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Tilings of the hyperbolic space and Lipschitz functions

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Abstract. We use a special tiling for the hyperbolic d-space $\mathbb{H}^d$ for $d = 2, 3, 4$ to construct an (almost) explicit isomorphism between the Lipschitz-free space $\mathcal{F}(\mathbb{H}^d)$ and $\mathcal{F}(P) \oplus \mathcal{F}(\mathcal{N})$, where P is a polytope in $\mathbb{R}^d$ and $\mathcal{N}$ a net in $\mathbb{H}^d$ coming from the tiling. This implies that the spaces $\mathcal{F}(\mathbb{H}^d)$ and $\mathcal{F}(\mathbb{R}^d) \oplus \mathcal{F}(\mathcal{M})$ are isomorphic for every net $\mathcal{M}$ in $\mathbb{H}^d$. In particular, we obtain that, for $d = 2, 3, 4$, $\mathcal{F}(\mathbb{H}^d)$ has a Schauder basis. Moreover, using a similar method, we also give an explicit isomorphism between $\mathrm{Lip}(\mathbb{H}^d)$ and $\mathrm{Lip}(\mathbb{R}^d)$.

1 Introduction

Given a metric space M with a distinguished point $0_M \in M$, the Lipschitz-free space $\mathcal{F}(M)$, together with an isometric mapping $\delta\colon M \to \mathcal{F}(M)$, is the uniquely determined (up to linear isometry) Banach space with the following universal property: for every Lipschitz mapping $f\colon M \to X$ to a Banach space X with $f(0_M) = 0$, there is a unique bounded linear operator $F\colon \mathcal{F}(M) \to X$ with $\|F\| = \mathrm{Lip}(f)$ such that the diagram

$$\begin{tikzcd} \mathcal{F}(M) \arrow[r, "F"] & X \\ M \arrow[u, "\delta"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. See, for instance, [26] for an approach to Lipschitz-free spaces via the universal property. The dual space of $\mathcal{F}(M)$ is the space $\mathrm{Lip}_0(M)$ of Lipschitz functions $f\colon M \to \mathbb{R}$ with $f(0_M) = 0$ equipped with the Lipschitz constant as norm, (i.e., $\|f\| := \mathrm{Lip}(f)$). Note that the condition $f(0_M) = 0$ eliminates the constant functions and hence ensures that the Lipschitz constant is indeed a norm.

The name Lipschitz-free spaces was introduced by G. Godefroy and N.J. Kalton in [31] where, among others, these spaces are used to construct canonical examples of nonseparable Banach spaces which are bi-Lipschitz equivalent but not linearly isomorphic. Such spaces have been studied by several authors in different contexts and with different terminology, and we refer to [47] and [40, Section 1.6] for some terminological and historical remarks. The appearance of [31] resulted in a new impetus to their study – in particular, in connection with nonlinear functional analysis, metric

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geometry, and theoretical computer science; let us refer to [1, 2, 4, 5, 23, 25, 43, 46] for a nonexhaustive list of some recent results. Let us also refer to [29] for a recent nice survey on some aspects of the theory of Lipschitz-free spaces. A detailed exposition of the spaces $\mathrm{Lip}_0(M)$ and $\mathcal{F}(M)$ can also be found in N. Weaver's book [47]. There, in addition to the Banach space $\mathrm{Lip}_0(M)$, also the Banach space $\mathrm{Lip}(M)$ of bounded real-valued Lipschitz functions is introduced and studied in detail. This space has the additional important property to be a Banach algebra.

In metric geometry, up to dimension and a scaling of the metric, there only exist three model spaces (see, for example, [14, Chapter I.2]): the sphere $\mathbb{S}^d$ with the intrinsic (geodesic) metric, the Euclidean space $\mathbb{R}^d$, and the hyperbolic space $\mathbb{H}^d$. The structure of $\mathcal{F}(\mathbb{R}^d)$ is well studied, both from the isometric [21] and from the isomorphic point of view [3, 13, 20, 33, 41]. Quite recently, it was proved that the spaces $\mathcal{F}(\mathbb{S}^d)$ and $\mathcal{F}(\mathbb{R}^d)$ are isomorphic, [1, Theorem 4.21] (see also [25] for a more general result). However, much less seems to be known about the structure of the space $\mathcal{F}(\mathbb{H}^d)$ and its dual space $\mathrm{Lip}_0(\mathbb{H}^d)$; some results concerning $\mathcal{F}(\mathbb{H}^d)$ can be found in [23] where, among others, the authors pose the questions of whether $\mathcal{F}(\mathbb{H}^d)$ has a Schauder basis and whether it is isomorphic to $\mathcal{F}(\mathbb{R}^d)$. While our work was under review, the preprint [27] by C. Gartland actually gave a positive answer to the latter question.

The aim of this paper is to explain how the local structure of $\mathbb{H}^d$ for $d = 2, 3, 4$ together with a macroscopic view of $\mathbb{H}^d$ determine the Banach space structure of $\mathcal{F}(\mathbb{H}^d)$ and $\mathrm{Lip}_0(\mathbb{H}^d)$. Since the space $\mathrm{Lip}_0(\mathbb{H}^d)$ is more tangible than $\mathcal{F}(\mathbb{H}^d)$, it will be more convenient for us to work with the space $\mathrm{Lip}_0(\mathbb{H}^d)$ and then transfer the results to the predual. More precisely, we construct an (almost) explicit isomorphism

$$\Phi\colon \mathrm{Lip}_0(\mathbb{H}^d) \simeq \mathrm{Lip}_0(P) \oplus \mathrm{Lip}_0(\mathcal{N}),$$

where P is a polytope (with nonempty interior) in $\mathbb{R}^d$ and $\mathcal{N}$ is a suitable net in $\mathbb{H}^d$. In particular, we build an explicit isomorphism between $\mathrm{Lip}_0(\mathbb{H}^d)$ and $Z \oplus \mathrm{Lip}_0(\mathcal{N})$, where Z is a direct sum of certain subspaces of $\mathrm{Lip}_0(P)$. This gives us an explicit procedure to reduce the study of $\mathrm{Lip}_0(\mathbb{H}^d)$ to the discrete case of $\mathrm{Lip}_0(\mathcal{N})$ and to a space of Lipschitz functions on $\mathbb{R}^d$. Since our main focus is on the hyperbolic case, we allow ourselves to use nonexplicit arguments, such as Lee–Naor extension results [39] and Pełczyński decomposition method, in the proof that Z is isomorphic to $\mathrm{Lip}_0(P)$ (in Section 4). Let us, however, point out that this part of the argument could also be made entirely explicit by using a variant of the arguments from Section 5.

Since Φ is weak*-to-weak* continuous, it is the adjoint of an isomorphism

$$\mathcal{F}(\mathbb{H}^d) \simeq \mathcal{F}(P) \oplus \mathcal{F}(\mathcal{N}).$$

Moreover, by the results of [21], $\mathcal{F}(P)$ is isomorphic to $\mathcal{F}(\mathbb{R}^d)$, which yields

$$\mathcal{F}(\mathbb{H}^d) \simeq \mathcal{F}(\mathbb{R}^d) \oplus \mathcal{F}(\mathcal{N}) \qquad \text{and} \qquad \mathrm{Lip}_0(\mathbb{H}^d) \simeq \mathrm{Lip}_0(\mathbb{R}^d) \oplus \mathrm{Lip}_0(\mathcal{N})$$

for $d = 2, 3, 4$. Since both the space $\mathcal{F}(\mathbb{R}^d)$ and $\mathcal{F}(\mathcal{N})$ have a Schauder basis, the former result being due to P. Hájek and E. Pernecká in [33] and the latter to M. Doucha and P. Kaufmann in [23], we conclude that, for $d = 2, 3, 4$, $\mathcal{F}(\mathbb{H}^d)$ also admits a Schauder basis, thereby giving a partial positive answer to the first question mentioned above. Let us recall that if one is aiming for weaker structural properties, such as the bounded approximation property, or the (π)-property, then more general results are available; see, for instance, [30, 37] and the references therein.

Using the same methods, we also show that the space $\mathrm{Lip}(\mathbb{H}^d)$ is isomorphic to $\mathrm{Lip}(\mathbb{R}^d)$, this time by giving an entirely explicit isomorphism (Remark 5.11). Combining this result with standard arguments, in Remark 5.12, we conclude that

$$\mathrm{Lip}(\mathbb{H}^d) \simeq \mathrm{Lip}(\mathbb{R}^d) \simeq \mathrm{Lip}(\mathbb{S}^d)$$

(i.e., the spaces of bounded Lipschitz functions on the model spaces of metric geometry are all isomorphic).

At the core of our argument, we have to decompose a Lipschitz function on $\mathbb{H}^d$ into a Lipschitz function on a net $\mathcal{N}$ and a sequence of Lipschitz functions on a convex subset of $\mathbb{H}^d$. In order to do this, we consider a suitable tiling of $\mathbb{H}^d$ by polytopes; the existence of such tilings depends upon classical results from the theory of reflection groups; see, for example, [22, Chapter 6] or [45, Chapter 5]. More precisely, given a right-angled polytope P (namely, such that all dihedral angles are exactly $\pi/2$), by reflecting across the faces of P, we obtain a tessellation of $\mathbb{H}^d$ by isometric copies of P. In [44] (see also [42, Section 2]), the author shows that such right-angled polytopes exist only if $d \leqslant 4$; explicit constructions in dimensions $d = 2, 3, 4$ were already known to exist (see Section 3.1). This justifies why we are able to prove our results only in dimension $d = 2, 3, 4$. In order to emphasize the subtlety of hyperbolic tilings results, let us mention two more results: in dimension $d \geqslant 6$, there exists no regular tiling of $\mathbb{H}^d$ [19, p. 206], and, more generally, for $d \geqslant 30$, there are no hyperbolic reflection groups at all (see, for example, [22, Theorem 6.11.8]).

Given a tiling of $\mathbb{H}^d$ by right-angled polytopes, we first use an extension operator from the net, given by choosing a distinguished point inside the polytope, to decompose a Lip_0-function on $\mathbb{H}^d$ into a function on the net and a *bounded* Lipschitz function on $\mathbb{H}^d$. Then, using an extension operator for Lipschitz functions on P, we decompose the bounded Lipschitz function in a bounded sequence of Lipschitz functions on P. The latter construction is inspired by a decomposition method for $(C^\infty$-)smooth functions on $\mathbb{R}^d$ into sequences of functions on the unit cube in [8] and [9].

Let us close this section with a brief description of the structure of the paper. In Section 2, we recall basic notions on Lipschitz-free spaces and metric geometry. A self-contained revision of hyperbolic geometry is the content of Section 3; in particular, we explain the properties of the tilings that we need in Section 3.1. As we mentioned already, Section 4 is dedicated to the local problem, and we study the space $\mathrm{Lip}_0(P)$, for a polytope P in $\mathbb{R}^d$. Finally, the core of our paper with the proof of the main results is Section 5.

2 Preliminary material

Given a pointed metric space (M, d) with distinguished point $0_M \in M$, we consider the Banach space $\mathrm{Lip}_0(M)$ of all Lipschitz functions $f\colon M \to \mathbb{R}$ such that $f(0_M) = 0$, endowed with the norm

$$\|f\|_{\mathrm{Lip}_0} \coloneqq \mathrm{Lip}(f) \coloneqq \sup\left\{\left|\frac{f(x)-f(y)}{d(x,y)}\right| : x \neq y \in M\right\}.$$

Moreover, when (M, d) is a metric space, we consider the vector space of all bounded Lipschitz functions $f: M \to \mathbb{R}$ that, following [47, Chapter 2], we denote by $\mathrm{Lip}(M)$. $\mathrm{Lip}(M)$ becomes a Banach space when equipped with the norm

$$\|f\|_{\mathrm{Lip}} := \|f\|_{\mathrm{Lip}_0} + \|f\|_\infty.$$

The pointwise multiplication induces an algebra structure on $\mathrm{Lip}(M)$ due to the basic inequality

$$\mathrm{Lip}(fg) \leqslant \mathrm{Lip}(f)\|g\|_\infty + \mathrm{Lip}(g)\|f\|_\infty.$$

When M is bounded, the same product also gives an algebra structure on $\mathrm{Lip}_0(M)$ because $\|f\|_\infty \leqslant \mathrm{diam}(M)\|f\|_{\mathrm{Lip}_0}$. Actually, a different product turns every $\mathrm{Lip}_0(M)$ into a Banach algebra, [2].

For $p \in M$, the evaluation functional $\delta_p \in \mathrm{Lip}_0(M)^*$ is defined by $\langle \delta_p, f\rangle := f(p)$. It is easy to see that $\|\delta_p\| = d(p, 0_M)$. Then one can define $\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta_p : p \in M\} \subset \mathrm{Lip}_0(M)^*$ and verify that $\mathcal{F}(M)$ satisfies the universal property stated in the Introduction. In particular, $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$.

As mentioned in the Introduction, our argument will proceed in $\mathrm{Lip}_0(M)$, and only at the very end, we will pass to preduals and deduce results for $\mathcal{F}(M)$. Therefore, we need information on the weak* topology of $\mathrm{Lip}_0(M)$ induced by the predual $\mathcal{F}(M)$. By definition, the set $\{\delta_p : p \in M\}$ of elementary molecules is linearly dense in $\mathcal{F}(M)$. Thus, on bounded sets, the weak* topology coincides with the weak topology induced by the functionals $\{\delta_p : p \in M\}$. In other words, it agrees with the topology of pointwise convergence on M. When combined with the Banach–Dieudonné theorem, this fact has important consequences. First, a subspace $X \subset \mathrm{Lip}_0(M)$ is weak* closed if and only if it is pointwise closed. Second, a bounded operator $L: \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(N)$ is weak*-to-weak* continuous if and only if it is pointwise-to-pointwise continuous (see, for example. [24, Exercise 3.65] or [47, Corollary 2.33]). These facts will be freely used multiple times in our arguments. Moreover, if X is a weak* closed subspace of $\mathrm{Lip}_0(M)$, then it is the dual to some quotient Z of $\mathcal{F}(M)$, and the weak* topology of X induced by Z coincides with the restriction to X of the weak* topology of $\mathrm{Lip}_0(M)$; see, for example, [18, Corollary V.2.2].

A ubiquitous role in our proofs will be played by linear extension operators. If N is a subset of M with $0_M \in N$, a *linear extension operator* $E: \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$ is a linear operator such that Ef is an extension of f for every $f \in \mathrm{Lip}_0(N)$. Plainly, the restriction operator $f \mapsto f|_N$ is a left-inverse to E. Thus, E defines an isomorphic embedding of $\mathrm{Lip}_0(N)$ as a complemented subspace of $\mathrm{Lip}_0(M)$. Moreover, if E is pointwise-to-pointwise continuous, both the isomorphic embedding and the projection are weak*-to-weak* continuous. The construction of linear extension operators for Lipschitz functions has been a topic of great interest in recent years; see, for example, [1, 6, 15, 16, 28, 35, 39].

In most cases, our extensions operators will be based on direct and explicit constructions. However, in one place we shall use the following deep extension result due to Lee and Naor [39]: there is a universal constant C such that for every pointed metric space (M, d) and every subspace N with $0_M \in N$ that is λ-doubling, there is a linear extension operator $E: \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$ with $\|E\| \leqslant C \log(\lambda)$. For a

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more recent simpler proof, see [16, Theorem 4.1]; importantly, the operator E is also weak*-to-weak* continuous, as it is explained for example in [37].

Next, we need to mention some basics on metric geometry. Recall that a metric space (M,d) is *geodesic* if for every two points $x,y \in M$, there is an isometry $\gamma\colon [0,d(x,y)] \to M$ such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. In case such an isometry γ is unique, (M,d) is said to be *uniquely geodesic*. Intuitively speaking, geodesics are the metric analogue of segments. Accordingly, the image $\gamma([0,d(x,y)])$ of a geodesic connecting x to y is sometimes called *metric segment* and denoted $[xy]$. Moreover, we say that a subset C of M is *(geodesically) convex* if every metric segment $[xy]$ with $x,y \in C$ is entirely contained in C. Let us refer, for example, to [14] for more on geodesic metric spaces.

An important property of geodesic metric spaces is that the Lipschitz condition becomes a local property, in the sense of the following lemma. For convex subsets of Banach spaces, it is due to D. J. Ives and D. Preiss in [34].

Lemma 2.1 [10, Lemma 2.1] *Let M be a geodesic space, N a metric space, and $\{Z_i\}_{i=1}^{\infty}$ a countable family of sets covering M. Let $f\colon M \to N$ be a continuous mapping whose restrictions to the sets Z_i are Lipschitz and satisfy* $\mathrm{Lip}(f|_{Z_i}) \leqslant L$ *for some $L > 0$. Then f is Lipschitz with* $\mathrm{Lip}(f) \leqslant L$.

3 Hyperbolic geometry

This section is dedicated to a brief introduction to the *hyperbolic d-space* $\mathbb{H}^d$. The shortest way to introduce it is to define $\mathbb{H}^d$ as the unique complete, simply-connected Riemannian d-manifold with constant sectional curvature -1. The uniqueness of a Riemannian d-manifold with such properties is a consequence of the Killing–Hopf theorem; see, for example, [38, Theorem 12.4]. However, for our purposes, it will be more convenient to have an explicit description of a model for $\mathbb{H}^d$; we shall now recall two such models and later use whichever model is more convenient for our purpose.

Let us start by recalling the *hyperboloid model*: consider the non-degenerate bilinear form

$$\langle x,y\rangle \coloneqq \sum_{i=1}^{d} x_i y_i - x_{d+1} y_{d+1}$$

on $\mathbb{R}^{d+1}$ and define

$$\mathbb{H}^d \coloneqq \left\{ x \in \mathbb{R}^{d+1} \colon \langle x,x\rangle = -1,\ x_{d+1} > 0 \right\}.$$

A metric on $\mathbb{H}^d$ can be defined by

$$\rho(x,y) = \operatorname{arcosh}(-\langle x,y\rangle).$$

As it turns out, $\langle\cdot,\cdot\rangle$ is positive definite on the tangent bundle of $\mathbb{H}^d$ and ρ is exactly the Riemannian distance induced by the Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^d$. Geodesic lines are defined as the intersections of $\mathbb{H}^d$ with 2-dimensional subspaces of $\mathbb{R}^{d+1}$. In particular, $\mathbb{H}^d$ is uniquely geodesic, and every geodesic can be uniquely extended to a geodesic line. Angles are also defined in terms of $\langle\cdot,\cdot\rangle$: given two geodesics that meet at a point $\xi \in \mathbb{H}^d$, one takes unit tangent vectors u and v to the geodesics at ξ and defines the angle between the geodesics as the unique $\alpha \in [0, \pi]$ such that $\cos\alpha = \langle u, v \rangle$. Note that, in particular, if the point in question is the origin, then $u_{d+1}v_{d+1} = 0$; thus, $\langle u, v \rangle_{\mathbb{H}^d} = \langle u, v \rangle_{\mathbb{R}^{d+1}}$, and hence, the angle is the Euclidean one. For the same reason, if any geodesic subspace containing the origin meets a geodesic at an orthogonal angle in $\mathbb{R}^{d+1}$, then the (two possible) tangent vectors u of said geodesic will satisfy $u_{d+1} = 0$, and the hyperbolic angle will be orthogonal as well.

An *isometry* of $\mathbb{H}^d$ is a bijection of $\mathbb{H}^d$ that preserves the distance ρ; it then follows that such isometries also preserve angles. Let us recall that the group of isometries acts transitively on $\mathbb{H}^d$; moreover, each isometry of $\mathbb{H}^d$ can be obtained as a composition of at most $d+1$ reflections through hyperplanes. More on isometries of $\mathbb{H}^d$ can be found in [11, Chapter 19], [14, Chapter I.2], or [17, Section 10].

We next briefly describe the *Beltrami–Klein model*. It is represented by points in the Euclidean open unit ball $\mathbb{B}^d \coloneqq \{x \in \mathbb{R}^d : \|x\|_2 < 1\}$. Geodesics are simply intersections of Euclidean lines with $\mathbb{B}^d$. Likewise, hyperplanes are intersections of Euclidean hyperplanes with $\mathbb{B}^d$. The metric for $\mathbb{H}^d_{BK}$ can be defined as follows: given points x and y in $\mathbb{B}^d$, let x_∞ and y_∞ be the intersections of the line through x and y with the boundary of $\mathbb{B}^d$ (arranged so that x_∞, x, y, y_∞ appear in order). Then

$$\rho_{BK}(x, y) \coloneqq \frac{1}{2} \log \frac{\|x - y_\infty\|_2 \|y - x_\infty\|_2}{\|x - x_\infty\|_2 \|y - y_\infty\|_2}.$$

Further information on this and more models can be found in [11, Chapter 19], or [14, Chapter I.6]. Let us notice that, by the Killing–Hopf theorem, all such models are mutually isometric. More importantly, simple and explicit isometries between the models are available, thus allowing explicit transfer of properties; see, for example, [11, Chapter 19], or [38, Theorem 3.7]. Finally, for a gentle and elementary introduction to hyperbolic geometry, mainly in the plane, we refer the interested reader to [7].

The last result we require is a particular case of the well-known fact from Riemannian geometry that differentiable maps between Riemannian manifolds are locally Lipschitz. A direct, computational proof of this particular case can be given by using the explicit formula for the metric.

Lemma 3.1 *Let $\mathbb{H}^d_{BK}$ be the Beltrami-Klein model of $\mathbb{H}^d$ and let $\mathbb{B}^d$ be the Euclidean open unit ball in $\mathbb{R}^d$. Then the identity function* $\mathrm{id}\colon \mathbb{H}^d_{BK} \to \mathbb{B}^d$ *is locally bi-Lipschitz.*

3.1 Tilings of $\mathbb{H}^d$

We call a subset of $\mathbb{R}^d$ or $\mathbb{H}^d$ a *polyhedron* if it is a finite intersection of closed half-spaces. A bounded (or equivalently compact) polyhedron is called a *polytope*. Note that polyhedra are convex sets, as is, for example, obvious in the Beltrami–Klein model. Hence, the nearest point projection – namely, the mapping which assigns to a point x the point in the polyhedron which minimizes the distance to x – is well defined and 1-Lipschitz; see Proposition 2.4 in [14, p. 176]. In particular, polyhedra are 1-Lipschitz retracts of both $\mathbb{R}^d$ and $\mathbb{H}^d$.

In order to investigate the local and global structure of the Lipschitz functions on $\mathbb{H}^d$ separately, we will need a suitable tiling of the hyperbolic space $\mathbb{H}^d$. A sequence

$(P_n)_{n\in\mathbb{N}}$ of polytopes is a *tiling*, or *tessellation*, of $\mathbb{H}^d$ if $\bigcup_{n\in\mathbb{N}} P_n = \mathbb{H}^d$, and the intersection of any two distinct polytopes is either empty or a face of both polytopes (in particular, the interiors of the polytopes P_n are mutually disjoint).

In our arguments, we will use the existence in $\mathbb{H}^d$ for $d = 2, 3, 4$ of a *regular orthogonal tiling* – namely, a tiling $(P_n)_{n\in\mathbb{N}}$ consisting of mutually isometric polytopes and where each P_n satisfies the following definitions (see, for example, [22, Chapter 6] for more details):

(T1) A polytope P in $\mathbb{H}^d$ is *regular* if the group of isometries of P is flag-transitive. More precisely, a *flag* in P is a chain $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$, where F_0 is a vertex of P and each F_k is a k-dimensional face of P. Then P is regular if for every two flags $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ and $F'_0 \subset F'_1 \subset \cdots \subset F'_{d-1}$, there is an isometry of P that maps one flag onto the other.

(T2) A polytope is *right-angled*, or *orthogonal*, if all dihedral angles are exactly $\pi/2$.

Before passing to the explanation of the existence of the tiling, let us mention two more properties that follow from (T1) and (T2) (two further properties will be proved in Lemma 3.2 and Lemma 3.3 below).

(T3) Every polytope P_n has an inscribed circle, whose center we denote by p_n and whose radius is by definition the *in-radius* of P_n.

Indeed, every regular polytope P admits a center, whose existence can be shown as follows. If F is a maximal face of P, then $x \mapsto \operatorname{dist}(x, F)$ is a convex function (as follows easily from Proposition 2.2 in [14, p. 176]). Thus, letting $\{F_1, \ldots, F_k\}$ be the maximal faces of P, the map $x \mapsto \operatorname{dist}(x, F_1)^2 + \cdots + \operatorname{dist}(x, F_k)^2$ is strictly convex and hence has a unique minimum p. Since the above map is defined only by metric properties, every isometry of P must fix p. Finally, by regularity, p has the same distance to all maximal faces. For further details, we refer to [14, pp. 178–179].

(T4) There is a number $N(d)$ such that every P_n intersects at most $N(d)$ polytopes from the tiling.

Indeed, at each vertex of P_n, there are exactly 2^d polytopes intersecting P_n, by the right-angle property. So a (rough) upper bound for $N(d)$ is 2^d times the number of vertices of P_n (such a number of vertices is independent of n, as the polytopes are mutually isometric).

We now pass to explaining the existence of regular orthogonal tilings $(P_n)_{n\in\mathbb{N}}$ for $d = 2, 3, 4$. By Proposition 6.3.2 and 6.3.9 in [22], every polytope P whose dihedral angles are of the form π/m for some integer $m \geqslant 2$ (these polytopes are sometimes called *Coxeter polytopes*) is simple. Hence, by Theorem 6.4.3 in [22], P is a strict fundamental domain of the reflection group generated by reflections across the faces of P. In other words, P intersects every orbit in exactly one point. This implies in particular that the images of P by the elements of the reflection group tessellate $\mathbb{H}^d$. Thus, every element of the reflection group maps the faces of a polytope P_n to the faces of some other polytope of the tiling.

If we restrict our attention to right-angled polytopes, by the above observations, the existence of a right-angled polytope P directly implies that there is a tessellation of $\mathbb{H}^d$ by isometric copies of P. Several explicit constructions of such polytopes for dimensions $d = 2, 3, 4$ are available in the literature, and we mention a few of them

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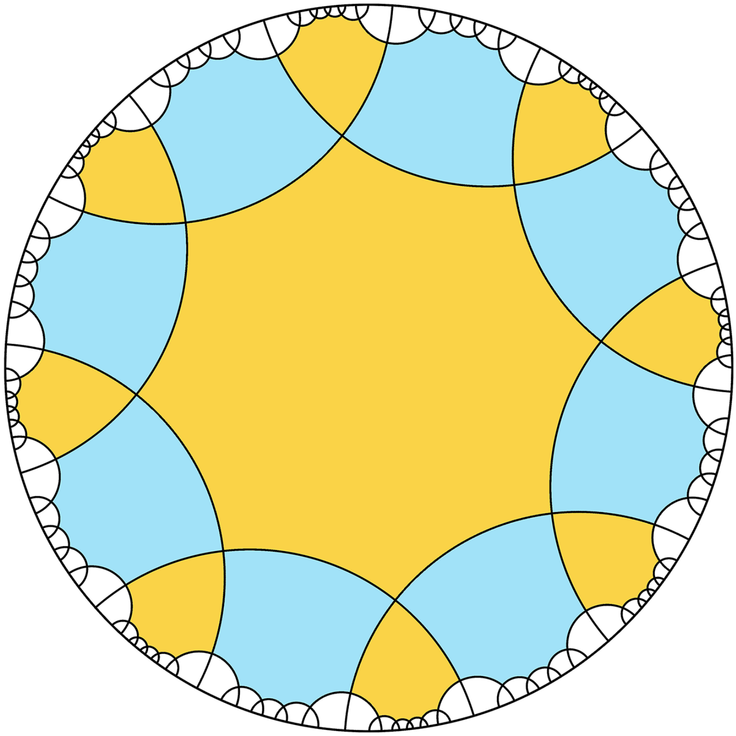

Figure 1: The first seventeen isometric octagons tiling the hyperbolic plane.

in the next paragraph. On the other hand, E. B. Vinberg proved in [44] that there is no right-angled polytope in $\mathbb{H}^d$ when $d \geqslant 5$; a short proof of this fact can also be found in [42, Section 2]. Consequently, an orthogonal tiling of $\mathbb{H}^d$ exists precisely for dimensions $d = 2, 3, 4$.

As it turns out, in dimension $d = 2, 3, 4$, an orthogonal tiling can be constructed that is additionally regular. In dimension $d = 2$, there are infinitely many regular orthogonal tilings, one for each integer $p \geqslant 5$. They are obtained by taking a regular right-angled p-gon in $\mathbb{H}^2$ and gluing together 4 of them at each vertex; by means of the so-called *Schläfli symbol*, such a configuration is denoted by $\{p, 4\}$. For instance, $\{8, 4\}$ is a tiling via four regular octagons meeting at each vertex, illustrated in Figure 1. In dimension $d = 3$, a tiling exists described by $\{5, 3, 4\}$, which is to be read as follows: start with a regular right-angled pentagon in $\mathbb{H}^2$ and glue three of them at each vertex. This gives a (hyperbolic) dodecahedron $\{5, 3\}$ and gluing 4 on each edge gives the tiling. In dimension $d = 4$, instead, one glues together 3 dodecahedra on each edge to obtain a 4-dimensional polytope $\{5, 3, 3\}$ whose maximal faces are the dodecahedra $\{5, 3\}$. Then, gluing 4 such polytopes at each 2-dimensional face, one gets the tiling $\{5, 3, 3, 4\}$. A list of all regular tilings of $\mathbb{H}^d$ can be found in [45, Section 5.3.3] or [19, Sections 2–4] (the orthogonal ones among them are exactly those with 4 as the last number in their Schläfli symbol).

One notable feature of orthogonal tilings, which we will use in the next lemma as well as Section 5, is the following:

Lemma 3.2 *Whenever two hyperplanes $H_1, H_2 \subset \mathbb{H}^d$ meet orthogonally, the reflections R_{H_1} and R_{H_2} across them commute and $R_{H_1}R_{H_2} = R_{H_2}R_{H_1} = R_{H_1 \cap H_2}$. Further, $R_{H_2}[H_1] = H_1$ and R_{H_2} preserves the two half-spaces defined by H_1.*

Proof We will prove this statement using the hyperboloid model of $\mathbb{H}^d$. Since for every point in $\mathbb{H}^d$, there exists an automorphism mapping it to the origin, we can assume that $0 \in H_1 \cap H_2$. Moreover, up to a rotation around 0 of $\mathbb{H}^d$ (i.e., a rotation of $\mathbb{R}^{d+1}$ around the x_{d+1} axis), we may assume that $H_1 = \{y \in \mathbb{H}^d : y_1 = 0\}$ and $H_2 = \{y \in \mathbb{H}^d : y_2 = 0\}$ (recall that angles at the origin $0_{\mathbb{H}^d}$ are Euclidean angles).

Then it follows that $R_{H_1}(x) = (-x_1, x_2, \ldots, x_{d+1})$ and $R_{H_2}(x) = (x_1, -x_2, \ldots, x_{d+1})$ for $x \in \mathbb{H}^d$. Clearly, these two mappings commute, and their composition is the mapping $x \mapsto (-x_1, -x_2, x_3, \ldots, x_{d+1})$, which is the reflection across $H_1 \cap H_2$. The last clause also follows directly from these formulas. ∎

As one can already glean from Figure 1, the edges of the octagons combine into geodesic lines, and this is true in general for any regular orthogonal tiling.

Lemma 3.3 *Let $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$ be a regular orthogonal tiling of $\mathbb{H}^d$, $d \leqslant 4$. Then, there exists a sequence $(H_k)_{k \in \mathbb{N}}$ of hyperplanes in $\mathbb{H}^d$ such that*

$$\bigcup_{n \in \mathbb{N}} \partial P_n = \bigcup_{k \in \mathbb{N}} H_k.$$

These H_k are exactly all the supporting hyperplanes for any maximal face of any P_n. As a consequence, if R is a reflection across some maximal face,

$$R\left[\bigcup_{k \in \mathbb{N}} H_k\right] = \bigcup_{k \in \mathbb{N}} H_k.$$

Proof Let $(H_k)_{k \in \mathbb{N}}$ be the sequence listing all the supporting hyperplanes for any maximal face of any polytope P_n. Then the validity of the '$\subset$' inclusion is clear.

For the converse inclusion, take a hyperplane $H \in \{H_k\}_{k \in \mathbb{N}}$. By definition, there is a polytope $P \in \mathcal{P}$ such that $M := H \cap P$ is a maximal face of P. Then M is a regular right-angled polytope in H (which is isometric to $\mathbb{H}^{d-1}$); hence, the reflection group generated by reflections across the maximal faces of M induces a tiling of H, by copies of M. However, a reflection across a maximal face of M is the restriction to H of a reflection across a hyperplane $\tilde{H}$ in $\mathbb{H}^d$, orthogonal to H (in fact, the reflection across $\tilde{H}$ maps H to H, by Lemma 3.2). Moreover, $\tilde{H}$ supports a maximal face of P, as P is right-angled. Therefore, the images of M under the reflection group of M are images of maximal faces of P under the reflection group of P. Hence, $H \subset \bigcup_{n \in \mathbb{N}} \partial P_n$, as desired.

Finally, the last claim follows because R preserves the set $\bigcup_{n \in \mathbb{N}} \partial P_n$ of faces. ∎

4 Lipschitz functions on polytopes

This section is dedicated to the local part of our construction, where we study the Banach space of Lipschitz functions on a single polytope. The result we prove in this section asserts that the space of all Lipschitz functions that vanish on certain subsets

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of the boundary of a polytope P is weak*-to-weak* isomorphic to $\mathrm{Lip}_0(P)$. Since $\mathbb{H}^d$ is locally bi-Lipschitz equivalent to $\mathbb{R}^d$ by Lemma 2.1, it is irrelevant if we consider Euclidean or hyperbolic polytopes; therefore, in all the section, we consider polytopes in $\mathbb{R}^d$. As we explained in the Introduction, our focus being on the hyperbolic case, we treat the Euclidean case as a black-box; this explains why in this section we will use nonexplicit arguments and we do not have an explicit formula for the isomorphism.

We begin by introducing the following notation.

Definition 4.1 If S is a subset of a pointed metric space M, we denote by

$$\mathrm{Lip}_{0,S}(M) := \{f \in \mathrm{Lip}_0(M) : f|_S = 0\}$$

the space of all Lipschitz functions on M vanishing on S (in addition to vanishing on 0_M).

It is easy to see that $\mathrm{Lip}_{0,S}(M)$ is pointwise closed, and hence weak* closed as well. Thus, $\mathrm{Lip}_{0,S}(M)$ is the dual of a quotient of $\mathcal{F}(M)$, and the corresponding weak* topology is the restriction of the weak* topology of $\mathrm{Lip}_0(M)$; hence, on bounded sets, it coincides with the pointwise one.

Before the main result of the section, we collect in the following lemma a basic construction of a linear extension operator.

Lemma 4.2 *Let (M,d) be a pointed metric space and $N \subset M$ be a bounded Lipschitz retract of M with $0_M \in N$. Then for every $\varepsilon > 0$, there is a bounded, weak*-to-weak* continuous, linear extension operator*

$$E_{N,\varepsilon} : \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M), \qquad f \mapsto E_{N,\varepsilon} f$$

such that, for every $f \in \mathrm{Lip}_0(N)$, the support of $E_{N,\varepsilon} f$ is contained in the set $\overline{B}(N,\varepsilon) = \{x \in M : \mathrm{dist}(x,N) \leqslant \varepsilon\}$.

Proof Fix a Lipschitz retraction r of M onto N and define the mapping $\lambda : M \to [0,1]$ by $\lambda(x) = \max\{1 - \frac{\mathrm{dist}(x,N)}{\varepsilon}, 0\}$. λ is clearly $1/\varepsilon$-Lipschitz and vanishes outside $\overline{B}(N,\varepsilon)$. Then the mapping

$$E_{N,\varepsilon} : \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M), \qquad f \mapsto (f \circ r)\lambda$$

is well defined and bounded since

$$\begin{aligned}\mathrm{Lip}\left((f \circ r)\lambda\right) &\leqslant \mathrm{Lip}(f \circ r)\|\lambda\|_\infty + \mathrm{Lip}(\lambda)\|f \circ r\|_\infty \\ &\leqslant \mathrm{Lip}(f)\mathrm{Lip}(r) + \frac{1}{\varepsilon}\|f\|_\infty \leqslant \left(\mathrm{Lip}(r) + \frac{\mathrm{diam}(N)}{\varepsilon}\right)\|f\|_{\mathrm{Lip}_0}.\end{aligned}$$

Finally, the pointwise-to-pointwise continuity of $E_{N,\varepsilon}$ is clear from the definition. ∎

Proposition 4.3 *Let P be a polygon in $\mathbb{R}^d$ with $0 \in \mathrm{int}(P)$ and let $S \subset \partial P$ be the union of one or more faces (of any dimension) of P. Then,*

$$\mathrm{Lip}_0(P) \simeq \mathrm{Lip}_{0,S}(P).$$

Moreover, the isomorphism is weak-to-weak* continuous.*

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Proof Let us first prove that $\mathrm{Lip}_0(P)$ is weak*-to-weak* isomorphic to a weak*-complemented subspace of $\mathrm{Lip}_{0,S}(P)$. For this, note that, for $\alpha \in (0,1)$, $\mathrm{Lip}_0(P)$ and $\mathrm{Lip}_0(\alpha P)$ are weak*-to-weak* isomorphic (P and αP are bi-Lipschitz equivalent). Hence, it is enough to prove that $\mathrm{Lip}_0(\alpha P)$ is weak*-to-weak* isomorphic to a weak*-complemented subspace of $\mathrm{Lip}_{0,S}(P)$. Now, αP is a Lipschitz retract of P (via the nearest point projection), and it has positive distance to the boundary ∂P of P. So, by Lemma 4.2 for $\varepsilon = \mathrm{dist}(\partial P, \alpha P)/2 > 0$, there is a bounded, weak*-to-weak* continuous, linear extension operator

$$E_{\alpha P,\varepsilon}\colon \mathrm{Lip}_0(\alpha P) \to \mathrm{Lip}_0(P)$$

such that $E_{\alpha P,\varepsilon} f(x) = 0$ for all $x \in \partial P$ (so the image of $E_{\alpha P,\varepsilon}$ is contained in $\mathrm{Lip}_{0,S}(P)$). As we observed in Section 2, this gives the desired embedding of $\mathrm{Lip}_0(\alpha P)$ into $\mathrm{Lip}_{0,S}(P)$.

Next, we show that $\mathrm{Lip}_{0,S}(P)$ is a weak*-complemented subspace of $\mathrm{Lip}_0(P)$. Consider the subset $S \cup \{0\}$ of P and observe that P is a doubling metric space (as the doubling property passes to subspaces and $\mathbb{R}^d$ is doubling). Therefore, we can apply Lee's and Naor's extension result [39] (see Section 2) and find a weak*-to-weak* continuous linear extension operator $E\colon \mathrm{Lip}_0(S \cup \{0\}) \to \mathrm{Lip}_0(P)$. Then the map

$$Q\colon \mathrm{Lip}_0(P) \to \mathrm{Lip}_{0,S}(P)\colon f \mapsto f - E(f|_{S\cup\{0\}})$$

is a linear projection onto $\mathrm{Lip}_{0,S}(P)$ since by definition, Qf vanishes on S and for each $f \in \mathrm{Lip}_{0,S}(P)$, $E(f|_{S\cup\{0\}}) = 0$. Hence, $\mathrm{Lip}_{0,S}(P)$ is a complemented subspace of $\mathrm{Lip}_0(P)$. In addition, Q is weak*-to-weak* continuous since E is.

Finally, by standard duality, the assertions proved in the previous paragraphs yield that $\mathrm{Lip}_{0,S}(P)$ is the dual to a complemented subspace Z of $\mathcal{F}(P)$ and $\mathcal{F}(P)$ is isomorphic to a complemented subspace of Z. Moreover, P has nonempty interior in $\mathbb{R}^d$; hence, combining [36, Corollary 3.5] and [36, Theorem 3.1], we obtain that $\mathcal{F}(P)$ is isomorphic to its ℓ_1-sum. Therefore, Pełczyński decomposition method assures us that $\mathcal{F}(P)$ is isomorphic to Z and passing to the duals we reach the sought conclusion. ∎

5 Extension operators and proof of the main result

In this section, we construct a number of extension operators which are the central tools for the proof of the main result. We are using a regular orthogonal tiling, and hence, we consider $\mathbb{H}^d$ only for $d = 2, 3, 4$. Given a regular orthogonal tiling $\mathcal{P} := (P_n)_{n\in\mathbb{N}}$ by isometric copies of a single polytope, we denote by p_n the center point of P_n in the sense of (T3) of Section 3.1 and consider the net $\mathcal{N} := \{p_n\colon n \in \mathbb{N}\}$. For the space $\mathrm{Lip}_0(P_n)$, we use p_n as the distinguished point, and for the sake of simplicity, we assume that $p_1 = 0$.

Using these data, we use the following strategy to prove the main result: In Section 5.1, we use an extension operator from the net $\mathcal{N}$ to decompose Lipschitz functions on $\mathbb{H}^d$ into Lipschitz functions on $\mathcal{N}$ and (bounded) Lipschitz functions vanishing on $\mathcal{N}$. In Section 5.2, we decompose the latter functions into a sequence of Lipschitz functions on the tiles. In Section 5.3, we use extension operators from the tiles to construct the inverse operator of this decomposition operator. Finally,

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in Section 5.4, we combine these arguments to finish the proof and state some consequences.

5.1 Extension from the net $\mathcal{N}$

In this part, we construct a bounded, weak*-to-weak* continuous linear extension operator from the net $\mathcal{N}$ to $\mathbb{H}^d$. In order to achieve this, we will exploit a Lipschitz partition of unity in the spirit of [1, 16, 35]. Notice that said results cannot be applied directly because $\mathbb{H}^d$ is not a doubling metric space. However, the specific shape of the net, inherited from the tiling, allows a very simple construction of the partition of unity.

Since the polytopes P_n are mutually isometric, the in-radius $\operatorname{dist}(p_n, \partial P_n)$ of P_n is the same for all n; thus, we can set $\delta \coloneqq \operatorname{dist}(p_n, \partial P_n) > 0$. Therefore, when $k \neq n$, $\operatorname{dist}(p_k, P_n) \geqslant \delta$. Define functions $\rho_n \colon \mathbb{H}^d \to \mathbb{R}$ by $\rho_n(x) \coloneqq \max\left\{1 - \frac{\operatorname{dist}(x, P_n)}{\delta}, 0\right\}$. Then $\rho_n(p_k) = \delta_{k,n}$ and ρ_n is $1/\delta$-Lipschitz. Moreover, the restriction of ρ_n to P_k is nonzero only when P_k intersects P_n. Hence, by (T4), the series $\sum_{n=1}^{\infty} \rho_n$ is locally a sum of at most $N(d)$ terms, and therefore, it defines a bounded Lipschitz function. Additionally, we have $\sum_{n=1}^{\infty} \rho_n(x) \geqslant 1$ for all $x \in \mathbb{H}^d$, as ρ_n equals 1 on P_n. Consequently, the functions

$$\varphi_n \coloneqq \frac{\rho_n}{\sum_{n=1}^{\infty} \rho_n}$$

are uniformly Lipschitz, with constant, say, $L_{\mathcal{N}}$. Clearly, $(\varphi_n)_{n=1}^{\infty}$ is the desired partition of unity and $\varphi_n(p_k) = \delta_{k,n}$. We can now proceed to the construction of the extension operator from the net $\mathcal{N}$.

Lemma 5.1 *The operator $E_{\mathcal{N}}$ defined as*

$$E_{\mathcal{N}} \colon \operatorname{Lip}_0(\mathcal{N}) \to \operatorname{Lip}_0(\mathbb{H}^d) \colon f \mapsto \sum_{n=1}^{\infty} f(p_n)\varphi_n$$

is a bounded linear extension operator from the net $\mathcal{N}$ to $\mathbb{H}^d$ which is weak-to-weak* continuous.*

Proof The sum defining $E_{\mathcal{N}} f$ is locally finite; thus, $E_{\mathcal{N}}$ is clearly a well-defined linear extension operator, and $E_{\mathcal{N}} f$ is continuous. In order to bound the Lipschitz constant of $E_{\mathcal{N}} f$, we use Lemma 2.1. Thus, fix $f \in \operatorname{Lip}_0(\mathcal{N})$ and $x, y \in P_n$, for some $n \in \mathbb{N}$. Let $\{n_1, \ldots, n_{N(d)}\}$ be the set of indices corresponding to the polytopes P_k that intersect P_n. Hence, we have

$$\begin{aligned}
|E_{\mathcal{N}} f(x) - E_{\mathcal{N}} f(y)| &= \left| \sum_{j=1}^{N(d)} \left(\varphi_{n_j}(x) - \varphi_{n_j}(y)\right) f(p_{n_j}) \right| \\
&= \left| \sum_{j=1}^{N(d)} \left(\varphi_{n_j}(x) - \varphi_{n_j}(y)\right)\left(f(p_{n_j}) - f(p_n)\right) \right| \\
&\leqslant \sum_{j=1}^{N(d)} L_{\mathcal{N}}\, \rho(x, y) \operatorname{Lip}(f)\, \rho(p_{n_j}, p_n) \leqslant C \operatorname{Lip}(f)\, \rho(x, y).
\end{aligned}$$

Here, $C = 2\,\mathrm{diam}(P_n)\, L_{\mathcal{N}}\, N(d)$, and we used the fact that $\rho(p_{n_j}, p_n) \leqslant 2\,\mathrm{diam}(P_n)$, since $P_{n_j} \cap P_n \neq \varnothing$. According to Lemma 2.1, this inequality proves that $\mathrm{Lip}(E_{\mathcal{N}} f) \leqslant C\mathrm{Lip}(f)$, whence the boundedness of $E_{\mathcal{N}}$.

Lastly, to prove weak*-to-weak* continuity, we need only show pointwise-to-pointwise continuity. So assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence of Lipschitz functions converging pointwise to some function $f \in \mathrm{Lip}_0(\mathcal{N})$. Then, using again local finiteness of the sum,

$$\lim_{k\to\infty} E_{\mathcal{N}}(f_k)(x) = \sum_{n=1}^{\infty} \lim_{k\to\infty} f_k(p_n)\varphi_n(x) = \sum_{n=1}^{\infty} f(p_n)\varphi_n(x) = E_{\mathcal{N}}(f)(x). \qquad \blacksquare$$

5.2 Decomposition into functions on tiles

The aim of this section is to decompose a Lipschitz function on $\mathbb{H}^d$ into a sequence of functions defined on the tiles of our tiling. As a first step, we use the results of the previous section to remove the part of the function defined on the net $\mathcal{N}$. More precisely, Lemma 5.1 implies that the mapping

$$\mathrm{Lip}_0(\mathbb{H}^d) \to \mathrm{Lip}_0(\mathcal{N}) \oplus \mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)\colon \qquad f \mapsto \big(f|_{\mathcal{N}}, f - E_{\mathcal{N}}(f|_{\mathcal{N}})\big)$$

is a weak*-to-weak* continuous isomorphism (with inverse $(g,h) \mapsto E_{\mathcal{N}}(g) + h$).

Hence, our goal now is to decompose a function in $\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$ into a sequence of functions defined on the tiles. We start with some geometric preliminaries.

Definition 5.2 Let H be a hyperplane in $\mathbb{H}^d$ not containing the origin 0. The two connected components of $\mathbb{H}^d \backslash H$ are called the *open half-spaces* defined by H. We denote their closures by H^+ and H^-, where H^+ is chosen so that 0 is in the interior of H^+.

Let us now fix some notation and parameters. Let $\mathcal{P}$ be our tiling of $\mathbb{H}^d$ and $(H_m)_{m\in\mathbb{N}}$ be the sequence of hyperplanes supporting the faces of the tiles P_n as in Lemma 3.3. Let $\varepsilon > 0$ be a fixed number which is smaller than the in-radius of P_m and smaller than half of the minimal distance of nonadjacent faces of P_m. For the hyperplane H_m, we denote by $R_m\colon \mathbb{H}^d \to \mathbb{H}^d$ the reflection across H_m.

In order to start the construction of the decomposition, for each hyperplane H_n, we define a cutoff function

$$\psi_n\colon \mathbb{H}^d \to [0,1]\colon \qquad x \mapsto \max\left\{1 - \frac{\mathrm{dist}(x, H_n)}{\varepsilon}, 0\right\},$$

which clearly satisfies $\mathrm{Lip}(\psi_n) \leqslant 1/\varepsilon$, $\psi_n(x) = 1$ if and only if $x \in H_n$, and $\psi_n(x) = 0$ if and only if $\mathrm{dist}(x, H_n) \geqslant \varepsilon$.

We now construct operators on the space $(\bigoplus_{m\in\mathbb{N}} \mathrm{Lip}_0(P_m))_{\ell_\infty}$. In order to simplify the notation, we often interpret this sum as the space of all functions g defined on $\bigcup_{m\in\mathbb{N}} \mathrm{int}(P_m) \subset \mathbb{H}^d$ which are Lipschitz on every $\mathrm{int}(P_m)$ with $\sup_{m\in\mathbb{N}} \mathrm{Lip}(g|_{P_m}) < \infty$ and that vanish on the net $\mathcal{N}$. More precisely, to a sequence $(g_m)_{m\in\mathbb{N}}$, we associate the function g defined to be equal to g_m on $\mathrm{int}(P_m)$, for every $m \in \mathbb{N}$; vice versa, to a function g, we associate the sequence $(g_m)_{m\in\mathbb{N}}$, where g_m is the unique continuous extension of $g|_{\mathrm{int}(P_m)}$ to P_m. The important advantage of this interpretation is that we do

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not consider values on the boundaries of the tiles, which simplifies several formulas. As a first instance of this, this interpretation allows us to identify $\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$ with the subspace of all functions admitting a continuous extension to $\mathbb{H}^d$.

With these preparations, we are now able to define the operator

$$\chi_n\colon \Big(\bigoplus_{m\in\mathbb{N}} \mathrm{Lip}_0(P_m)\Big)_{\ell_\infty} \to \Big(\bigoplus_{m\in\mathbb{N}} \mathrm{Lip}_0(P_m)\Big)_{\ell_\infty}$$

by setting

$$(\chi_n g)(x) = \begin{cases} g(x) & \text{for } x \in H_n^-, \\ g(x) - \psi_n(x) g(R_n(x)) & \text{for } x \in H_n^+. \end{cases}$$

Notice that in the definition of χ_n, we used the interpretation explained above. Also, according to Lemma 3.3, R_n maps $\bigcup_{m\in\mathbb{N}} \mathrm{int}(P_m)$ to itself; thus, we can legitimately evaluate g at the point $R_n(x)$.

Since reflections are isometries and g is bounded, the operator χ_n is bounded and it is clearly also weak*-to-weak* continuous. Also note that since χ_n does not change, the values of g on H_n^- and $R_n(x) \in H_n^-$ whenever $x \in H_n^+$, its inverse is the operator defined by

$$(\chi_n^{-1} g)(x) = \begin{cases} g(x) & \text{for } x \in H_n^-, \\ g(x) + \psi_n(x) g(R_n(x)) & \text{for } x \in H_n^+. \end{cases}$$

We denote by $\chi_{1,\dots,n} := \chi_n \circ \cdots \circ \chi_1$ the composition of $\chi_1, \dots, \chi_n$ and note that it satisfies $\chi_{1,\dots,n}^{-1} = \chi_1^{-1} \circ \cdots \circ \chi_n^{-1}$.

Given a function $g \in \mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$, we define a function $g_m\colon P_m \to \mathbb{R}$ by

$$g_m(x) := \lim_{n\to\infty} \chi_{1,\dots,n}(g)(x) \qquad (x \in P_m).$$

Note that for each m, there is an N_m such that $\chi_{1,\dots,n}(g)(x)$ does not change for $n \geqslant N_m$. Hence, the above limit exists since the sequence is eventually constant, and the index where it becomes constant only depends on P_m and not on the individual point $x \in P_m$.

Let us now consider the linear mapping

$$\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d) \to \Big(\bigoplus_{m\in\mathbb{N}} \mathrm{Lip}_0(P_m)\Big)_{\ell_\infty}\colon \quad g \mapsto (g_m)_{m\in\mathbb{N}},$$

which will eventually be the main ingredient for our isomorphism. Since it is not surjective between the above spaces, we have to determine a suitable codomain. With this in mind, for each $m \in \mathbb{N}$, we define

$$S_m := \bigcup_{\substack{n\in\mathbb{N} \\ P_m \subset H_n^+}} H_n \cap P_m. \tag{5.1}$$

Now we are able to define the desired operator

$$\Phi\colon \mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d) \to \Big(\bigoplus_{m\in\mathbb{N}} \mathrm{Lip}_{0,S_m}(P_m)\Big)_{\ell_\infty} \colon g \mapsto (g_m)_{m\in\mathbb{N}}$$

and start checking that it has the desired properties.

Lemma 5.3 *The linear operator Φ maps $\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$ into $\left(\bigoplus_{m\in\mathbb{N}}\mathrm{Lip}_{0,S_m}(P_m)\right)_{\ell_\infty}$. Moreover, it is bounded and weak*-to-weak* continuous.*

Proof We first check that $\Phi(g)$ is a sequence of Lipschitz functions whose Lipschitz constants are bounded by a multiple of the Lipschitz constant of g. Since $g\circ R_n$ and ψ_n are Lipschitz and bounded, the standard product formula gives us that χ_n is bounded with $\|\chi_n\| = \|\chi_1\|$ for all n. Moreover, χ_n only changes its argument on an ε-neighborhood of H_n, and P_m intersects said neighborhood if and only if H_n supports a face of P_m. Hence, the number of faces $N(d)$ of the polytopes P_m gives us the number of times the function g can at most be changed on that tile during the limit process that produces $\Phi(g)$. Thus, the Lipschitz constant of $\Phi(g)_m$ is at most

$$\mathrm{Lip}(\Phi(g)_m) \leqslant \|\chi_1\|^{N(d)}\mathrm{Lip}(g).$$

Also, $\Phi(g)(p_m) = g(p_m) = 0$ because $p_m \in \mathcal{N}$ and the center-point p_m of any P_m is not in an ε-neighborhood of any H_n. It follows that $\Phi(g)_m \in \mathrm{Lip}_0(P_m)$. Note that the above inequality also shows the boundedness of Φ. By the same reason, the weak*-to-weak* continuity of χ_n implies that Φ is also weak*-to-weak* continuous.

To conclude, we need to check that $(\Phi(g))_m$ vanishes on S_m. For this aim, assume $x \in P_m \cap H_n$ and $P_m \subset H_n^+$ and observe that

$$\chi_{1,\dots,n}(g)(x) = \chi_{1,\dots,n-1}(g)(x) - \underbrace{\psi_n(x)}_{=1}(\chi_{1,\dots,n-1}(g)(x))(\underbrace{R_n x}_{=x}) = 0.$$

So at the n-th step of our limit process, the values of our function $\chi_{1,\dots,n}g$ are set to zero everywhere along H_n on the tiles that lie within H_n^+. To express this briefly, we say that $\chi_{1,\dots,n}g$ vanishes on ∂H_n^+. We now have to show that this value of zero is retained by all the subsequent functions $\chi_{1,\dots,k}g$, $k > n$. We will do this by induction.

So fix $k > n$ and assume that $\chi_{1,\dots,k-1}g(x) = 0$ for $x \in \partial H_n^+$. To show that $\chi_{1,\dots,k}g(x) = 0$ as well, we distinguish two cases. If H_k and H_n are not orthogonal, it follows from our choice of ε that ψ_k is zero on H_n. Hence, $\chi_{1,\dots,k}g(x) = \chi_{1,\dots,k-1}g(x) = 0$ for $x \in \partial H_n^+$. However, if H_k is orthogonal to H_n, Lemma 3.2 implies that $R_k[H_n] = H_n$ and $R_k[H_n^+] = H_n^+$. Therefore, $\chi_{1,\dots,k-1}g(R_k x) = 0$ when $x \in \partial H_n^+$. As before, it follows that $\chi_{1,\dots,k}g$ vanishes on ∂H_n^+, which finishes the proof. ∎

In the following subsection, we will construct an inverse operator for Φ, thus showing that Φ is an isomorphism.

5.3 Extension operators from the tiles and inverse of Φ

The goal of this section is to construct a linear extension operator from $\mathrm{Lip}_{0,S_m}(P_m)$ to $\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$ which works well with the decomposition considered in the previous section and allows us to show that the operator Φ is an isomorphism. The interpretation of $(\bigoplus_{k\in\mathbb{N}}\mathrm{Lip}_0(P_k))_{\ell_\infty}$ and its subspace $(\bigoplus_{k\in\mathbb{N}}\mathrm{Lip}_{0,S_k}(P_k))_{\ell_\infty}$ considered in the previous section allows us to view the space $\mathrm{Lip}_{0,S_m}(P_m)$ as a subset of $(\bigoplus_{k\in\mathbb{N}}\mathrm{Lip}_{0,S_k}(P_k))_{\ell_\infty}$, by setting the function equal to zero outside of P_m.

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For the construction of the extension operator for $\mathrm{Lip}_{0,S_m}(P_m)$, let $n_1 > \cdots > n_s$ be the natural numbers such that the hyperplane H_{n_j} supports a face of P_m which is not contained in S_m. As a first step, we define the operator

$$\mathrm{Lip}_{0,S_m}(P_m) \to \Big(\bigoplus_{m\in\mathbb{N}} \mathrm{Lip}_0(P_m)\Big)_{\ell_\infty} : \qquad h \mapsto (\chi_{n_s}^{-1} \circ \cdots \circ \chi_{n_1}^{-1})(h).$$

The next lemma shows that $(\chi_{n_s}^{-1} \circ \cdots \circ \chi_{n_1}^{-1})(h)$ admits a continuous extension to $\mathbb{H}^d$, and hence, we may define the extension operator

$$E_m \colon \mathrm{Lip}_{0,S_m}(P_m) \to \mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d): \qquad h \mapsto E_m h,$$

where $E_m h$ is the unique continuous extension of $(\chi_{n_s}^{-1} \circ \cdots \circ \chi_{n_1}^{-1})(h)$ to $\mathbb{H}^d$.

Lemma 5.4 *The operator E_m is well defined, bounded, linear, and weak*-to-weak* continuous. Moreover, $\|E_m\|$ is bounded by a constant depending only on ε, the number of faces of P_m and the diameter of P_m.*

Proof In order to show that E_m is well defined, we have to show that $g := (\chi_{n_s}^{-1} \dots \chi_{n_1}^{-1})(h)$ admits a continuous extension to $\mathbb{H}^d$. We show this in an inductive manner. Note that no hyperplane H_n intersects the interior of P_m and, by definition of S_m (see (5.1)), $H_{n_j}^+$ is the half-space separated from P_m by H_{n_j} (otherwise, H_{n_j} would support a face contained in S_m, which it does not by definition of $n_1, \dots, n_s$).

Since we think of $C_k := \bigcup_{j=k+1}^{s} \mathrm{int}(H_{n_j}^+)$ as the set to where we have not yet extended our function in the k-th step, we now show that $g_k := (\chi_{n_k}^{-1} \circ \cdots \circ \chi_{n_1}^{-1})(h)$ has an extension to $\mathbb{H}^d$ which is zero on C_k and continuous on $\mathbb{H}^d \backslash C_k$. Since $g = g_s$ and $C_s = \varnothing$ this will prove our claim.

The function $g_0 = h$ is zero outside of P_m and hence on C_0 and $(\mathbb{H}^d \backslash C_0) \backslash P_m$. Since the (relative) boundary of P_m in the set $\mathbb{H}^d \backslash C_0$ is precisely S_m and h_m vanishes there, g_0 has a continuous extension to $\mathbb{H}^d \backslash C_0$.

Let us assume now that we have already shown that g_{k-1} has an extension to $\mathbb{H}^d$ which is zero on C_{k-1} and continuous on $\mathbb{H}^d \backslash C_{k-1}$. By abuse of notation, we call this extension g_{k-1} as well. We now have to check that g_k also has the desired properties.

We first check that g_k vanishes on C_k and recall that $g_k(x) = g_{k-1}(x)$ for $x \in H_{n_k}^-$ and $g_k(x) = g_{k-1}(x) + \psi_n(x) g_{k-1}(R_{n_k} x)$ for $x \in H_{n_k}^+$. We distinguish between two cases: If for $j > k$ we have $H_{n_j} \cap H_{n_k} \neq \varnothing$, these hyperplanes are orthogonal, and hence, $x \in H_{n_j}^+$ if and only if $R_{n_k} x \in H_{n_j}^+$, and hence, $g_k(x) = 0$ for $x \in \mathrm{int}(H_{n_j}^+)$. For the other case, note that H_{n_k} cannot be in $H_{n_j}^+$ since otherwise the face of P_m contained in H_{n_k} would have to be in $H_{n_j}^+$, which is only possible if it was in H_{n_j}, which contradicts $H_{n_k} \cap H_{n_j} = \varnothing$. Hence, we have $\mathrm{dist}(H_{n_j}^+, H_{n_k}) > 2\varepsilon$, and therefore, $g_k(x) = g_{k-1}(x) = 0$ for $x \in H_{n_{s-k+1}}$.

In order to show that g_k has a continuous extension to $\mathbb{H}^d \backslash C_k$, let $z \in H_{n_k} \backslash C_k$ and note that $g_{k-1}(x) = 0$ for $x \in \mathrm{int}(H_{n_k}^-)$. Hence,

$$\lim_{x \to z} g_k(x) = \lim_{x \to z} \psi_{n_k}(x) g_{k-1}(R_{n_k}(x)) = g_{k-1}(z),$$

where x converges to z in the interior of $H_{n_k}^-$ since $\psi_{n_k}(z) = 1$ and $R_{n_k}(z) = z$. This finishes the proof of the well-definedness of E_m.

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Linearity and pointwise-to-pointwise continuity are clear. Since $\chi_{n_s}^{-1}\dots\chi_{n_1}^{-1}$ is the composition of bounded operators on $(\bigoplus_{m\in\mathbb{N}}\mathrm{Lip}_0(P_m))_{\ell_\infty}$ and $E_m h_m$ is continuous, boundedness of E_m follows from Lemma 2.1 and $\|E_m\| \leqslant \|\chi_{n_s}^{-1}\circ\cdots\circ\chi_{n_1}^{-1}\|$. Finally, note that this norm only depends on ε, s, and the diameter of P_m and that s is at most the number of faces of P_m. ■

Remark 5.5 Note that applying χ_n^{-1} for n distinct from $n_1,\dots,n_s$ in definition of E_m does not change anything, as the function we are working with is zero on the interior of H_n^-, so instead of using $\chi_{n_s}^{-1}\circ\cdots\circ\chi_{n_1}^{-1}$, we can also use $\chi_{1,\dots,n_1}^{-1}$ or even $\chi_{1,\dots,k}^{-1}$ for every $k\geqslant n_1$.

We are now in position to define the operator

$$\Psi\colon\Big(\bigoplus_{m\in\mathbb{N}}\mathrm{Lip}_{0,S_m}(P_m)\Big)_{\ell_\infty}\longrightarrow \mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d):\qquad (h_m)_{m\in\mathbb{N}}\mapsto\sum_{m\in\mathbb{N}}E_m h_m$$

(where the sum is meant to be taken pointwise) and show that it is the inverse of Φ.

Proposition 5.6 *Ψ is well defined, linear, bounded, weak*-to-weak* continuous, and the inverse of Φ.*

Proof The sum in the definition of Ψ is locally finite with a uniform bound $N(d)$ on the number of summands; thus, Ψ is a bounded linear operator by Lemma 5.4. Since weak*-to-weak* continuity of Ψ is easy to see, we are left to show that Ψ is the inverse of Φ.

We start by showing that $\Phi\circ\Psi = \mathrm{Id}$. For this, let $(h_m)_{m\in\mathbb{N}}\in(\bigoplus_{m\in\mathbb{N}}\mathrm{Lip}_{0,S_m}(P_m))_{\ell_\infty}$ and fix an arbitrary $k\in\mathbb{N}$. It is enough to check that $(\Phi(\Psi((h_m)_{m\in\mathbb{N}})))_k(x) = h_k(x)$ for x in the interior of P_k. Recall that in Section 5.2, we observed that there is an index $N_k\in\mathbb{N}$ such that $(\chi_{1,\dots,n}g)(x) = (\chi_{1,\dots,N_k}g)(x)$ for $n\geqslant N_k$, every $g\in\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$, and all $x\in P_k$. Using this observation together with the definition of these operators and Remark 5.5, we have

$$\begin{aligned}(\Phi(\Psi((h_m)_{m\in\mathbb{N}})))_k(x) &= \Big(\Phi\Big(\sum_{m\in\mathbb{N}}E_m h_m\Big)\Big)_k(x) = \sum_{m\in\mathbb{N}}\chi_{1,\dots,N_k}(E_m h_m)(x)\\ &= \sum_{m\in\mathbb{N}}(\chi_{1,\dots,N_k}(\chi_{1,\dots,N_k}^{-1}(h_m)))(x) = \sum_{m\in\mathbb{N}}h_m(x) = h_k(x)\end{aligned}$$

for all $x\in\mathrm{int}\,(P_k)$ (i.e. $(\Phi(\Psi((h_m)_{m\in\mathbb{N}})))_k = h_k$).

In order to show that $\Psi\circ\Phi = \mathrm{Id}$, let $g\in\mathrm{Lip}_{0,\mathcal{N}}(\mathbb{H}^d)$ be given. It is enough to check that $\Psi\circ\Phi(g)(x) = g(x)$ for all x in the interior of P_k and for all k. Since $\chi_n^{-1}((\Phi(g))_k)(x)$ for $k\neq n$ is only nonzero if $x\in H_n^+$ and within ε-distance of H_n and there are at most d hyperplanes $H_{n_1},\dots,H_{n_d}$ with this property, there are numbers $M,N\in\mathbb{N}$ such that

$$\begin{aligned}(\Psi(\Phi(g)))(x) &= \sum_{m=1}^{M}(E_m((\Phi(g))_m))(x) = \sum_{m=1}^{M}(\chi_{1,\dots,N}^{-1})(\Phi(g))_m)(x)\\ &= \chi_{1,\dots,N}^{-1}\Big(\sum_{m=1}^{M}\mathbb{1}_{\mathrm{int}\,P_m}\chi_{1,\dots,N}(g)\Big)(x)\\ &= \chi_{1,\dots,N}^{-1}\chi_{1,\dots,N}\Big(\sum_{m=1}^{M}\mathbb{1}_{\mathrm{int}\,P_m}g\Big)(x) = g(x)\end{aligned}$$

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since we may interpret the sum $\sum_{m=1}^{M} \mathbb{1}_{\operatorname{int} P_m} \chi_{1,\dots,N}(g)$ locally around x as a representation of $\chi_{1,\dots,N}(g)$ as a function similar to the previous sections, and the operator $\chi_{1,\dots,N}$ and its inverse are defined pointwise and only depend on "nearby" values of the function. ∎

5.4 Conclusion of the proof and consequences

In this section, we use the arguments of the previous sections and combine them with the results of Section 4 to prove our main result. Moreover, we state a number of direct consequences of the main result and compare it to the case of the space of bounded Lipschitz functions $\operatorname{Lip}(\mathbb{H}^d)$.

Theorem 5.7 *For $d = 2, 3, 4$, we have*

$$\operatorname{Lip}_0(\mathbb{H}^d) \simeq \Big(\operatorname{Lip}_0(\mathcal{N}) \oplus \bigoplus_{m\in\mathbb{N}} \operatorname{Lip}_0(P_m)\Big)_{\ell_\infty}$$

and

$$\mathcal{F}(\mathbb{H}^d) \simeq \Big(\mathcal{F}(\mathcal{N}) \oplus \bigoplus_{m\in\mathbb{N}} \mathcal{F}(P_m)\Big)_{\ell_1}.$$

Proof We consider the mapping

$$\operatorname{Lip}_0(\mathbb{H}^d) \to \operatorname{Lip}_0(\mathcal{N}) \oplus \Big(\bigoplus_{m\in\mathbb{N}} \operatorname{Lip}_{0,S_m}(P_m)\Big)_{\ell_\infty}: \qquad f \mapsto \big(f|_{\mathcal{N}}, \Phi(f - f|_{\mathcal{N}})\big),$$

which by Lemma 5.1, Lemma 5.3, and Proposition 5.6 is a weak*-to-weak* continuous isomorphism. By Lemma 3.1, P_m is bi-Lipschitz equivalent to the same polytope P'_m considered in $\mathbb{R}^d$. Combining this with Proposition 4.3 implies that $\operatorname{Lip}_{0,S_m}(P_m)$ and $\operatorname{Lip}_0(P_m)$ are weak*-to-weak* isomorphic, with a uniform bound on the distortion (because the polytopes P_m are mutually isometric). This gives the first of the claims above and shows that the isomorphisms are also weak*-to-weak* continuous. Hence, we may pass to the preduals and arrive at the second claim. ∎

Corollary 5.8 *Let $d = 2, 3, 4$ and $\mathcal{M}$ be any net in $\mathbb{H}^d$. Then $\mathcal{F}(\mathbb{H}^d)$ is isomorphic to $\mathcal{F}(\mathcal{M}) \oplus \mathcal{F}(\mathbb{R}^d)$.*

Proof As proved by Bogopolskii in [12], all nets in $\mathbb{H}^d$ are bi-Lipschitz equivalent; thus $\mathcal{F}(\mathcal{M}) \simeq \mathcal{F}(\mathcal{N})$, where $\mathcal{N}$ is the net from the above theorem. Moreover, by Lemma 3.1, P_1 is bi-Lipschitz equivalent to a polytope in $\mathbb{R}^d$ (with nonempty interior), so $\mathcal{F}(P_1) \simeq \mathcal{F}(\mathbb{R}^d)$ due to [36, Corollary 3.5]. Finally, each $\mathcal{F}(P_n)$ is isometric to $\mathcal{F}(P_1)$; thus, $(\bigoplus_n \mathcal{F}(P_n))_{\ell_1} \simeq (\bigoplus_n \mathcal{F}(\mathbb{R}^d))_{\ell_1} \simeq F(\mathbb{R}^d)$, where the last isomorphism follows from [36, Theorem 3.1]. ∎

Remark 5.9 Alternatively, instead of [12], we could have used the following result from [32]: if $\mathcal{N}$ and $\mathcal{M}$ are nets in a metric space, both of cardinality the density of the metric space, then $\mathcal{F}(\mathcal{N}) \simeq \mathcal{F}(\mathcal{M})$.

Corollary 5.10 *For $d = 2, 3, 4$, $\mathcal{F}(\mathbb{H}^d)$ has a Schauder basis.*

Proof We will show that $\mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathbb{R}^d)$ has a Schauder basis. $\mathcal{F}(\mathcal{N})$ does as proved by Doucha and Kaufmann in [23], whereas $\mathcal{F}(\mathbb{R}^d)$ does by [33]. Thus, their direct sum has a Schauder basis as well. ∎

Remark 5.11 Note that a similar construction using a tessellation with cubes works in $\mathbb{R}^d$. Additionally, if we look at the spaces of *bounded* Lipschitz functions $\mathrm{Lip}(\mathbb{R}^d)$ and $\mathrm{Lip}(\mathbb{H}^d)$ rather than the ones which are zero at a given base point, then we can use an analogous construction to the one we employ in Section 5.2 and Section 5.3 to decompose any function $f \in \mathrm{Lip}(\mathbb{H}^d)$ and any $g \in \mathrm{Lip}(\mathbb{R}^d)$ into a sequence of bounded Lipschitz functions $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ on the tiles $(P_n)_{n\in\mathbb{N}}$ tessellating $\mathbb{H}^d$ and the cubes $(Q_n)_{n\in\mathbb{N}}$ tesselating $\mathbb{R}^d$, respectively. Since the functions in $\mathrm{Lip}(\mathbb{R}^d)$ and $\mathrm{Lip}(\mathbb{H}^d)$ are already bounded and functions in $\mathrm{Lip}(M)$ do not require a base point within the domain on which they are zero, removing the values on a net is not needed in this case. Thus, for $d = 2, 3, 4$,

$$\mathrm{Lip}(\mathbb{H}^d) \simeq \Big(\bigoplus_{n\in\mathbb{N}} \mathrm{Lip}_{S_n}(P_n)\Big)_{\ell_\infty} \qquad \text{and} \qquad \mathrm{Lip}(\mathbb{R}^d) \simeq \Big(\bigoplus_{n\in\mathbb{N}} \mathrm{Lip}_{T_n}(Q_n)\Big)_{\ell_\infty},$$

where S_n is defined as in Definition 5.1 and T_n, analogously, is the union of all boundary (hyper-)surfaces of the (hyper-)cubes Q_n which are supported by a hyperplane which does not separate Q_n from the origin $0_{\mathbb{R}^d}$. Equivalently, the union of all faces of Q_n which contain the point(s) of Q_n that is (are) furthest from the origin.

From this, we are able to obtain that $\mathrm{Lip}(\mathbb{H}^d) \simeq \mathrm{Lip}(\mathbb{R}^d)$ – this time, in a completely explicit manner.

Indeed, since S_n is the part of the boundary of P_n which is invisible from the origin (i.e., the geodesics connecting a point from S_n with the origin intersect the polytope in other points) an elementary geometric argument shows that it is simply connected. Similarly, T_n is a simply connected subset of Q_n. It follows that one can find explicit bi-Lipschitz mappings between any pair (P_n, S_n) and (Q_m, T_m). The only two tiles for which this is not true are the two tiles $P_1 \subset \mathbb{H}^d$ and $Q_1 \subset \mathbb{R}^d$ which contain their spaces' respective base points 0, since those are the only two tiles whose "boundary condition" S_n is the entire boundary ∂P_1 and ∂Q_1, respectively. However, this simply means that we map P_1 to Q_1 in a bi-Lipschitz way, and then carry on with the rest of the sequence arbitrarily since any other (P_n, S_n) can be bi-Lipschitz mapped to any, say, (Q_m, T_m).

Finally, there are only finitely many different pairs (P_n, S_n), which are not pairwise congruent (and similarly, finitely many pairs (Q_m, T_m)). Therefore, these bi-Lipschitz maps have uniformly bounded distortion and hence induce and isomorphism $\mathrm{Lip}(\mathbb{H}^d) \simeq \mathrm{Lip}(\mathbb{R}^d)$.

Remark 5.12 The isomorphism $\mathrm{Lip}(\mathbb{S}^d) \simeq \mathrm{Lip}(\mathbb{R}^d)$ can be deduced from the corresponding isomorphism for the Lip_0-spaces. First note that since $\mathbb{S}^d$ has a finite diameter, the space $\mathrm{Lip}_0(\mathbb{S}^d)$ is a hyperplane in $\mathrm{Lip}(\mathbb{S}^d)$. Moreover, we have

$$\mathrm{Lip}_0(\mathbb{S}^d) \simeq \mathrm{Lip}_0(\mathbb{R}^d) \simeq \mathrm{Lip}_0([0,1]^d)$$

by [1] or [25]. As for the sphere, the space $\mathrm{Lip}_0([0,1]^d)$ is a hyperplane in $\mathrm{Lip}([0,1]^d)$. By [20, Theorem 5] all the Lip_0-spaces above and hence also all spaces of bounded Lipschitz functions contain a (complemented) copy of ℓ_∞; hence, they are isomorphic to their hyperplanes. Therefore, we may conclude that $\mathrm{Lip}(\mathbb{S}^d) \simeq \mathrm{Lip}([0,1]^d)$. Using Proposition 4.3 together with an argument similar to the one in Remark 5.11, we obtain that $\mathrm{Lip}(\mathbb{S}^d) \simeq \mathrm{Lip}(\mathbb{R}^d)$. In contrast to the case of the hyperbolic space, this argument works for arbitrary d.

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References

[1] F. Albiac, J. L. Ansorena, M. Cúth and M. Doucha, *Lipschitz free spaces isomorphic to their infinite sums and geometric applications*. Trans. Amer. Math. Soc. **374**(2021), 7281–7312.
[2] F. Albiac, J. L. Ansorena, M. Cúth and M. Doucha, *Lipschitz algebras and Lipschitz-free spaces over unbounded metric spaces*. Int. Math. Res. Not. IMRN 2022(**2022**), 16327–16362.
[3] F. Albiac, J. L. Ansorena, M. Cúth and M. Doucha, *Structure of the Lipschitz free p-spaces* $\mathcal{F}_p(\mathbb{Z}^d)$ *and* $\mathcal{F}_p(\mathbb{R}^d)$ *for* $0 < p \leq 1$. Collect. Math. **73**(2022), 337–357.
[4] R. J. Aliaga, C. Gartland, C. Petitjean and A. Procházka, *Purely* 1*-unrectifiable metric spaces and locally flat Lipschitz functions*. Trans. Amer. Math. Soc. **375**(2022), 3529–3567.
[5] R. J. Aliaga, E. Pernecká, C. Petitjean and A. Procházka, *Supports in Lipschitz-free spaces and applications to extremal structure*. J. Math. Anal. Appl. **489**(2020), 124128.
[6] L. Ambrosio and D. Puglisi, *Linear extension operators between spaces of Lipschitz maps and optimal transport*. J. Reine Angew. Math. **764**(2020), 1–21.
[7] J. W. Anderson, *Hyperbolic geometry*, Springer-Verlag, London, 1999.
[8] C. Bargetz, *Commutativity of the Valdivia–Vogt table of representations of function spaces*. Math. Nachr. **287**(2014), 10–22.
[9] C. Bargetz, *Explicit representations of spaces of smooth functions and distributions*. J. Math. Anal. Appl. **424**(2015), 1491–1505.
[10] C. Bargetz, S. Reich and D. Thimm, *Generic properties of nonexpansive mappings on unbounded domains*. J. Math. Anal. Appl. **526**(2023), 127179.
[11] M. Berger, ***Geometry II***, Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Berlin, 1996.
[12] O. V. Bogopolskii, *Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent*. Algebra and Logic **36**(1997), 155–163.
[13] L. Borel-Mathurin, *Approximation properties and non-linear geometry of Banach spaces*. Houston J. Math. **38**(2012), 1135–1148.
[14] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.
[15] A. Brudnyi and Y. Brudnyi, *Metric spaces with linear extensions preserving Lipschitz condition*. Amer. J. Math. **129**(2007), 217–314.
[16] E. Bruè, S. Di Marino and F. Stra, *Linear Lipschitz and* C^1 *extension operators through random projection*. J. Funct. Anal. **280**(2021), 108868.
[17] J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry, *Hyperbolic geometry*, Flavors of geometry, Math. Sci. Res. Inst. Publ., Vol. 31, Cambridge University Press, Cambridge, 1997, 59–115.
[18] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, Vol. 96, Springer, New York, 1990.
[19] H. S. M. Coxeter, *Regular honeycombs in hyperbolic space*. In Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III. Erven P. Noordhoff N. V., Groningen, 1956, pp. 155–169.
[20] M. Cúth, M. Doucha and P. Wojtaszczyk, *On the structure of Lipschitz-free spaces*. Proc. Amer. Math. Soc. **144**(2016), 3833–3846.
[21] M. Cúth, O. F. K. Kalenda and P. Kaplický, *Isometric representation of Lipschitz-free spaces over convex domains in finite-dimensional spaces*. Mathematika **63**(2017), 538–552.
[22] M. W. Davis, *The geometry and topology of Coxeter groups*, London Math. Soc. Monogr. Ser., Vol. 32, Princeton University Press, Princeton, NJ, 2008.

[23] M. Doucha and P. Kaufmann, *Approximation properties in Lipschitz-free spaces over groups.* J. London Math. Soc. **105**(2022), 1681–1701.
[24] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.
[25] D. M. Freeman and C. Gartland, *Lipschitz functions on unions and quotients of metric spaces.* Studia Math. **273**(2023), 29–61.
[26] E. García-Sánchez, D. de Hevia and P. Tradacete, *Free objects in Banach space theory*. In Proceedings of the Biennial Conference of the Spanish Royal Mathematical Society 2022, RSME-Springer, (to appear).
[27] C. Gartland, *Hyperbolic metric spaces and stochastic embeddings*. Preprint, 2024, arXiv:2406.10986.
[28] G. Godefroy, *Extensions of Lipschitz functions and Grothendieck's bounded approximation property*. North-West. Eur. J. Math. **1**(2015), 1–6.
[29] G. Godefroy, *A survey on Lipschitz-free Banach spaces*. Comment. Math. **55**(2015), 89–118.
[30] G. Godefroy, *Lipschitz-free spaces and approximating sequences of projections*. Banach J. Math. Anal. **18**(2024), 23.
[31] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*. Studia Math. **159**(2003), 121–141.
[32] P. Hájek and M. Novotný, *Some remarks on the structure of Lipschitz-free spaces*. Bull. Belg. Math. Soc. Simon Stevin **24**(2017), 283–304.
[33] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*. J. Math. Anal. Appl. **416**(2014), 629–646.
[34] D.J. Ives and D. Preiss, *Not too well differentiable Lipschitz isomorphisms*. Israel J. Math. **115**(2000), 343–353.
[35] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *Extensions of Lipschitz maps into Banach spaces*. Israel J. Math. **54**(1986), 129–138.
[36] P. Kaufmann, *Products of Lipschitz-free spaces and applications*. Studia Math. **226** (2015), 213–227.
[37] G. Lancien and E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*. J. Funct. Anal. **264**(2013), 2323–2334.
[38] J. M. Lee, *Introduction to Riemannian manifolds*, Graduate Texts in Mathematics, Vol. 176, Springer, Cham, 2018.
[39] J. R. Lee and A. Naor, *Extending Lipschitz functions via random metric partitions*. Invent. Math. **160**(2005), 59–95.
[40] S. Ostrovska and M. I. Ostrovskii, *Generalized transportation cost spaces*. Mediterr. J. Math. **16**(2019), 157.
[41] E. Pernecká and R. J. Smith, *The metric approximation property and Lipschitz-free spaces over subsets of* $\mathbb{R}^N$. J. Approx. Theory **199**(2015), 29–44.
[42] L. Potyagailo and E. B. Vinberg, *On right-angled reflection groups in hyperbolic spaces*. Comment. Math. Helv. **80**(2005), 63–73.
[43] T. Veeorg, *Characterizations of Daugavet points and delta-points in Lipschitz-free spaces*. Studia Math. **268**(2023), 213–233.
[44] E. B. Vinberg, *Absence of crystallographic reflection groups in Lobachevsky space of large dimension*. Trudy Moskov. Mat. Obshch. **47**(1984), 68–102. English transl. Trans. Mosc. Math. Soc. (1985), 75–112.
[45] E. B. Vinberg and O. V. Shvartsman, *Discrete groups of motions of spaces of constant curvature*. Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya **29**(1988), 147–259. English transl. in *Geometry II. Spaces of constant curvature*, Encyclopaedia of Math. Sci., Vol. 29, Springer-Verlag, Berlin 1993, 139–248.
[46] N. Weaver, *On the unique predual problem for Lipschitz spaces*. Math. Proc. Cambridge Philos. Soc. **165**(2018), 467–473.
[47] N. Weaver, *Lipschitz algebras*, second edition, World Scientific Publishing Company, 2018.

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