-2}
\]
for some \( c > 0 \) by taking \( A = (\log \theta)/(c_2(1+\varepsilon)^2 \wedge \varepsilon) \). Hence, from the Borel-Cantelli lemma we obtain
\[
\limsup_{k \to \infty} \sup_{-\infty < s \leq k_0} \sup_{0 \leq \theta \leq \theta_0} \frac{|X_m(t + s) - X_m(t)|}{b_{m_2}^{1/2}(t \vee \theta^j)^{m-1/2} \theta_j (2 \log \theta^{k-j})^{1/2}} \leq (1 + \varepsilon)^2 \text{ a.s.}
\]
which, in combination with (4.9), implies (4.3) by arbitrariness of \( \theta > 1 \).

Next we show (4.6). Let \( A_j = j^{(\log \log j)^{-1}} \) again, and let \( B_0 = 0 \), \( B_j = j^{A_j} \), \( j = 1, 2, \ldots \), \( C_k = \{ T : B_{k-1} < T \leq B_k, B_{j-1} < a_T \leq B_j \} \). By condition (4.4), for any \( A > 0 \), there exists an integer \( j_0 \) such that for \( j \geq j_0 \)
\[
(4.10) \quad \log(B_k/B_j) \geq (\log B_k)^{(\log \log k)^{-A}} \geq A_k^{(\log \log k)^{-A}}.
\]
On the other hand, by the derivative calculus for the function \( g(x) = \log B_x \), we have
\[
\log B_k - \log B_j \leq 2(k - j) \frac{A_k \log k}{k \log \log k},
\]
which, in combination with (4.10), implies that
\[
j \leq k - \left[ \frac{k \log \log k}{2 \log k} A_k^{1+ (\log \log k)^{-A}} \right] := k_2.
\]
Noting that \( b_{m_2}^{1/2}(t \vee a_T)^{m-1/2} a_T (2 \log (T/a_T))^{1/2} \) is an increasing function of both \( T \) and \( a_T \) we can write
\[
(4.11) \quad \liminf_{T \to \infty} \sup_{0 < t \leq T-a_T} \frac{|X_m(t + a_T) - X_m(t)|}{b_{m_2}^{1/2}(t \vee a_T)^{m-1/2} a_T (2 \log (T/a_T))^{1/2}}
\]
\[
\geq \liminf_{k \to \infty} \inf_{1 \leq k_2} \inf_{T \in C_{k_2}} \sup_{0 < t \leq T-a_T} \frac{|X_m(t + a_T) - X_m(t)|}{b_{m_2}^{1/2}(t \vee a_T)^{m-1/2} a_T (2 \log (T/a_T))^{1/2}}
\]
\[
\geq \liminf_{k \to \infty} \inf_{1 \leq k_2} \sup_{0 < t \leq B_k - 1/2} \frac{|X_m(t + B_j) - X_m(t)|}{b_{m_2}^{1/2}(t \vee B_j)^{m-1/2} B_j (2 \log (B_k/B_j))^{1/2}} \times \frac{\log \theta^{k-j}}{(2 \log (B_k/(B_j - B_{j-1})))^{1/2} (t \vee B_j)^{m-1/2}}
\]
\[
=: J_1 - J_2.
\]
By the derivative calculus for the function \( h(x) = B_x \), we have
\[
\frac{B_j - B_{j-1}}{B_j} \leq \frac{2A_j \log j}{j \log \log j}.
\]
The last inequality and condition (4.5) imply that, as $k \to \infty$,
\[
\log B_k \leq (1 + o(1)) \log B_{k-1} \leq 2(\log B_j)^2(1-\varepsilon)\log\log B_j \leq 2(A_j \log j)^2(1-\varepsilon)\log\log j.
\]
Hence
\[
(t \vee (B_j - B_{j-1}))^{m-1/2}(B_j - B_{j-1})(\log(B_k/(B_j - B_{j-1})))^{1/2}
\]
\[
\leq \frac{B_j - B_{j-1}}{B_j} (\log B_k)^{1/2} \leq \frac{2\sqrt{2}A_j \log j}{j \log\log j} \cdot (A_j \log j)^{1-\varepsilon}\log\log j
\]
\[
= \frac{2\sqrt{2}A_j (\log j)^{1+1-\varepsilon}\log\log j}{j \varepsilon \log\log j} \to 0 \quad \text{as } j \to \infty.
\]

Then by (4.3) and (4.12) we obtain
\[
J_2 = 0 \quad \text{a.s.}
\]
Consider $J_1$ and for fixed $k$, define
\[
Y_j(i) = \frac{X_m((i + 1)B_j) - X_m(iB_j)}{(iB_j)^{m-1/2}}, \quad 0 < i \leq B_k/B_j - 1, \ j = 0, 1, \ldots, k_2.
\]
Furthermore, let $Z_j(i) = Y_j(e^i), \ i = 0, 1, \ldots, k_3 - 1$ with $k_3 = [\log(B_k/B_j)]$. Similarly to (3.5), we have
\[
EZ_j(i_1)Z_j(i_2) \leq c'_m(\log\log k)^{-A-2}B_j^2
\]
for some $c'_m > 0$ and any $i_1 \geq k_3/3, \ i_2 - i_1 \geq D_k' := 3(A + 2)\log\log k$. Let $\{\xi_{ij}, \ i \geq 0\}$ and $\zeta_j$ be independent normal random variables with means zero and $E\xi_{ij}^2 = EZ_j(i)^2 - c'_m(\log\log k)^{-A-2}B_j^2, \ E\zeta_j^2 = c'\log\log k)^{-A-2}B_j^2$. Then, similarly to (3.6), using (4.10) with $A > 6/\varepsilon$ we obtain for all large $k$
\[
P\left\{ \inf_{0 \leq j \leq k_2} \max_{\substack{i \leq k_3/3 \leq i \leq k_3 - 1 \mod D_k'} Z_j(i) \leq (1 - \varepsilon)b_{m_2}^{1/2}B_j(2\log\log(B_k/B_j))^{1/2} \right\}
\]
\[
\leq \sum_{j=0}^{k_2} \left( \exp \left\{ -\frac{(\log(B_k/B_j))^{1-\varepsilon/2}k_3}{2D_k'(8\pi \log\log(B_k/B_j))^{1/2}} \right\} \right)
\]
\[
+ \exp \left\{ -\frac{\varepsilon^2b_{m_2}}{4c_m' \log\log k)^{-A-2} \log\log(B_k/B_j)} \right\}
\]
\[
\leq c \sum_{j=0}^{k_1} \left( \exp \left\{ -\frac{(\log(B_k/B_j))^{\varepsilon/2}}{D_k'(8\pi \log\log(B_k/B_j))^{1/2}} \right\} \right)
\]
It is easy to see that
\[ D'_k = o(\log(B_k/B_j)), \quad \log \log(B_k/B_j) = o(\log(B_k/B_j)). \]
So for large \( k \),
\[ \exp \left\{ -\frac{(\log(B_k/B_j))^{1/2}}{D'_k(8\pi \log \log(B_k/B_j))^{1/2}} \right\} \leq \exp\{-(\log(B_k/B_j))^{1/3}\}. \]
Combining it with (4.14) implies
\[ \sum_{k=1}^{\infty} P \left\{ \inf_{0 \leq j \leq k_2} \max_{0 \leq i \leq k_1-1} Z_j(i) \leq (1-\varepsilon)b_{m2}^{1/2}B_j(2\log \log(B_k/B_j))^{1/2} \right\} < \infty. \]
Hence
\[ J_1 \geq 1 - \varepsilon \quad \text{a.s.} \]
Combining (4.15) with (4.13) we conclude that (4.4) holds. This completes the proof of Theorem 4.1.

Acknowledgements

The author would like to thank the referee for valuable suggestions. The project was supported by NSFC (19571021) and NSFZP (199016).

References


Department of Mathematics  
Zhejiang University, Xixi Campus  
Hangzhou  
Zhejiang 310028  
P. R. China  
e-mail: zlin@mail.hz.zj.cn