PAPER 2

MAGNETO-HYDRODYNAMIC WAVES IN COMPRESSIBLE FLUIDS WITH FINITE VISCOSITY AND HEAT CONDUCTIVITY*

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ABSTRACT

The general theory of magneto-hydrodynamic waves in an ideal conducting fluid embedded in a uniform field of magnetic induction, and the application of the theory to the systematic analysis of the various modes of propagation in incompressible and compressible fluids have been presented by the author in two earlier papers [1, 2]. In these papers, however, no effort was made to include the thermodynamics of the situation, which amounts to the tacit assumption that the fluid is of zero heat conductivity. In this case the resulting modes are of two kinds: isothermal (v-modes) and adiabatic (p-modes).

In this paper we first establish the conservation laws of momentum and energy for a (macroscopic) compressible fluid with finite viscosity and finite thermal and electrical conductivities, which is embedded in a uniform field of magnetic induction, and we then derive quite generally the exact (non-linearized) equation governing the distribution of temperature in such a fluid. Next, making use of the linearized magneto-hydrodynamic wave equation in the fluid velocity, combined with the resulting heat diffusion equation and with the equation of state of the fluid, and applying the mathematical techniques developed earlier, we obtain a higher order partial differential equation in the fluid temperature from which ensue all the temperature modes.

In particular, we examine in detail the behavior of plane homogeneous waves, and it is shown that a compressible fluid with the indicated properties sustains altogether six different modes, two of which are pure shear modes, devoid of density, pressure, and hence temperature fluctuations (v-modes), while the remaining four are shear-compression waves accompanied necessarily by density, pressure, and temperature fluctuations (p-modes). The two shear modes, which are isothermal, comprise a slightly attenuated Alfvén wave, and a highly attenuated viscous mode, sometimes referred to as a vorticity mode. The four shear-compression modes have in general very complex properties, but in the low frequency and low heat conductivity case they are easily identified as (1) a modified (adiabatic) sound wave slightly attenuated;

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(2) a slightly attenuated modified Alfvén p-wave; (3) a highly attenuated viscous wave; and (4) a highly attenuated thermal wave governed in the main by the thermal properties of the medium.

1. INTRODUCTION

The underlying fundamental notions in the theory of magneto-hydrodynamic waves in incompressible fluids were originally due to Alfvén and his co-workers [3], but there had been lacking for some time a systematic analysis of the linearized, unbounded media, and boundary value problems in the field of magneto-hydrodynamic waves in incompressible and compressible fluids. To this end we undertook to give, in two earlier papers [1, 2], hereinafter to be referred to as I and II respectively, such a systematic study. The first paper deals mainly with the general theory of plane homogeneous waves and of time harmonic cylindrical waves propagating in a homogeneous and isotropic conducting fluid of infinite extent embedded in a uniform field of magnetic induction. The medium is assumed to consist of an ideal fluid devoid of viscosity and expansive friction, which is characterized (in rationalized mks units) by the rigorously constant macroscopic parameters μ, ϵ, and σ, where $\mu\epsilon = c^{-2}$ and σ is the (ohmic) conductivity. The second paper deals with the application of the general theory to the determination of the modes of propagation and to the computation of the corresponding propagation constants in incompressible and compressible fluids.

However, in these two papers no effort was made to include the thermodynamics of the situation, which amounts to the tacit assumption that the fluid is of zero heat conductivity. In this case the resulting modes are of two kinds: isothermal (v-modes) and adiabatic (p-modes). In this paper we continue with the purely macroscopic approach, for we believe that the results obtained are of considerable value and may serve as a guide to the more complicated problems in which the macroscopic approach is no longer tenable. The medium is now assumed to be a conducting compressible fluid endowed with finite viscosity and heat conductivity, embedded in a uniform field of magnetic induction.

First, we examine anew the conservation laws of momentum and energy and we then derive quite generally the exact (non-linearized) equation governing the distribution of temperature in such a fluid. Next, making use of the linearized magneto-hydrodynamic wave equation in the fluid velocity, combined with the linearized form of the heat diffusion equation and with the equation of state of the fluid, and applying the mathematical techniques developed in II, we obtain a higher order partial differential equation in the fluid temperature from which ensue all the temperature modes.

We examine, in particular, the structure of plane homogeneous waves in a conducting compressible fluid with finite viscosity and heat conductivity, and it is shown that a fluid with the indicated properties sustains altogether six different modes, two of which are pure *shear* waves, devoid of density, pressure, and temperature fluctuations (v-modes), while the remaining four are *shear-compression* modes which of necessity are accompanied by density, pressure, and temperature fluctuations (p-modes).

The two shear modes, which are isothermal, consist of a slightly attenuated Alfvén wave and a highly attenuated viscous wave, sometimes referred to as a vorticity mode. The four shear-compression modes have in general very complex properties, but in the low frequency and low heat conductivity case they are readily identified as: (1) a slightly attenuated modified (adiabatic) sound wave; (2) a slightly attenuated modified Alfvén p-wave; (3) a highly attenuated viscous wave, and (4) a highly attenuated temperature wave governed in the main by the thermal properties of the medium.

2. CONSERVATION LAWS

In order to establish the equation governing the distribution of temperature in an unbounded magneto-hydrodynamic field we need to examine first the conservation laws of momentum and energy as they apply to a rigid volume V within a bounding surface S rigorously fixed in the observer's inertial frame of reference. The heat diffusion equation then ensues quite generally by combining the two conservation laws as indicated below.

Conservation of momentum

The law of conservation of momentum states simply that the time rate of change of the total mechanical plus electromagnetic momentum contained within the fixed volume V is equal to the mechanical force acting across the bounding surface on the fluid contained within the volume, plus the *influx* of both electromagnetic and mechanical momentum across the surface S. Expressed in tensor notation the law becomes

$$(d/dt)\int_V (\rho v_i + g_i)\, d\tau = \int_S P_{in}\, da + \int_S T_{in}\, da - \int_S (\rho v_i)\, v_n\, da, \qquad (1)$$

in which the subscript n refers to the outward normal. In the volume integral ρv_i denotes the mechanical momentum density and $g_i = \mu\epsilon S_i$ the electromagnetic momentum density.

The first surface integral on the right of (1) denotes the total mechanical force acting on the fluid contained within the bounding surface as deduced from the mechanical stress tensor [4]

$$P_{ik} = -\left(p + \tfrac{2}{3}\bar{\mu}\,\frac{\partial v_\alpha}{\partial x_\alpha}\right)\delta_{ik} + \bar{\mu}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right), \tag{2}$$

wherein p is the pressure and $\bar{\mu}$ the coefficient of viscosity, and whose (tensor) divergence leads to the familiar Stokes–Navier equation. The second surface integral represents the rate at which electromagnetic momentum is flowing into the volume and is computed in terms of the *total* Maxwell's electromagnetic stress tensor, Eq. (I-8), which in the present notation becomes

$$T_{ik} = \epsilon(e_i e_k - \tfrac{1}{2}e^2\delta_{ik}) + \mu(H_i H_k - \tfrac{1}{2}H^2\delta_{ik}), \tag{3}$$

and in which it is recalled $\mu\epsilon = c^{-2}$. Finally, the third surface integral on the right of (1) represents merely the influx of mechanical momentum transported across the surface S by the moving fluid.

To obtain from (1) the differential form of the law of conservation of momentum we first convert all four integrals into simple volume integrals by transposing under the sign of integration the time derivative acting on the volume integral and by applying the (tensor) divergence theorem to the remaining surface integrals. Then, making use of the law of conservation of mass (equation of continuity)

$$\partial\rho/\partial t + \partial(\rho v_\alpha)/\partial x_\alpha = 0, \tag{4}$$

we obtain from (1) the differential form

$$\rho(dv_i/dt) + \partial g_i/\partial t = \partial P_{i\alpha}/\partial x_\alpha + \partial T_{i\alpha}/\partial x_\alpha. \tag{5}$$

Finally, introducing into (5) the Lorentz force density of electromagnetic origin which, according to Eq. (I-10), can be written in the form

$$f_i = \partial T_{i\alpha}/\partial x_\alpha - \partial g_i/\partial t, \tag{6}$$

we obtain the Eulerian equations of motion,

$$\rho(dv_i/dt) = f_i + \partial P_{i\alpha}/\partial x_\alpha, \tag{7}$$

for a compressible fluid with finite viscosity.

Conservation of energy

The law of conservation of energy states in the present instance that the time rate of change of the *total* (kinetic plus internal plus electromagnetic) energy stored within the fixed volume V is equal to the sum of three terms: the rate of doing work of the mechanical forces acting on the fluid within the surface S, the influx of kinetic plus internal energy transported across the bounding surface by the moving fluid, and the influx of heat plus electromagnetic energy across the surface S. Expressed in tensor notation the law becomes

$$(d/dt)\int_V [\tfrac{1}{2}\rho v^2+\rho U+(\tfrac{1}{2}\epsilon e^2+\mu H^2)]\,d\tau=\int_S v_\alpha P_{\alpha n}\,da$$
$$-\int_S(\tfrac{1}{2}\rho v^2+\rho U)\,v_n\,da-\int_S(q_n+S_n)\,da, \tag{8}$$

in which U denotes the intrinsic internal energy of the fluid, q_i the heat flow vector, and S_i the familiar Poynting's vector. Once again the subscript n refers to the outward normal.

To obtain the differential form of the law of conservation of energy we proceed as before by transforming every integral in (8) into a simple volume integral. Thus, applying the divergence theorem to the surface integrals and again making use of the equation of continuity (4), we obtain the law in the form

$$\rho v_\alpha\frac{dv_\alpha}{dt}+\rho\frac{dU}{dt}+\frac{\partial}{\partial t}(\tfrac{1}{2}\epsilon e^2+\tfrac{1}{2}\mu H^2)=\frac{\partial(v_\alpha P_{\alpha\beta})}{\partial x_\beta}-\frac{\partial q_\alpha}{\partial x_\alpha}-\frac{\partial S_\alpha}{\partial x_\alpha}. \tag{9}$$

To reduce this equation further we note from (7) that

$$\rho v_\alpha(dv_\alpha/dt)=f_\alpha v_\alpha+v_\alpha(\partial P_{\alpha\beta}/\partial x_\beta) \tag{10}$$

and we recall that, according to Eq. (I-7), we have in the present notation

$$f_\alpha v_\alpha=-J^2/\sigma-(\partial/\partial t)(\tfrac{1}{2}\epsilon e^2+\tfrac{1}{2}\mu H^2)-\partial S_\alpha/\partial x_\alpha. \tag{11}$$

Hence, replacing $f_\alpha v_\alpha$ in (10) by (11) and making use of the resulting expression in (9), we obtain the much simpler expression

$$\rho\frac{dU}{dt}=-\frac{\partial q_\alpha}{\partial x_\alpha}+\frac{\partial v_\alpha}{\partial x_\beta}P_{\alpha\beta}+\frac{J^2}{\sigma} \tag{12}$$

which expresses in differential form the principle of conservation of energy for a conducting compressible fluid with finite viscosity.

Heat diffusion equation

To deduce from (12) the equation governing the distribution of temperature in a magneto-hydrodynamic field we assume first that the fluid is endowed with a *constant* heat conductivity K. Thus, the heat flow vector q_i may be written as

$$q_i=-K(\partial T/\partial x_i), \tag{13}$$

where T is the absolute temperature, whence the divergence of the heat flow vector becomes

$$\partial q_\alpha / \partial x_\alpha = -K(\partial^2 T / \partial x_\alpha^2) = -K\nabla^2 T. \qquad (14)$$

Next, we observe that the second term on the right of (12) may be resolved into two terms

$$\frac{\partial v_\alpha}{\partial x_\beta} P_{\alpha\beta} = -p\frac{\partial v_\alpha}{\partial x_\alpha} + \Phi, \qquad (15)$$

where the first term denotes the rate at which work is done by the pressure p in compressing the fluid inside the surface S, and where

$$\Phi = \bar{\mu}\left[\frac{\partial v_\alpha}{\partial x_\beta}\left(\frac{\partial v_\beta}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_\beta}\right) - \frac{2}{3}\left(\frac{\partial v_\alpha}{\partial x_\alpha}\right)^2\right] \qquad (16)$$

is the viscous dissipation function (Goldstein, 1943 [5]), a quadratic function in the velocity components which is always positive definite [6].

Substituting (14) and (15) into (12) and reverting to Gibbsian vector notation we obtain

$$\rho(dU/dt) + p(\nabla \cdot \mathrm{v}) = K\nabla^2 T + \Phi + J^2/\sigma, \qquad (17)$$

which is the equation governing the distribution of temperature in a conducting compressible fluid with finite viscosity and heat conductivity embedded in a uniform field of magnetic induction. We note that the equation contains two quadratic source terms: the dissipation function Φ due to finite viscosity and the electromagnetic dissipation function J^2/σ due to finite electrical conductivity. From a purely macroscopic point of view Eq. (17) is exact, having assumed that the fluid is endowed with a constant thermal conductivity K and a constant ohmic conductivity σ. To apply (17) to a specific case it is of course necessary to invoke an equation of state linking the intrinsic energy U to the temperature and to other pertinent thermodynamic variables.

3. LINEARIZED EQUATIONS

The foregoing discussion is quite general and in order to apply the theory to the determination of the plane wave modes in a compressible fluid with finite viscosity and heat conductivity we must of course linearize (17) and relate it to the magneto-hydrodynamic wave equation applicable to the present case.

Assuming at the outset that we can neglect the electric displacement current ($\epsilon = 0$), and confining our attention exclusively to time harmonic waves, we obtain from Eq. (I-66) the linearized magneto-hydrodynamic wave equation

$$\nabla^2(\mathbf{F}/\sigma + B_0^2 \mathbf{v}_t) = -i\omega\mu\mathbf{F} + B_0^2(\nabla_t^2 \mathbf{v}_t - \nabla_t \nabla_t . \mathbf{v}_t), \tag{18}$$

in which B_0 denotes the externally applied uniform field of magnetic induction and in which the vector $\mathbf{F}$, as deduced from Eqs. (I-21) and (7), becomes in the present instance

$$\mathbf{F} = -i\omega\rho_0 \mathbf{v} + \nabla p - \tfrac{1}{3}\rho_0 \nu \nabla\nabla . \mathbf{v} - \rho_0 \nu \nabla^2 \mathbf{v}, \tag{19}$$

where ρ_0 denotes the equilibrium density and ν the kinematic viscosity.

Next, we take up the linearization of the heat diffusion equation (17). Although it is possible to proceed quite generally with an arbitrary equation of state for the fluid, we find it convenient to assume initially that the fluid obeys the law of perfect gases,

$$p = (\gamma - 1)\, c_v \rho T, \tag{20}$$

where c_v is the specific heat at constant volume and γ the ratio of specific heats, $\gamma = c_p/c_v$. In this case the internal energy depends only on the absolute temperature, $dU = c_v dT$.

Hence, letting ρ_0, p_0, and T_0 denote the constant values of the chosen thermodynamic variables corresponding to the equilibrium state, and letting ρ, p, and T denote from now on the small departures from the equilibrium state, we obtain from (17), upon dropping all quadratic terms, the linearized form

$$(K\nabla^2 + i\omega\rho_0 c_v)\, T = p_0 \nabla . \mathbf{v}, \tag{21}$$

which yields $\nabla . \mathbf{v}$ once we know the temperature distribution. Finally, to make the system determinate in the three dependent variables p, T, and $\nabla . \mathbf{v}$ we need, in addition to (18) and (21), the expression

$$i\omega p = p_0 \nabla . \mathbf{v} + (\gamma - 1)\, i\omega\rho_0 c_v T, \tag{22}$$

which is readily deduced by eliminating the (excess) density ρ with the aid of the linearized forms of (4) and (20).

It now remains to make use of the foregoing equations to determine the structure of the plane wave modes which can exist in the presence of finite viscosity and heat conductivity. For the purpose, we adopt here the elementary plane wave solutions illustrated in Figs. I-1 and II-12 and described in detail in §§ II-2 and II-4·1. It is shown that the solutions generated by the velocity vector $\mathbf{v}_1$, Eq. (II-11), lead in the present instance to two distinct pure shear *velocity* modes, which are devoid of density, pressure, and temperature fluctuations, and which are therefore *isothermal*. Similarly, the solutions generated by the linear combination $\mathbf{v}$, Eq. (II-33), lead in this case to four distinct shear-compression *pressure* modes, which are necessarily accompanied by density, pressure, and temperature fluctuations, and which therefore will henceforth be referred to as *temperature* modes.

4. ISOTHERMAL MODES

Following the techniques outlined in § II-1, we first insert (19) into (18), and we then proceed to the reduction of the resulting vector equation (18) to three scalar equations by seeking the z-component, the divergence, and the z-component of the curl. This last procedure yields the equation

$$\{[1 - ia(1 - iq\nabla^2)]\ \nabla^2 + k_a^2 - \nabla_t^2\}\ (\hat{e}_z . \nabla \times \mathbf{v}) = 0 \tag{23}$$

in which the unit vector $\hat{e}_z$ denotes the direction of the externally applied magnetic field. Here, k_a is the wave number associated with Alfvén's phase velocity

$$k_a = \omega / V_a = \omega(\mu\rho_0)^{1/2}/B_0 \tag{24}$$

and a and q are two convenient parameters,

$$a = \omega\rho_0/\sigma B_0^2 \quad \text{and} \quad q = \nu/\omega, \tag{25}$$

which measure respectively the hydromagnetic coupling and the viscous damping. The parameter a, which vanishes in the limit of infinite conductivity, is dimensionless, whereas q has the dimensions of a cross-section and vanishes when the kinematic viscosity goes to zero. As a check, it is observed that (23), after putting $q = 0$, becomes identical to Eq. (II-4) upon placing $\epsilon = 0$.

Next, we observe that for plane waves the velocity vector $\mathbf{v}_1$, Eq. (II-11), is perpendicular to the plane defined by the direction of the magnetic field and the vector propagation constant $\mathbf{k}$. Therefore, as shown in Eq. (II-12), this vector is divergenceless (pure shear) and has no z component; furthermore, assuming that the vector $\mathbf{k}$ does not coincide with the direction of the magnetic field, we have in (23) that $\hat{e}_z . \nabla \times \mathbf{v}$ is non-zero. Hence, to satisfy (23) we need merely replace ∇ by $i\mathbf{k}$ and, equating the bracket to zero, we obtain the quadratic in k^2

$$aqk^4 + (\cos^2\theta - ia)\ k^2 - k_a^2 = 0, \tag{26}$$

in which θ denotes the angle between the vector $\mathbf{k}$ and the direction of the magnetic field (Fig. I-1). As a check we note that, putting $q = 0$ in (26), yields immediately the limiting ($\epsilon \to 0$) form of Eq. (II-13).

Equation (26) has two distinct roots in k^2 and therefore leads to two shear modes. The exact roots of (26) can be readily written down, but we prefer to examine the limiting form of the roots when $aqk_a^2 \ll 1$, which corresponds to the case of high electrical conductivity, low viscosity, and moderately low frequencies. In this case, the roots of (26) are given approximately, to first order of small quantities, by

$$k_+^2 \approx \frac{k_a^2}{\cos^2\theta - ia}\left\{1 - \frac{aqk_a^2}{(\cos^2\theta - ia)^2}\right\} \tag{27}$$

and

$$k_-^2 \approx -\frac{\cos^2\theta - ia}{aq}\left\{1 + \frac{aqk_a^2}{(\cos^2\theta - ia)^2}\right\}, \tag{28}$$

from which we can readily deduce the corresponding phase velocities and attenuation factors. The first mode, characterized by the wave number k_+, is an ordinary Alfvén wave slightly attenuated by the presence of finite conductivity and finite viscosity. The second mode, governed by k_-, is a highly attenuated pure shear or vorticity mode characteristic of viscous layer phenomena. Both modes are solenoidal and, according to (21), isothermal. Therefore, as pointed out before, these modes are entirely devoid of density, pressure, and temperature fluctuations. Finally, we note in passing that, if $\cos^2\theta \ll a$, then the leading terms of (27) and (28) become respectively

$$k_+^2 \approx i\omega\mu\sigma \quad \text{and} \quad k_-^2 \approx i/q = i\omega/\nu, \tag{29}$$

indicating that in this limit the Alfvén wave degenerates into a 'skin' wave governed in the main by the electromagnetic properties of the medium, whereas the viscous mode becomes a true vorticity mode characterized mainly by the kinematic viscosity.

5. TEMPERATURE MODES

Continuing with the method of attack outlined in the preceding section, we first insert (19) into (18) and then proceed to compute the z-component and the divergence of the resulting vector equation. In this manner we obtain two scalar equations involving the variables v_z, $\nabla\cdot\mathbf{v}$, and the (excess) pressure p. Combining these two equations we first eliminate v_z, obtaining a single equation in $\nabla\cdot\mathbf{v}$ and p. Next, making use of (22) we eliminate p in terms of $\nabla\cdot\mathbf{v}$ and T to obtain finally, instead of Eqs. (II-7) and (II-8), the more involved expressions

$$\gamma k_s^2(1 - iq\nabla^2)\, v_z = -(1 - \tfrac{1}{3}iq\gamma k_s^2)\,(\partial/\partial z)\,(\nabla\cdot\mathbf{v}) - (i\omega/T_0)\,(\partial T/\partial z), \tag{30}$$

$$\{[\nabla^2 + (k_a^2 - ia\nabla^2)\,(1 - iq\nabla^2)]\,[\nabla^2 + \gamma k_s^2(1 - \tfrac{4}{3}iq\nabla^2)] - (1 - \tfrac{1}{3}iq\gamma k_s^2)\,\nabla^2\nabla_t^2\}$$
$$\times(\nabla\cdot\mathbf{v}) + [\partial^2/\partial z^2 + (k_a^2 - ia\nabla^2)\,(1 - iq\nabla^2)]\,(i\omega/T_0)\,\nabla^2 T = 0, \tag{31}$$

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in which we now have the additional variable T. In these equations we have introduced, in addition to the wave number k_a and the parameters a and q, as defined by (24) and (25), the wave number associated with the (adiabatic) velocity of sound in the medium $k_s = \omega/V_s$, $V_s^2 = \gamma p_0/\rho_0$. As a check we observe that, in the absence of viscosity ($q = 0$) and in the adiabatic limit of vanishing heat conductivity ($K \to 0$), making use of (21) to eliminate T, equations (30) and (31) reduce respectively to the limiting ($\epsilon \to 0$) form of Eqs. (II-7) and (II-8). Finally, to obtain the higher order partial differential equation governing the distribution of temperature we need only substitute (21) into (31) to eliminate $\nabla . \mathbf{v}$; however, no purpose is served by writing down this more complicated equation, since we wish to examine plane waves at once.

For the purpose, we choose a velocity vector $\mathbf{v}$ which lies in the plane of the wave normal and the direction of the externally applied field in accordance with Eqs. (II-33) and (II-34), as illustrated in Fig. II-12. We observe that these shear-compression modes, Eqs. (II-33) and (II-34), are such that $\hat{e}_z . \nabla \times \mathbf{v} = 0$; hence, (23) is identically satisfied, and we must now make use of (31) and (21) to determine the various temperature modes. To this end we substitute (21) into (31), and replacing ∇ by $i\mathbf{k}$, we obtain finally a fourth-order algebraic equation in k^2,

$$\begin{aligned} &\{[k^2 - (k_a^2 + iak^2)\ (1 + iqk^2)]\ [k^2 - \gamma k_s^2(1 + \tfrac{4}{3}iqk^2)] \\ &\quad - (1 - \tfrac{1}{3}iq\gamma k_s^2)\ k^2 k_x^2\}\ (k^2 - i\omega\rho_0 c_v/K) \\ &\quad - \{[k^2 - (k_a^2 + iak^2)\ (1 + iqk^2)]\ c - k_x^2\}\ (\gamma - 1)\ i\omega\rho_0 c_v k^2/K = 0, \end{aligned} \tag{32}$$

which has four distinct roots and which, therefore, yields four shear-compression modes accompanied by density, pressure, and temperature fluctuations; that is, the counterpart of the *pressure* modes discussed in § II-4.

No attempt will be made here to examine in detail the exact roots of (32), which probably can only be handled numerically, but we can apply to (32) various tests of its validity and we can examine the limiting form of the roots in various cases of practical interest. As a first test, let us make the externally applied field vanish; i.e. let us remove all hydromagnetic coupling ($B_0 = 0$). In this case, both k_a^2 and a become infinite as B_0^{-2}, which reduces (32) to the simpler equation

$$[k^2 - \gamma k_s^2(1 + \tfrac{4}{3}iqk^2)]\ (k^2 - i\omega\rho_0 c_v/K) - (\gamma - 1)\ i\omega\rho_0 c_v k^2/K = 0, \tag{33}$$

from which ensue the acoustic, vorticity, and thermal modes characteristic of an acoustic field with finite thermal conductivity. As an example, let us examine (33) in the limit of small frequencies ($\omega \to 0$); in this case we obtain from (33), as long as K remains finite, a mode with the wave number

$$k^2 \approx k_s^2(1 - \tfrac{4}{3}iqk_s^2)^{-1}, \tag{34}$$

which corresponds to an *adiabatic* sound wave slightly attenuated by the presence of finite viscosity. On the other hand, in the limit of very large frequencies ($\omega \to \infty$), we deduce from (33) a mode with the wave number

$$k^2 \approx \gamma k_s^2(1 - \tfrac{4}{3}iq\gamma k_s^2)^{-1}, \tag{35}$$

which represents, for small viscosities, a slightly attenuated sound wave propagating with the *isothermal* phase velocity.

As a second test, let us examine the limiting form of (32) in the case of infinite electrical conductivity ($a = 0$), zero viscosity ($q = 0$), and zero heat conductivity ($K = 0$). In this case (32) reduces to

$$(k^2 - k_a^2)\ (k^2 - k_s^2) = k^2 k_x^2, \tag{36}$$

which fully confirms the limiting ($\epsilon \to 0$) form of Eq. (II-36) and from which we deduced in § II-4 the properties of the ideal magneto-acoustic modes.

Finally, to illustrate with one example the application of (32) to special cases of practical interest, we propose to examine the limiting form of (32) in the case of vanishingly small heat conductivity. Thus, letting $K \to 0$ in (32), we obtain the cubic in k^2

$$[k^2 - (k_a^2 + iak^2)\ (1 + iqk^2)]\ [k^2 - k_s^2(1 + \tfrac{4}{3}iqk^2)] = (1 - \tfrac{1}{3}iqk_s^2)\ k^2 k_x^2, \tag{37}$$

which now supersedes our earlier Eqs. (II-41) and (II-43), and from which ensue three shear-compression temperature modes: a modified (adiabatic) sound wave, a modified Alfvén pressure wave, and a modified vorticity mode. The fourth temperature mode, which has disappeared from (32) by putting $K = 0$, is seen to be governed mainly by the thermal properties of the medium and is, therefore, a highly attenuated wave.

Other cases of interest that can be examined profitably include infinitely high heat conductivity, which according to (21) leads to an isothermal temperature distribution, and the cases of both high and low frequencies. In all cases the computations can be greatly simplified if we can assume that the fluid possesses extremely high electrical conductivity ($a \ll 1$) and very low viscosity ($qk_a^2 \ll 1$), for then familiar perturbation methods such as were employed in II are available to us. Finally, to complete the discussion we observe that, once the wave number k has been determined from (32) or from any of its limiting forms for a particular shear-compression mode, then the corresponding angular parameter ϕ which defines the chosen linear combination (II-33) can be readily obtained by applying the technique outlined in § 4·1.

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Discussion

Spitzer: How many modes vanish if the coefficient of viscosity goes to zero?

Baños: With finite viscosity I get six modes, two shear modes and four temperature modes. With vanishing viscosity one shear mode and three temperature modes remain.

Swann: You have referred to the thermodynamics involved in the derivations. When the mechanism is expressed explicitly in terms of viscosity one needs no thermodynamics in the ordinary sense of the word except when you use the equation of a perfect gas. Am I right in saying that you do not use thermodynamics except in that case?

Baños: Yes.

Schatzman: What would come out of the equations if the conductivity depends on the temperature and the density?

Baños: This is a difficult question which I cannot answer immediately.

Swann: A perturbation method could perhaps give the answer.

Spitzer: Professor Baños has given a very complete and elegant analysis of infinitesimal waves in a fluid—van de Hulst's category I B. An interesting result on finite hydromagnetic waves in a plasma has been obtained by Kruskal, Rosenbluth and others in the U.S.A.; this provides at least one result under category II C. The analysis considers a solitary hydromagnetic disturbance, traveling perpendicular to the magnetic field, in a plasma in which no collisions occur. The orbits of the charged particles in the time variable magnetic field are taken into account; the gas temperature is assumed zero. The analysis goes through without difficulty for a velocity up to twice the Alfvén velocity for infinitesimal disturbances. At this critical velocity the magnetic field rises to three times its value in front of and behind the pulse.