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ABSTRACT CHARACTERIZATIONS OF FIXED POINT SUBALGEBRAS OF THE ROTATION ALGEBRA

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ABSTRACT. We determine abstract characterizations of the fixed point subalgebras of the rotation algebra $\mathcal{A}_\theta$ under the automorphisms $U \mapsto V$, $V \mapsto U^{-1}$, $U \mapsto e^{-\pi i\theta}U^{-1}V$, $V \mapsto U^{-1}$ and $U \mapsto V$, $V \mapsto e^{-\pi i\theta}U^{-1}V$.

0. **Introduction.** Recently considerable progress has been made in the study of certain C^*-algebras abstractly characterized by generators and relations. One of the most well-known examples of such a C^*-algebra is the rotation algebra, $\mathcal{A}_\theta$, the universal C^*-algebra generated by two unitaries U and V satisfying the relation $VU = \rho UV$ with $\rho = e^{2\pi i\theta} = e^{i\phi}$, $\theta \in [0, 1)$, $\phi = 2\pi\theta$, which has been shown to be a limit of two direct sums of circle algebras for θ irrational [7]. It is known that $\mathrm{SL}(2, \mathbb{Z})$ acts naturally on $\mathcal{A}_\theta$ [5], [8], and for θ irrational with suitable Diophantine properties, any diffeomorphism of $\mathcal{A}_\theta$ is the product of an inner automorphism and diffeomorphisms coming from the actions of $\mathbb{T}^2$ and $\mathrm{SL}(2, \mathbb{Z})$ on $\mathcal{A}_\theta$ [6] (when θ does not have such properties this decomposition can fail [13]). Consequently it is natural to investigate the fixed point subalgebras of $\mathcal{A}_\theta$ associated to the $\mathrm{SL}(2, \mathbb{Z})$ action and in [8] we gave a complete classification. The infinite order elements of $\mathrm{SL}(2, \mathbb{Z})$ give rise to 'trivial' fixed point subalgebras [6] while the finite order elements ($\neq I_2$) of $\mathrm{SL}(2, \mathbb{Z})$ have Trace $= 0, \pm 1, -2$ with their conjugacy classes represented by four elements and therefore, up to isomorphism, giving four fixed point subalgebras [8]. To be consistent with earlier papers we wish to introduce the following notation for these four special cases. Let σ, τ, ζ and η be the automorphisms of $\mathcal{A}_\theta$ defined by,

$$\begin{aligned} \sigma(U) &= U^{-1}, & \tau(U) &= V, & \zeta(U) &= e^{-\pi i\theta}U^{-1}V, & \eta(U) &= V, \\ \sigma(V) &= V^{-1}, & \tau(V) &= U^{-1}, & \zeta(V) &= U^{-1}, & \eta(V) &= e^{-\pi i\theta}U^{-1}V, \end{aligned}$$

Note that $\sigma^2 = \zeta^3 = 1$, $\tau^2 = \eta^3 = \sigma$ and $\eta^2 = \zeta$. We will denote the corresponding fixed point subalgebras by $\mathcal{A}_\theta^\sigma$, $\mathcal{A}_\theta^\tau$, $\mathcal{A}_\theta^\zeta$ and $\mathcal{A}_\theta^\eta$ respectively. Clearly $\mathcal{A}_\theta^\tau \subseteq \mathcal{A}_\theta^\sigma$, $\mathcal{A}_\theta^\eta \subseteq \mathcal{A}_\theta^\sigma$ and $\mathcal{A}_\theta^\eta \subseteq \mathcal{A}_\theta^\zeta$.

The study of these fixed point subalgebras was initiated by Bratteli, Elliott, Evans and Kishimoto who characterized $\mathcal{A}_\theta^\sigma$ in terms of generators and relations for general θ [1], with Kumjian computing the K-theory [14], and also as a C^*-bundle over a 2-sphere orbifold with four singular points for θ rational [2]. We studied the three remaining cases

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$\mathcal{A}_\theta^\gamma$, $\gamma = \tau, \zeta, \eta$ when θ is rational [10], [11], [12], proving that they are all C^*-bundles over a 2-sphere orbifold with three singular points. We now wish to consider arbitrary θ and in particular θ irrational. In this paper we give abstract characterizations of $\mathcal{A}_\theta^\tau$ for all $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ and $\frac{3}{4}$, $\mathcal{A}_\theta^\zeta$ for all $\theta \neq 0, \frac{1}{2}, \frac{1}{3}$ and $\frac{2}{3}$ and $\mathcal{A}_\theta^\eta$ for all θ irrational (Theorem 5.1.7). Specifically they can all be written as the universal C^*-algebra generated by two self-adjoint elements satisfying three relations. Moreover, if θ and θ' are irrational $\mathcal{A}_\theta^\gamma$ and $\mathcal{A}_{\theta'}^\gamma$ are isomorphic if and only if $\theta' = \theta$ or $1 - \theta$, for $\gamma = \tau, \zeta$ and η. The proofs owe much in spirit to the case of the automorphism σ, also referred to as the flip, considered by Bratteli, Elliott, Evans and Kishimoto [1] although the computations involved are now more technical. Recently Bratteli and Kishimoto [3] have proved that $\mathcal{A}_\theta^\sigma$ is AF when θ is irrational, by building on a characterization of $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$ given by Kumjian [14] and work of Putnam [16]. Unfortunately the techniques employed do not directly apply to the other three cases but it would certainly be interesting to know if these algebras are AF as well. So far not even their K-theory is determined (when θ is irrational), which one might conjecture is the same as in the rational case, as is true for $\mathcal{A}_\theta^\sigma$ [14]. We hope that our abstract characterizations, though rather complex in form, can shed some light onto these questions. Also worth noticing is that the almost Mathieu operator $U + U^* + V + V^*$ is an element of $\mathcal{A}_\theta^\tau$ and in this way we might get information about its properties. We would like to take this opportunity to thank Prof. Elliott for several discussions throughout the course of this work and also the referee for helpful comments on an earlier version of this paper.

In detail the contents of this paper are as follows. In Sections 1 and 2 we characterize the fixed point polynomial subalgebras corresponding to the automorphisms τ, ζ and η in terms of vector space bases. In Section 3 we derive relations that certain elements of these polynomial algebras satisfy and proceed to characterize the algebras in terms of generators and relations. In Section 4 we prove some technical decomposition lemmas. Section 5 contains the main theorems which are proved assuming a certain uniform bound (Theorem 5.1.2). Sections 6, 7 and 8 contain the proof of this bound for the fixed point algebras $\mathcal{A}_\theta^\tau$, $\mathcal{A}_\theta^\zeta$ and $\mathcal{A}_\theta^\eta$ respectively.

1. **Preliminaries.**

DEFINITION 1.1.1. Let $\mathcal{P}_\theta = \mathrm{Span}\{U^n V^m : n, m \in \mathbb{Z}\} \subset \mathcal{A}_\theta$ be the $*$-subalgebra of polynomials in the unitaries U and V. If $\gamma \in \mathrm{Aut}(\mathcal{A}_\theta)$ let $\mathcal{A}_\theta^\gamma$ and $\mathcal{P}_\theta^\gamma$ denote the corresponding fixed point subalgebras of $\mathcal{A}_\theta$ and $\mathcal{P}_\theta$ respectively.

PROPOSITION 1.1.2. *If $\gamma, \delta \in \mathrm{Aut}(\mathcal{A}_\theta)$ with $\delta^r = \gamma$, $r \in \mathbb{N}$, then $r^{-1} \sum_{i=0}^{r-1} \delta^i$ restricted to $\mathcal{A}_\theta^\gamma$ is an idempotent with image $\mathcal{A}_\theta^\delta$ which maps $\mathcal{P}_\theta^\gamma$ onto $\mathcal{P}_\theta^\delta$.*

DEFINITION 1.1.3. For $\delta \in \mathrm{Aut}(\mathcal{A}_\theta)$ with $\delta^r = 1$, $r \in \mathbb{N}$, define $\delta(n,m) = \rho^{nm/2} \sum_{i=0}^{r-1} \delta^i(U^n V^m)$ and $\mathcal{P}_\theta^\delta = \mathrm{Span}\{\delta(n,m) : n, m \in \mathbb{Z}\}$.

For the special cases τ, ζ and η we wish to consider, using Definition 1.1.3 and the relation $VU = \rho UV$, it is straightforward to prove:

PROPOSITION 1.1.4. The following relations hold for every $n, m, k, \ell \in \mathbb{Z}$.

In $\mathcal{P}_\theta^\tau$,

$$\tau(n,m)^* = \tau(n,m), \tag{τ1}$$

$$\tau(n,m) = \tau(-n,-m), \tag{τ2}$$

$$\tau(n,m) = \tau(m,-n), \tag{τ3}$$

$$\begin{aligned}\tau(n,m)\tau(k,\ell) = {}& \rho^{(mk-n\ell)/2}\tau(n+k,m+\ell) + \rho^{-(mk-n\ell)/2}\tau(n-k,m-\ell)\\ &+ \rho^{-(m\ell+nk)/2}\tau(n-\ell,m+k) + \rho^{(m\ell+nk)/2}\tau(n+\ell,m-k).\end{aligned} \tag{τ4}$$

In $\mathcal{P}_\theta^\zeta$,

$$\zeta(n,m)^* = \zeta(-n,-m), \tag{ζ1}$$

$$\zeta(n,m) = \zeta\big(-(n+m),n\big), \tag{ζ2}$$

$$\zeta(n,m) = \zeta\big(m,-(n+m)\big), \tag{ζ3}$$

$$\begin{aligned}\zeta(n,m)\zeta(k,\ell) = {}& \rho^{(mk-n\ell)/2}\zeta(n+k,m+\ell) + \rho^{-(nk+m(k+\ell))/2}\zeta(n-(k+\ell),m+k)\\ &+ \rho^{(m\ell+n(k+\ell))/2}\zeta\big(n+\ell,m-(k+\ell)\big).\end{aligned} \tag{ζ4}$$

In $\mathcal{P}_\theta^\eta$,

$$\eta(n,m)^* = \eta(n,m), \tag{η1}$$

$$\eta(n,m) = \eta(-n,-m), \tag{η2}$$

$$\eta(n,m) = \eta\big(-(n+m),n\big), \tag{η3}$$

$$\eta(n,m) = \eta(-m,n+m), \tag{η4}$$

$$\begin{aligned}\eta(n,m)\eta(k,\ell) = {}& \rho^{(mk-n\ell)/2}\eta(n+k,m+\ell) + \rho^{-(nk+m(k+\ell))/2}\eta\big(n-(k+\ell),m+k\big)\\ &+ \rho^{-(mk-n\ell)/2}\eta(n-k,m-\ell) + \rho^{(nk+m(k+\ell))/2}\eta\big(n+(k+\ell),m-k\big)\\ &+ \rho^{(m\ell+n(k+\ell))/2}\eta\big(n+\ell,m-(k+\ell)\big)\\ &+ \rho^{-(m\ell+n(k+\ell))/2}\eta\big(n-\ell,m+(k+\ell)\big).\end{aligned} \tag{η5}$$

Define the following distinguished elements of $\mathcal{A}_\theta$.

DEFINITION 1.1.5. Let $C = \tau(1,0)$, $D = \tau(1,1)$, $\hat{E} = \zeta(0,1)$, $\hat{F} = \zeta(-1,1)$, $E = \frac{1}{2}(\hat{E}+\hat{F})$, $F = \frac{1}{2i}(\hat{E}-\hat{F})$, $G = \eta(1,0)$ and $H = \eta(1,1)$. Note that C, D, E, F, G and H are self-adjoint and that $\hat{F} = \hat{E}^*$.

2. The structure of $\mathcal{P}_\theta^\tau$, $\mathcal{P}_\theta^\zeta$ and $\mathcal{P}_\theta^\eta$.

2.1 *The polynomial fixed point subalgebra of τ.*

REMARK 2.1.1. From the relations (τ2) and (τ3), for all $N \in \mathbb{N}$, $\operatorname{Span}\{\tau(n,m) : |n|,|m| \le N\} = \operatorname{Span}\{\tau(n,m) : 0 \le n \le N, 0 \le m \le N\}$ and $\operatorname{Span}\{\tau(n,m) : n,m \in \mathbb{Z}\} = \operatorname{Span}\{\tau(n,m) : n,m \ge 0\}$.

PROPOSITION 2.1.2. *For a fixed $N \in \mathbb{N}$ and $\rho \neq \pm 1$:*

$$\operatorname{Span}\{\tau(n,m) : |n|, |m| \leq N\} = \operatorname{Span}\left\{ \begin{matrix} C^n D^m \\ C^{r-1} DCD^{s-1} \end{matrix} : \begin{matrix} n, m \geq 0, & m+n \leq N \\ r, s > 0, & r+s \leq N \end{matrix} \right\}.$$

PROOF. To facilitate the proof we introduce some notation. Let $S_N = \operatorname{Span}\{\tau(n,m) : |n|, |m| \leq N\}$ and $R_N = \operatorname{Span}\left\{ \begin{matrix} C^n D^m \\ C^{r-1} DCD^{s-1} \end{matrix} : \begin{matrix} n, m \geq 0, & m+n \leq N \\ r, s > 0, & r+s \leq N \end{matrix} \right\}$. We will prove that $S_N \subset R_N$ by induction on the number N. For $N = 0$, $\tau(0,0) = 4I$ and the result is obvious. When $N = 1$, using the definitions of C and D, Remark 2.1.1 and $\tau(1,0) = \tau(0,1)$ the result is true. If $N = 2$ using,

$$\begin{aligned} CD = \tau(1,0)\tau(1,1) &= \rho^{-1/2}\tau(2,1) + \rho^{1/2}\tau(1,2) + (\rho^{1/2} + \rho^{-1/2})C, \\ DC = \tau(1,1)\tau(1,0) &= \rho^{1/2}\tau(2,1) + \rho^{-1/2}\tau(1,2) + (\rho^{1/2} + \rho^{-1/2})C, \\ C^2 = \tau(1,0)\tau(1,0) &= \tau(2,0) + 4I + (\rho^{1/2} + \rho^{-1/2})D, \\ D^2 = \tau(1,1)\tau(1,1) &= \tau(2,2) + (\rho + \rho^{-1})\tau(2,0) + 4I, \end{aligned}$$

and Remark 2.1.1 together with $\tau(0,2) = \tau(2,0)$ we see that the result is again true, provided $\rho \neq \pm 1$.

Now suppose that $S_N \subset R_N$ for all $N \leq k$ ($k \geq 2$) and consider $N = k+1$. Firstly observe that $\operatorname{Span}\{C\tau(n,m), \tau(n,m)D : |n|, |m| \leq k\} \subset R_{k+1}$. Using this and the equalities below,

$$\begin{aligned} C\tau(\pm k, m) &= \rho^{-m/2}\tau(\pm k+1, m) + \rho^{m/2}\tau(\pm k-1, m) + \rho^{\mp k/2}\tau(\pm k, m-1) + \rho^{\pm k/2}\tau(\pm k, m+1), \\ C\tau(n, \pm k) &= \rho^{\mp k/2}\tau(n+1, \pm k) + \rho^{\pm k/2}\tau(n-1, \pm k) + \rho^{-n/2}\tau(n, \pm k-1) + \rho^{n/2}\tau(n, \pm k+1), \\ \tau(k,k)D &= \tau(k+1, k+1) + \tau(k-1, k-1) + \rho^{-k}\tau(k-1, k+1) + \rho^{k}\tau(k+1, k-1), \\ \tau(k, k-1)D &= \rho^{-1/2}\tau(k+1, k) + \rho^{1/2}\tau(k-1, k-2) \\ &\quad + \rho^{-(k-1/2)}\tau(k-1, k) + \rho^{(k-1/2)}\tau(k+1, k-2), \\ \tau(k-1, k)D &= \rho^{1/2}\tau(k, k+1) + \rho^{-1/2}\tau(k-2, k-1) \\ &\quad + \rho^{-(k-1/2)}\tau(k-2, k+1) + \rho^{(k-1/2)}\tau(k, k-1), \end{aligned}$$

we observe that $\tau(k+1, m), \tau\big(-(k+1), m\big), \tau(n, k+1)$ and $\tau\big(n, -(k+1)\big)$ with $|n|, |m| \leq k-1$ together with $\tau(k+1, k+1), \tau(k, k+1)$ and $\tau(k+1, k)$ are contained in R_{k+1} which is sufficient by Remark 2.1.1. Thus we have shown $S_N \subset R_N$. Finally observe that the dimension of S_N is $N(N+1) + 1 = N^2 + N + 1$ (Remark 2.1.1 and $\tau(0,r) = \tau(r,0)$) while that of R_N is at most $\frac{(N+1)(N+2)}{2} + \frac{N(N-1)}{2} = N^2 + N + 1$, so $S_N = R_N$. ∎

2.2 *The polynomial fixed point subalgebra of ζ.*

REMARK 2.2.1. From the relations (ζ2), (ζ3), for all $N \in \mathbb{N}$, $\operatorname{Span}\{\zeta(n,m) : |n|, |m|, |n+m| \leq N\} = \operatorname{Span}\{\zeta(n,m) : -N \leq n \leq 0 \leq m \leq N\}$ and $\operatorname{Span}\{\zeta(n,m) : n, m \in \mathbb{Z}\} = \operatorname{Span}\{\zeta(n,m) : n \leq 0 \leq m\}$.

PROPOSITION 2.2.2. *For a fixed $N \in \mathbb{N}$ and $\rho \neq \pm 1$:*

$$\operatorname{Span}\{\zeta(n,m) : |n|,|m|,|n+m| \leq N\} = \operatorname{Span}\left\{ \begin{array}{c} \hat{E}^n\hat{F}^m \\ \hat{E}^{r-1}\hat{F}\hat{E}\hat{F}^{s-1} \end{array} : \begin{array}{cc} n,m \geq 0, & m+n \leq N \\ r,s > 0, & r+s \leq N \end{array} \right\}.$$

PROOF. This proof follows the same lines of Proposition 2.1.2 so we will be reasonably brief. We let $S_N = \operatorname{Span}\{\zeta(n,m) : |n|,|m|,|n+m| \leq N\}$ and

$$R_N = \operatorname{Span}\left\{ \begin{array}{c} \hat{E}^n\hat{F}^m \\ \hat{E}^{r-1}\hat{F}\hat{E}\hat{F}^{s-1} \end{array} : \begin{array}{cc} n,m \geq 0, & m+n \leq N \\ r,s > 0, & r+s \leq N \end{array} \right\}$$

and show $S_N \subset R_N$ by induction on N. The result is clear for $N = 0$ and 1. If $N = 2$, using,

$$\begin{aligned} \hat{E}\hat{F} &= \zeta(0,1)\zeta(-1,1) = \rho^{-1/2}\zeta(-1,2) + \rho^{1/2}\zeta(-2,1) + 3I, \\ \hat{F}\hat{E} &= \zeta(-1,1)\zeta(0,1) = \rho^{1/2}\zeta(-1,2) + \rho^{-1/2}\zeta(-2,1) + 3I, \\ \hat{E}^2 &= \zeta(0,1)\zeta(0,1) = \zeta(0,2) + (\rho^{1/2} + \rho^{-1/2})\hat{F}, \\ \hat{F}^2 &= \zeta(-1,1)\zeta(-1,1) = \zeta(-2,2) + (\rho^{1/2} + \rho^{-1/2})\hat{E}, \end{aligned}$$

and Remark 2.2.1 we see that the result is again true, provided $\rho \neq \pm 1$.

Now suppose that $S_N \subset R_N$ for all $N \leq k$ ($k \geq 2$) and consider $N = k + 1$. Firstly observe that $\operatorname{Span}\{\hat{E}\zeta(n,m), \zeta(n,m)\hat{F} : |n|,|m|,|n+m| \leq k\} \subset R_{k+1}$. Using this, Remark 2.2.1 and the equalities below we observe $S_{k+1} \subset R_{k+1}$.

$$\begin{aligned} \hat{E}\zeta(-k,m) &= \rho^{-k/2}\zeta(-k,m+1) + \rho^{m/2}\zeta(-(k+1),m) + \rho^{(k-m)/2}\zeta(-k+1,m-1), \\ \hat{E}\zeta(n,k) &= \rho^{n/2}\zeta(n,k+1) + \rho^{k/2}\zeta(n-1,k) + \rho^{-(n+k)/2}\zeta(n+1,k-1), \\ \zeta(-k,k)\hat{F} &= \zeta\big(-(k+1),k+1\big) + \rho^{-k/2}\zeta(-k,k-1) + \rho^{k/2}\zeta(-k+1,k), \\ \zeta(-k+1,k)\hat{F} &= \rho^{-1/2}\zeta(-k,k+1) + \rho^{-(k-1)/2}\zeta(-k+1,k-1) + \rho^{k/2}\zeta(-k+2,k), \\ \zeta(-k,k-1)\hat{F} &= \rho^{1/2}\zeta\big(-(k+1),k\big) + \rho^{-k/2}\zeta(-k,k-2) + \rho^{(k-1)/2}\zeta(-k+1,k-1). \end{aligned}$$

Using a similar dimension argument to Proposition 2.1.2 we obtain the result. ∎

2.3 *The polynomial fixed point subalgebra of η.*

REMARK 2.3.1. By the relations (ηi), $i = 2,3,4$, for all $N \in \mathbb{N}$, $\operatorname{Span}\{\eta(n,m) : |n|,|m|,|n+m| \leq N\} = \operatorname{Span}\{\eta(n,m) : n,m \geq 0, n+m \leq N\}$ and $\operatorname{Span}\{\eta(n,m) : n,m \in \mathbb{Z}\} = \operatorname{Span}\{\eta(n,m) : n,m \geq 0\}$.

PROPOSITION 2.3.2. *For a fixed $N \in \mathbb{N}$ and $\rho \neq \pm 1$:*

$$\operatorname{Span}\left\{\eta(n,m) : \begin{array}{cc} |n|,|m| & \leq N \\ |n+m| & \leq N \end{array}\right\} = \operatorname{Span}\left\{ \begin{array}{c} G^nH^m \\ G^{r-1}HGH^{s-1} \end{array} : \begin{array}{cc} n,m \geq 0, & m+2n \leq N \\ r,s > 0, & r+2s \leq N \end{array} \right\}.$$

PROOF. The proof essentially follows the same argument as Propositions 2.1.2 and 2.2.2. so once again we will be brief. As usual we introduce the notation

$$S_N = \operatorname{Span}\left\{\eta(n,m) : \begin{array}{cc} |n|,|m| & \leq N \\ |n+m| & \leq N \end{array}\right\}$$

and

$$R_N = \text{Span}\left\{ \begin{matrix} G^n H^m \\ G^{r-1}HGH^{s-1} \end{matrix} : \begin{matrix} n, m \geq 0, & m + 2n \leq N \\ r, s > 0, & r + 2s \leq N \end{matrix} \right\}$$

and we show $S_N \subset R_N$ by induction on N. This is clear for $N = 0$ and 1. When $N = 2$ using Remark 2.3.1 and,

$$G^2 = \eta(1,0)\eta(1,0) = \eta(2,0) + (\rho^{1/2} + \rho^{-1/2})(G + H) + 6I,$$

we see that the result is again true and when $N = 3$ using,

$$\begin{aligned} GH &= \eta(1,0)\eta(1,1) = \rho^{-1/2}\eta(2,1) + \rho^{1/2}\eta(1,2) + (\rho^{1/2} + \rho^{-1/2})G + (\rho + \rho^{-1})\eta(2,0), \\ HG &= \eta(1,1)\eta(1,0) = \rho^{1/2}\eta(2,1) + \rho^{-1/2}\eta(1,2) + (\rho^{1/2} + \rho^{-1/2})G + (\rho + \rho^{-1})\eta(2,0), \\ G^3 &= \eta(3,0) + \rho\eta(2,1) + \rho^{-1}\eta(1,2) + (\rho^{1/2} + \rho^{-1/2})G(G + H) + 7G + (\rho + \rho^{-1})H, \end{aligned}$$

together with Remark 2.3.1 we see that the result is true, provided $\rho \neq \pm 1$.

Now suppose that $S_N \subset R_N$ for all $N \leq k$ ($k \geq 3$) and consider $N = k + 1$. Firstly observe that $\text{Span}\{G\eta(n,m) : |n|, |m|, |n + m| \leq k\} \subset R_{k+1}$ and $\text{Span}\{\eta(r,s)H : |r|, |s|, |r + s| \leq k - 1\} \subset R_{k+1}$. Then using ($\eta$5) to determine expressions for $\eta(n - 1, k - n)H$, $G\eta(n, k - n)$ and $G\eta(k - m, m)$ it can be shown that $S_{k+1} \subset R_{k+1}$. The usual dimension argument then proves the result. ∎

REMARK 2.3.3. When $\rho = \pm 1$ the results for τ, ζ and η given above all fail since the linear span of the $\tau(n,m)$, $\zeta(n,m)$ and $\eta(n,m)$'s is strictly larger than that of the respective polynomials. This can be seen by noting that when $\rho = \pm 1$ the generators C, D; $\hat{E}, \hat{F}$ and G, H essentially commute or anticommute so the dimension of R_N is strictly less than that of S_N.

3. Abstract characterizations of $\mathcal{P}_\theta^\tau$, $\mathcal{P}_\theta^\zeta$ and $\mathcal{P}_\theta^\eta$.

3.1 *Abstract characterization of* $\mathcal{P}_\theta^\tau$. We wish to determine polynomial identities, *i.e.*, relations, between the respective generators of the above polynomial algebras. We shall explain in detail the situation for for $\mathcal{P}_\theta^\tau$ and then just record the others. We have (*cf.* Proposition 2.1.2),

$$\begin{aligned} \tau(2,0) &= C^2 - (\rho^{1/2} + \rho^{-1/2})D - 4I, \\ \tau(2,1) &= (\rho - \rho^{-1})^{-1}\{\rho^{1/2}DC - \rho^{-1/2}CD\} - C, \\ \tau(1,2) &= (\rho - \rho^{-1})^{-1}\{\rho^{1/2}CD - \rho^{-1/2}DC\} - C, \\ \tau(2,2) &= D^2 - (\rho + \rho^{-1})[C^2 - (\rho^{1/2} + \rho^{-1/2})D - 4I] - 4I. \end{aligned}$$

Also, by considering the 3rd order entries, we get,

$$\begin{aligned} \tau(3,0) &= C^3 - (\rho + 1 + \rho^{-1})(\rho^{1/2} + \rho^{-1/2})^{-1}[DC + CD] + (\rho - 5 + \rho^{-1})C, \\ \tau(3,1) &= (\rho^2 - \rho^{-2})^{-1}\{\rho DC^2 - \rho^{-1}C^2D - (\rho - \rho^{-1})[(\rho^{1/2} + \rho^{-1/2})D^2 + (\rho + 4 + \rho^{-1})D]\}, \\ \tau(1,3) &= (\rho^2 - \rho^{-2})^{-1}\{\rho C^2D - \rho^{-1}DC^2 - (\rho - \rho^{-1})[(\rho^{1/2} + \rho^{-1/2})D^2 + (\rho + 4 + \rho^{-1})D]\}, \end{aligned}$$

$$\tau(3,2) = (\rho^2-\rho^{-2})^{-1}\{\rho D^2C - \rho^{-1}CD^2 - (\rho^2-\rho^{-2})C^3 + \rho^{3/2}(\rho+\rho^{-1})DC - \rho^{-3/2}(\rho+\rho^{-1})CD + (\rho-\rho^{-1})(5\rho-4+5\rho^{-1})C\},$$

$$\tau(2,3) = (\rho^2-\rho^{-2})^{-1}\{\rho CD^2 - \rho^{-1}D^2C - (\rho^2-\rho^{-2})C^3 - \rho^{-3/2}(\rho+\rho^{-1})DC + \rho^{3/2}(\rho+\rho^{-1})CD + (\rho-\rho^{-1})(5\rho-4+5\rho^{-1})C\}.$$

Moreover, CDC and DCD are given by,

$$CDC = \tau(3,1)+\tau(1,3)+(\rho^{3/2}+\rho^{-3/2})\tau(2,2)+(\rho^{1/2}+\rho^{-1/2})\big(\tau(2,0)+C^2\big)+(\rho+\rho^{-1})D,$$

$$DCD = \tau(3,2)+\tau(2,3)+(\rho^2+\rho^{-2})\tau(3,0)+\rho^{-1}\tau(1,2) + \rho\tau(2,1)+(\rho+\rho^{-1})C+(\rho^{1/2}+\rho^{-1/2})CD.$$

Thus, by substituting the expressions for the $\tau(n,m)$'s we obtain,

$$\begin{aligned}(\rho+\rho^{-1})CDC = C^2D + DC^2 &+ (\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})D^2 \\ &+ 4(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho+\rho^{-1})(\rho^2-\rho^{-2})I \\ &+ (\rho-\rho^{-1})^2(\rho^2+\rho+4+\rho^{-1}+\rho^{-2})D \\ &- (\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})(\rho^2-\rho^{-2})C^2,\end{aligned}\tag{1}$$

$$\begin{aligned}(\rho+\rho^{-1})DCD = CD^2 + D^2C &+ (\rho-\rho^{-1})(\rho^2-\rho^{-2})C^3 \\ &- (\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})(\rho^2-\rho^{-2})(DC+CD) \\ &+ (\rho-\rho^{-1})^2(\rho^2-5\rho+4-5\rho^{-1}+\rho^{-2})C.\end{aligned}\tag{2}$$

By considering the 4-th order entries we get,

$$\begin{aligned}&DCDC \\ &= (\rho^2+1+\rho^{-2})CDCD - (\rho+\rho^{-1})C^2D^2 - (\rho-\rho^{-1})(\rho^3-\rho^{-3})C^4 \\ &\quad - (\rho-\rho^{-1})(\rho^{\frac{5}{2}}-\rho^{-\frac{5}{2}})D^3 \\ &\quad + (\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho^3-\rho^{-3})(2\rho+3+2\rho^{-1})C^2D \\ &\quad + (\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})(\rho^2+\rho-1+\rho^{-1}+\rho^{-2})DC^2 \\ &\quad + (\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho-\rho^{-1})(\rho^{\frac{3}{2}}+\rho^{-\frac{3}{2}})(\rho^3-\rho^2-4\rho-8-4\rho^{-1}-\rho^{-2}+\rho^{-3})D^2 \\ &\quad - (\rho-\rho^{-1})^2(\rho^5+2\rho^3-8\rho^2+5\rho-8+5\rho^{-1}-8\rho^{-2}+2\rho^{-3}+\rho^{-5})C^2 \\ &\quad + (\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho-\rho^{-1})(\rho^6+2\rho^5+3\rho^4-8\rho^3-9\rho^2-10\rho-6 \\ &\quad - 10\rho^{-1}-9\rho^{-2}-8\rho^{-3}+3\rho^{-4}+2\rho^{-5}+\rho^{-6})D \\ &\quad + 4(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2(\rho-\rho^{-1})(\rho^2-\rho^{-2})(\rho^3+\rho^2+2\rho+2\rho^{-1}+\rho^{-2}+\rho^{-3})I.\end{aligned}\tag{3}$$

THEOREM 3.1.1. *Let $\mathcal{P}_\theta(C,D,I)$ denote the polynomial $*$-algebra in two self-adjoint elements C and D with the three relations (1), (2) and (3). If $\theta \neq 0, \frac{1}{2}$ $(\rho \neq \pm 1)$ then $\mathcal{P}_\theta(C,D,I)$ is canonically $*$-isomorphic to $\mathcal{P}_\theta^\tau$ by the correspondence:*

$$C \longmapsto U+U^{-1}+V+V^{-1}, \quad D \longmapsto \rho^{1/2}(UV+U^{-1}V^{-1}+VU^{-1}+V^{-1}U).$$

PROOF. By Proposition 2.1.2 we only need to show that any element of $\mathcal{P}_\theta(C,D,1)$ is a linear combination of elements of the form C^mD^n, m, $n \geq 0$ and $C^{r-1}DCD^{s-1}$, r, $s > 0$. We will prove this by induction, showing that any word in C and D of length N is a linear combination of such elements with $m+n \leq N$ and $r+s \leq N$. This is trivial for $N = 1$. Assume that the statement is true for $N \leq k$ and consider $N = k+1$. If W is a word of length $k+1$, then $W = C(W')$ or $D(W')$, where W' is a word of length k. Hence by the induction hypothesis W' is a linear combination of C^mD^n, m, $n \geq 0$, $m+n \leq k$ and $C^{r-1}DCD^{s-1}$, r, $s > 0$, $r+s \leq k$. Therefore $C(W')$ is of the required form while $D(W')$ is a linear combination of elements of type DC^mD^n, m, $n \geq 0$, $m+n \leq k$ and $DC^{r-1}DCD^{s-1}$, $r, s > 0, r+s \leq k$.

For words of type DC^mD^n, $m, n \geq 0$, $m+n \leq k$, we consider three cases. If $m = 0$, D^{n+1} is of the required form. If $m = 1$, DCD^n is also of the required form. If $m \geq 2$, we use (1) to write DC^2 as a linear combination of CDC, C^2D, D^2, D, C^2, I. Now, using $C(W')$ is of the required form if the length of W' is less than $k+1$ and the induction hypothesis, we are done.

For words of type $DC^{r-1}DCD^{s-1}$, $r, s > 0$, $r+s \leq k$, we use a similar argument. If $r = 1$, we can use (2) to write D^2C as a linear combination of DCD, CD^2, C^3, DC, CD, C. If $r = 2$, we use (3) to write $DCDC$ as a linear combination of $CDCD$, C^2D^2, D^3, C^4, C^2D, DC^2, D^2, D, C^2, I. For $r \geq 3$ we use (1) to write DC^2 as a linear combination of CDC, C^2D, D^2, D, C^2, I. ■

COROLLARY 3.1.2. *Let $\mathcal{P}_\theta(C,D)$ denote the polynomial $*$-algebra in two self-adjoint elements C and D, without constant term, with the three relations (1), (2) and (3). Then $\mathcal{P}_\theta(C,D) = \mathcal{P}_\theta(C,D,I)$ if and only if $\theta \neq \frac{n}{4}$, $n = 0,\ldots,3$ ($\rho \neq \pm 1, \pm i$).*

PROOF. Relation (1) can be solved for I if $\theta \neq \frac{n}{4}$, $n = 0,\ldots,3$. Conversely if $\theta = \frac{n}{4}$, $n = 0,\ldots,3$, we have the 1-dimensional representation, Π, of $\mathcal{P}_\theta(C,D,I)$ given by, $\Pi(C) = \Pi(D) = 0$ and $\Pi(I) = 1$, showing $\mathcal{P}_\theta(C,D) \subsetneq \mathcal{P}_\theta(C,D,I)$. ■

3.2 *Abstract characterization of $\mathcal{P}_\theta^\zeta$.* Reasoning as for $\mathcal{P}_\theta^T$, by computing the third and fourth order entries, we get in this case the relations,

$$(4)\qquad \begin{aligned}(\rho+\rho^{-1})\hat{E}\hat{F}\hat{E} = \hat{E}^2\hat{F} + \hat{F}\hat{E}^2 + (\rho-\rho^{-1})(\rho^{\frac{3}{2}} - \rho^{-\frac{3}{2}})\hat{F}^2 \\ - (\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2(\rho^2 + 3\rho + 1 + 3\rho^{-1} + \rho^{-2})\hat{E},\end{aligned}$$

$$(5)\qquad \begin{aligned}(\rho+\rho^{-1})\hat{E}^2\hat{F}^2 = {} & (\rho^2+1+\rho^{-2})\hat{E}\hat{F}\hat{E}\hat{F} \\ & + (\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})(2\rho+5+2\rho^{-1})(\rho-1+\rho^{-1})(\rho^{\frac{3}{2}} - \rho^{-\frac{3}{2}})\hat{E}\hat{F} - \hat{F}\hat{E}\hat{F}\hat{E} \\ & + (\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2(\rho^3 + 2\rho^2 + 2\rho - 1 + 2\rho^{-1} + 2\rho^{-2} + \rho^{-3})\hat{F}\hat{E} \\ & - (\rho-\rho^{-1})(\rho^{\frac{5}{2}} - \rho^{-\frac{5}{2}})\{\hat{E}^3 + \hat{F}^3\} - 3(\rho-1+\rho^{-1})(\rho^{\frac{3}{2}} - \rho^{-\frac{3}{2}})^2 I.\end{aligned}$$

THEOREM 3.2.1. *Let $\mathcal{P}_\theta(\hat{E},I)$ be the polynomial $*$-algebra in one non self-adjoint element $\hat{E}$ satisfying the two relations (4) and (5) where $\hat{F} = \hat{E}^*$. If $\theta \neq 0$, $\frac{1}{2}$ ($\rho \neq \pm 1$) then $\mathcal{P}_\theta(\hat{E},I)$ is canonically $*$-isomorphic to $\mathcal{P}_\theta^\zeta$ by the correspondence:*

$$\hat{E} \longmapsto V + U^{-1} + \rho^{-1/2}UV^{-1}.$$

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PROOF. This can be proved using the same method we used for Theorem 3.1.1, with relations (4), its adjoint (4)* and (5) instead of (1), (2) and (3). ∎

COROLLARY 3.2.2. *Let $\mathcal{P}_\theta(\hat{E})$ denote the polynomial $*$-algebra in the element $\hat{E}$, without constant term, with the relations (4) and (5). Then $\mathcal{P}_\theta(\hat{E}) = \mathcal{P}_\theta(\hat{E}, I)$ if and only if $\theta \neq \frac{n}{6}$, $n = 0, 1, 2, 4, 5$.*

If we split $\hat{E}$ into its self-adjoint parts, E and F, the relations (4) and (5) can be rewritten in terms of E and F as:

$$(6)\qquad \begin{aligned}(\rho^{\frac12}+\rho^{-\frac12})^2FEF = {}&(\rho^{\frac12}-\rho^{-\frac12})^2E^3+(\rho+\rho^{-1})(F^2E+EF^2)\\ &+(\rho^{\frac12}-\rho^{-\frac12})^2(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})E\\ &-(\rho-\rho^{-1})(\rho^{\frac32}-\rho^{-\frac32})(E^2-F^2),\end{aligned}$$

$$(7)\qquad \begin{aligned}(\rho^{\frac12}+\rho^{-\frac12})^2EFE = {}&(\rho^{\frac12}-\rho^{-\frac12})^2F^3+(\rho+\rho^{-1})(E^2F+FE^2)\\ &+(\rho^{\frac12}-\rho^{-\frac12})^2(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})F\\ &+(\rho-\rho^{-1})(\rho^{\frac32}-\rho^{-\frac32})(EF+FE),\end{aligned}$$

$$(8)\qquad \begin{aligned}0 = {}&(\rho^{\frac12}-\rho^{-\frac12})(\rho^{\frac32}-\rho^{-\frac32})\{E^4+FE^2F+EF^2E+F^4\}\\ &+(\rho^{\frac12}+\rho^{-\frac12})(\rho^{\frac32}+\rho^{-\frac32})\{F^2E^2+E^2F^2-FEFE-EFEF\}\\ &-2(\rho-\rho^{-1})(\rho^{\frac52}-\rho^{-\frac52})(E^3-F^2E-EF^2-FEF)\\ &+(\rho-\rho^{-1})^2(3\rho^2+\rho+1+\rho^{-1}+3\rho^{-2})(E^2+F^2)\\ &-3(\rho-1+\rho^{-1})(\rho^{\frac32}-\rho^{-\frac32})^2I.\end{aligned}$$

Therefore we have,

THEOREM 3.2.3. *Let $\mathcal{P}_\theta(E,F,I)$ be the polynomial $*$-algebra in two self-adjoint elements E, F satisfying the relations (6), (7) and (8) with $\theta \neq 0, \frac12$ ($\rho \neq \pm 1$). Then $\mathcal{P}_\theta(E,F,I)$ is canonically $*$-isomorphic to $\mathcal{P}_\theta(\hat{E},I)$ (hence to $\mathcal{P}_\theta^\zeta$) by the correspondence, $E \longmapsto \frac12(\hat{E}+\hat{F})$, $F \longmapsto \frac{1}{2i}(\hat{E}-\hat{F})$.*

3.3 *Abstract characterization of $\mathcal{P}_\theta^\eta$.* As for $\mathcal{P}_\theta^\tau$ and $\mathcal{P}_\theta^\zeta$ after very long, but straightforward, computations we get,

(9)

$$\begin{aligned}(\rho+\rho^{-1})GHG\\ = {}&G^2H+HG^2+6(\rho^{\frac12}-\rho^{-\frac12})(\rho^2-\rho^{-2})(\rho^3+4\rho-1+4\rho^{-1}+\rho^{-3})I\\ &+(\rho-\rho^{-1})(\rho^{\frac32}-\rho^{-\frac32})H^2+(\rho+\rho^{-1})(\rho^{\frac32}-\rho^{-\frac32})(\rho^2-\rho^{-2})(HG+GH)\\ &-(\rho^{\frac12}-\rho^{-\frac12})(\rho+\rho^{-1})(\rho-1+\rho^{-1})(\rho^3-\rho^{-3})G^2\\ &-(\rho^{\frac12}-\rho^{-\frac12})^2(\rho+\rho^{-1})(\rho^2+2\rho+1+2\rho^{-1}+\rho^{-2})G^3\\ &+(\rho-\rho^{-1})^2(\rho^4+\rho^3+4\rho^2+3\rho+9+3\rho^{-1}+4\rho^{-2}+\rho^{-3}+\rho^{-4})H\\ &+(\rho^{\frac12}-\rho^{-\frac12})^2(\rho^5+\rho^4+11\rho^3+19\rho^2+21\rho+38\\ &+21\rho^{-1}+19\rho^{-2}+11\rho^{-3}+\rho^{-4}+\rho^{-5})G,\end{aligned}$$

together with,

$$
\begin{aligned}
&(\rho+\rho^{-1})HGH\\
&\quad = GH^2 + H^2G + (\rho-\rho^{-1})(\rho^{\frac{5}{2}}-\rho^{-\frac{5}{2}})G^4\\
&\qquad -(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})(\rho^2+2\rho+1+2\rho^{-1}+\rho^{-2})(HG^2+G^2H)\\
&\qquad -(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})^2(\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})(\rho^2-3\rho+3-3\rho^{-1}+\rho^{-2})H^2\\
&\qquad -(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})^2(\rho^3+\rho^2+2\rho+3+2\rho^{-1}+\rho^{-2}\\
&\qquad +\rho^{-3})(GH+HG)+(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})(\rho-\rho^{-1})(\rho^7-\rho^6+\rho^5-2\rho^4-15\rho^2\\
&\qquad -9\rho-13-9\rho^{-1}-15\rho^{-2}-2\rho^{-4}+\rho^{-5}-\rho^{-6}+\rho^{-7})G^2\\
&\qquad -(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2(\rho^8+3\rho^7+5\rho^6+7\rho^5+8\rho^4-5\rho^3-33\rho^2\\
&\qquad -50\rho-63-50\rho^{-1}-33\rho^{-2}-5\rho^{-3}+8\rho^{-4}+7\rho^{-5}+5\rho^{-6}+3\rho^{-7}\\
&\qquad +\rho^{-8})H+(\rho-\rho^{-1})^2(\rho^5+\rho^3-\rho^2-4-\rho^{-2}+\rho^{-3}+\rho^{-5})G^3\\
&\qquad +6(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2(\rho+\rho^{-1})(\rho^{\frac{3}{2}}+\rho^{-\frac{3}{2}})(\rho^5+\rho^4+2\rho^3+2\rho^2\\
&\qquad +5\rho+3+5\rho^{-1}+2\rho^{-2}+2\rho^{-3}+\rho^{-4}+\rho^{-5})I\\
&\qquad -(\rho-\rho^{-1})^2(\rho^7-\rho^6+10\rho^5-2\rho^4+8\rho^3-7\rho^2+4\rho-35\\
&\qquad +4\rho^{-1}-7\rho^{-2}+8\rho^{-3}-2\rho^{-4}+10\rho^{-5}-\rho^{-6}+\rho^{-7})G,
\end{aligned}
$$

and,

$$
\begin{aligned}
HGHG = &-(\rho+\rho^{-1})G^2H^2-(\rho-\rho^{-1})(\rho^{\frac{5}{2}}-\rho^{-\frac{5}{2}})H^3+(\rho^2+1+\rho^{-2})GHGH\\
&+r_1(\rho)G^3H+r_2(\rho)G^2HG+r_3(\rho)GH^2+r_4(\rho)HGH+r_5(\rho)G^5\\
&+r_6(\rho)G^2H+r_7(\rho)GHG+r_8(\rho)G^4+r_9(\rho)H^2+r_{10}(\rho)GH\\
&+r_{11}(\rho)HG+r_{12}(\rho)G^3+r_{13}(\rho)H+r_{14}(\rho)G^2+r_{15}(\rho)G+r_{16}(\rho)I,
\end{aligned}
\tag{11}
$$

where $r_i(\rho)$ is a real-valued rational function of ρ having as denominator a factor of $\rho^{19}(1+\rho)$.

THEOREM 3.3.1. *Let $\mathcal{P}_\theta(G,H,I)$ be the polynomial $*$-algebra in two self-adjoint elements G and H satisfying the three relations (9), (10) and (11). If $\theta \neq 0, \frac{1}{2}$ ($\rho \neq \pm 1$) then $\mathcal{P}_\theta(G,H,I)$ is canonically $*$-isomorphic to $\mathcal{P}_\theta^\eta$ by the correspondence:*

$$
\begin{aligned}
G &\longmapsto U+V+\rho^{-1/2}U^{-1}V+U^{-1}+V^{-1}+\rho^{-1/2}UV^{-1},\\
H &\longmapsto \rho^{-1}(\rho^{3/2}UV+U^{-1}V^2+U^{-2}V+\rho^{3/2}U^{-1}V^{-1}+UV^{-2}+U^2V^{-1}).
\end{aligned}
$$

PROOF. This can be proved in a similar fashion to Theorem 3.1.1 with some slight adjustments given below. Firstly, we define the "length" of a word W in G and H to be $a+2b$, where a = number of G's in W, b = number of H's in W. Secondly, if W is a word of "length" $k+1$, then $W = (W')G$ or $W = (W'')H$ where W' is a word of "length" k

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and W'' is a word of "length" $k-1$. The theorem can now be proved using an induction argument on the "length" of a word together with relations (9), (10) and (11). ∎

COROLLARY 3.3.2. *Let $\mathcal{P}_\theta(G,H)$ denote the polynomial $*$-algebra in two self-adjoint elements G and H, without constant term, with the three relations (9), (10) and (11). Then $\mathcal{P}_\theta(G,H)=\mathcal{P}_\theta(G,H,I)$ if and only if $\theta\neq\frac{n}{4}$, $n=0,\ldots,3$ ($\rho\neq\pm1,\pm i$).*

4. **Decomposition lemmas.**

4.1 *A decomposition lemma for τ.* Consider, $\mathcal{P}_\theta^\sigma$, the polynomial fixed point subalgebra of the automorphism σ. We have $\mathcal{P}_\theta^\sigma=\operatorname{Span}\{\sigma(n,m):n,m\in\mathbb{Z}\}$ and $\mathcal{P}_\theta^\tau\subseteq\mathcal{P}_\theta^\sigma$. Note that $(\tau|_{\mathcal{P}_\theta^\sigma})^2=1$ and $\mathcal{P}_\theta^\tau=\{x\in\mathcal{P}_\theta^\sigma:\tau(x)=x\}$.

DEFINITION 4.1.1. Define $\tau_-(n,m)=(1-\tau)\big(\sigma(n,m)\big)$ for n, $m\in\mathbb{Z}$ and $\mathcal{P}_{(\theta,-)}^\tau=\{x\in\mathcal{P}_\theta^\sigma:\tau(x)=-x\}$. Since $\frac{1}{2}(1-\tau)$ is a projection from $\mathcal{P}_\theta^\sigma$ onto $\mathcal{P}_{(\theta,-)}^\tau$, $\mathcal{P}_{(\theta,-)}^\tau=\operatorname{Span}\{\tau_-(n,m):n,m\in\mathbb{Z}\}$ and it is straightforward to show that $\tau_-(n,m)$ satisfies, for all $n,m\in\mathbb{Z}$,

$$\begin{aligned}\tau_-(n,m)^*&=\tau_-(n,m)=\tau_-(-n,-m)=-\tau_-(m,-n),\\ \tau_-(n,m)\tau_-(k,\ell)&=\rho^{(mk-n\ell)/2}\tau(n+k,m+\ell)+\rho^{-(mk-n\ell)/2}\tau(n-k,m-\ell)\\ &\quad-\rho^{-(m\ell+nk)/2}\tau(n-\ell,m+k)-\rho^{(m\ell+nk)/2}\tau(n+\ell,m-k).\end{aligned}$$

Note that any $x\in\mathcal{P}_\theta^\sigma$ has a unique decomposition, $x=x_++x_-$, where $x_+\in\mathcal{P}_\theta^\tau$ and $x_-\in\mathcal{P}_{(\theta,-)}^\tau$. The following technical result will be used in the proof of the main theorem in Section 5.

LEMMA 4.1.2. *If $x\in\mathcal{P}_{(\theta,-)}^\tau$, $\theta\neq\frac{1}{4},\frac{3}{4}$, then $x=y\,\tau_-(1,0)+z\,\tau_-(1,1)$, where $y,z\in\mathcal{P}_\theta^\tau$.*

PROOF. Since $\mathcal{P}_{(\theta,-)}^\tau=\operatorname{Span}\{\tau_-(n,m):n,m\in\mathbb{Z}\}$ it is sufficient to prove the lemma for elements of the form $\tau_-(n,m)$ n, $m\in\mathbb{Z}$. Firstly $\tau_-(0,0)=0$, $\tau_-(1,0)=\tau_-(-1,0)=-\tau_-(0,1)=-\tau_-(0,-1)$ and $\tau_-(1,1)=\tau_-(-1,-1)=-\tau_-(-1,1)=-\tau_-(1,-1)$ are obviously of the required form. It is straightforward to prove,

$$\begin{aligned}\tau(n,m)\tau_-(k,\ell)&=\rho^{(mk-n\ell)/2}\tau_-(n+k,m+\ell)+\rho^{-(mk-n\ell)/2}\tau_-(n-k,m-\ell)\\ &\quad-\rho^{-(m\ell+nk)/2}\tau_-(n-\ell,m+k)-\rho^{(m\ell+nk)/2}\tau_-(n+\ell,m-k).\end{aligned}$$

Therefore we have,

$$\begin{aligned}\tau(1,0)\tau_-(1,0)&=\tau_-(2,0)+\tau_-(0,0)-(\rho^{1/2}-\rho^{-1/2})\tau_-(1,1),\\ \tau(1,1)\tau_-(1,1)&=\tau_-(2,2)+\tau_-(0,0)-(\rho-\rho^{-1})\tau_-(2,0),\\ \tau(1,1)\tau_-(1,0)&=\rho^{1/2}\tau_-(2,1)-\rho^{-1/2}\tau_-(1,2)-(\rho^{1/2}+\rho^{-1/2})\tau_-(1,0),\\ \tau(1,0)\tau_-(1,1)&=\rho^{-1/2}\tau_-(2,1)+\rho^{1/2}\tau_-(1,2)-(\rho^{1/2}-\rho^{-1/2})\tau_-(1,0).\end{aligned}$$

Thus $\tau_-(2,0)$ and $\tau_-(2,2)$ are of the required form (if $\tau_-(k,\ell)$ is of the required form then so is $\tau(n,m)\tau_-(k,\ell)$) and so are $\tau_-(2,1)$ and $\tau_-(1,2)$ provided $\rho\neq\pm i$ ($\theta\neq\frac{1}{4},\frac{3}{4}$).

Therefore $\tau_-(n,m)$ is of the required form for $|n|,|m| \leq 2$. The proof can be finished by an induction argument on $N = \max\{|n|,|m|\}$ (similar to the one in Proposition 2.1.2) which we omit to save repetition. ■

4.2 *A decomposition lemma for* ζ. We prove a decomposition result for $\mathcal{P}_\theta$ with respect to the automorphism ζ.

DEFINITION 4.2.1. Define $\zeta_\pm(n,m) = \rho^{nm/2}(1 + e^{\mp 2\pi i/3}\zeta + e^{\pm 2\pi i/3}\zeta^2)(U^nV^m)$ where $n,m \in \mathbb{Z}$ and $\mathcal{P}^\zeta_{(\theta,\pm)} = \{x \in \mathcal{P}_\theta : \zeta(x) = e^{\pm 2\pi i/3}x\} = \operatorname{Span}\{\zeta_\pm(n,m) : n,m \in \mathbb{Z}\}$. It is straightforward to show:

$$\zeta_\pm(n,m)^* = \zeta_\mp(-n,-m),$$
$$\zeta_\pm(n,m) = e^{\mp 2\pi i/3}\zeta_\pm\big(-(n+m),n\big) = e^{\pm 2\pi i/3}\zeta_\pm\big(m,-(n+m)\big).$$

Note that any $x \in \mathcal{P}_\theta$ can be decomposed as, $x = x_o + x_+ + x_-$, where $x_0 \in \mathcal{P}^\zeta_\theta$ and $x_\pm \in \mathcal{P}^\zeta_{(\theta,\pm)}$.

LEMMA 4.2.2. *If* $x_\pm \in \mathcal{P}^\zeta_{(\theta,\pm)}$, $\theta \neq \frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}$, *then* $x_\pm = y_\pm\zeta_\pm(-1,0) + z_\pm\zeta_\pm(-1,1)$, *where* $y_\pm, z_\pm \in \mathcal{P}^\zeta_\theta$.

PROOF. Since $\mathcal{P}^\zeta_{(\theta,\pm)} = \operatorname{Span}\{\zeta_\pm(n,m) : n,m \in \mathbb{Z}\}$ it is sufficient to prove the lemma for elements of the form $\zeta_\pm(n,m)$ $n,m \in \mathbb{Z}$. Firstly $\zeta_\pm(0,0) = 0$, $\zeta_\pm(-1,0) = e^{\pm 2\pi i/3}\zeta_\pm(0,1) = e^{\mp 2\pi i/3}\zeta_\pm(1,-1)$ and $\zeta_\pm(-1,1) = e^{\pm 2\pi i/3}\zeta_\pm(1,0) = e^{\mp 2\pi i/3}\zeta_\pm(0,-1)$ are obviously of the required form. It is straightforward to prove,

$$\begin{aligned}\zeta(n,m)\zeta_\pm(k,\ell) = \rho^{(mk-n\ell)/2}&\zeta_\pm(n+k,m+\ell)\\ &+\rho^{-(nk+m(k+\ell))/2}e^{\mp 2\pi i/3}\zeta_\pm\big(n-(k+\ell),m+k\big)\\ &+\rho^{(m\ell+n(k+\ell))/2}e^{\pm 2\pi i/3}\zeta_\pm\big(n+\ell,m-(k+\ell)\big).\end{aligned}$$

We can now follow the same type of argument as that given in the proof of Lemma 4.1.2. ■

REMARK 4.2.3. If $\theta = \frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}$ the above result is not true as $\zeta_\pm(-2,1)$ and $\zeta_\pm(-1,2)$ are not of the required form. In this case we have instead, $x_\pm = y_\pm\zeta_\pm(-1,0) + z_\pm\zeta_\pm(-1,1) + w_\pm\zeta_\pm(-2,1)$, with $y_\pm$, $z_\pm$ and $w_\pm \in \mathcal{P}^\zeta_\theta$. This is true in general but can be simplified to Lemma 4.2.2 for $\theta \neq \frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}$.

4.3 *A decomposition lemma for* η. We give a decomposition result for $\mathcal{P}^\zeta_\theta$ with respect to the automorphism η.

DEFINITION 4.3.1. Define $\eta_-(n,m) = (1-\eta)(\zeta(n,m))$ where $n,m \in \mathbb{Z}$ and also define $\mathcal{P}^\eta_{(\theta,-)} = \{x \in \mathcal{P}^\zeta_\theta : \eta(x) = -x\} = \operatorname{Span}\{\eta_-(n,m) : n,m \in \mathbb{Z}\}$. It is straightforward to show,

$$\eta_-(n,m) = \eta_-\big(-(n+m),n\big) = \eta_-\big(m,-(n+m)\big) = -\eta_-(-n,-m) = -\eta_-(n,m)^*.$$

Note that any $x \in \mathcal{P}^\zeta_\theta$ can be decomposed as, $x = x_o + x_-$, where $x_0 \in \mathcal{P}^\eta_\theta$ and $x_- \in \mathcal{P}^\eta_{(\theta,-)}$. We omit the proof of the following lemma since it uses the same techniques as those for Lemmas 4.1.2 and 4.2.2.

LEMMA 4.3.2. *If* $x \in \mathcal{P}^{\eta}_{(\theta,-)}$, $\theta \neq \frac{1}{4}, \frac{3}{4}$ *then* $x = y\,\eta_-(1,0) + z\,\eta_-(1,1)$, *where* y, $z \in \mathcal{P}^{\eta}_{\theta}$.

5. **Main results.**

5.1 *The* C^**-algebras* $C^*_\theta(C,D,I)$, $C^*_\theta(E,F,I)$ *and* $C^*_\theta(G,H,I)$.

DEFINITION 5.1.1. *For* $x \in \mathcal{P}_\theta(C,D,I)$ *define its universal* $*$*-norm by,*

$$|||x||| = \sup\{\|x\|_{(\theta,\Pi)}\},$$

where Π *ranges over all* $*$*-homomorphisms of* $\mathcal{P}_\theta(C,D,I)$ *into the bounded operators on a Hilbert space and* $\|x\|_{(\theta,\Pi)}$ *denotes the norm of* $\Pi(x)$ *in* $\Pi\big(\mathcal{P}_\theta(C,D,I)\big)$. *A priori* $|||x|||$ *is not finite. Analogously we can define the universal* $*$*-norm for an element in* $\mathcal{P}_\theta(E,F,I)$ *and* $\mathcal{P}_\theta(G,H,I)$.

To show that $|||\ |||$ is a C^*-norm on $\mathcal{P}_\theta(C,D,I)$ we need $|||C|||$ and $|||D|||$ to be finite. Similarly we require $|||E|||$, $|||F|||$, $|||G|||$ and $|||H|||$ to be finite. In order to prove the main theorem we will actually need the stronger results:

THEOREM 5.1.2. *There exists a constant* $K > 0$ *such that:*
(1) $|||\tau(n,m)||| \leq K$ *for any* m, $n \in \mathbb{Z}$ *and* $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ *and* $\frac{3}{4}$.
(2) $|||\zeta(n,m)||| \leq K$ *for any* m, $n \in \mathbb{Z}$ *and* $\theta \neq 0, \frac{1}{2}, \frac{1}{3}$ *and* $\frac{2}{3}$.
(3) $|||\eta(n,m)||| \leq K$ *for any* m, $n \in \mathbb{Z}$ *and* θ *irrational.*

The proof of Theorem 5.1.2 will be deferred to Sections 6, 7 and 8 for $\tau(n,m)$, $\zeta(n,m)$ and $\eta(n,m)$ respectively. In this section, assuming Theorem 5.1.2, we will prove that the completions of the polynomial algebras in the universal norm are the respective fixed point subalgebras.

DEFINITION 5.1.3. Define $C^*_\theta(C,D,I)$ to be the enveloping C^*-algebra of $\mathcal{P}_\theta(C,D,I)$ with respect to the universal norm $|||\ |||$. Analogously define $C^*_\theta(E,F,I)$ and $C^*_\theta(G,H,I)$.

LEMMA 5.1.4. *If* $\{\lambda_{n,m}\}$ *is a double sequence of scalars with* $\sum_{n,m} |\lambda_{n,m}| < \infty$, *then*
(1) $\sum_{n,m} \lambda_{n,m}\tau(n,m)$ *is an element of* $C^*_\theta(C,D,I)$ *for* $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ *and* $\frac{3}{4}$.
(2) $\sum_{n,m} \lambda_{n,m}\zeta(n,m)$ *is an element of* $C^*_\theta(E,F,I)$ *for* $\theta \neq 0, \frac{1}{2}, \frac{1}{3}$ *and* $\frac{2}{3}$.
(3) $\sum_{n,m} \lambda_{n,m}\eta(n,m)$ *is an element of* $C^*_\theta(G,H,I)$ *for* θ *irrational.*

PROOF. This is an obvious consequence of Theorem 5.1.2. ∎

LEMMA 5.1.5. *(1) The operator* $M = \begin{pmatrix} \tau_-(1,0)\tau_-(1,0)^* & \tau_-(1,0)\tau_-(1,1)^* \\ \tau_-(1,1)\tau_-(1,0)^* & \tau_-(1,1)\tau_-(1,1)^* \end{pmatrix}$ *is positive in* $C^*_\theta(C,D,I) \otimes M_2$.

(2) The operators

$$M_\pm = \begin{pmatrix} \zeta_\pm(-1,0)\zeta_\pm(-1,0)^* & \zeta_\pm(-1,0)\zeta_\pm(-1,1)^* & \zeta_\pm(-1,0)\zeta_\pm(-2,1)^* \\ \zeta_\pm(-1,1)\zeta_\pm(-1,0)^* & \zeta_\pm(-1,1)\zeta_\pm(-1,1)^* & \zeta_\pm(-1,1)\zeta_\pm(-2,1)^* \\ \zeta_\pm(-2,1)\zeta_\pm(-1,0)^* & \zeta_\pm(-2,1)\zeta_\pm(-1,1)^* & \zeta_\pm(-2,1)\zeta_\pm(-2,1)^* \end{pmatrix}$$

are positive in $C^*_\theta(E,F,I) \otimes M_3$.

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(3) The operator $M = \begin{pmatrix} \eta_-(1,0)\eta_-(1,0)^* & \eta_-(1,0)\eta_-(1,1)^* \\ \eta_-(1,1)\eta_-(1,0)^* & \eta_-(1,1)\eta_-(1,1)^* \end{pmatrix}$ *is positive in* $C^*_\theta(G,H,I) \otimes M_2$.

PROOF. (1) The spectrum of M in $\mathcal{A}^\sigma_\theta \otimes M_2$ is non-negative since,

$$M = \begin{pmatrix} \tau_-(1,0) \\ \tau_-(1,1) \end{pmatrix} (\tau_-(1,0)^* \quad \tau_-(1,1)^*) = zz^*, \quad \text{with } z = \begin{pmatrix} \tau_-(1,0) \\ \tau_-(1,1) \end{pmatrix},$$

and, $z^*z = \tau_-(1,0)^*\tau_-(1,0) + \tau_-(1,1)^*\tau_-(1,1)$, is obviously positive in $\mathcal{A}^\sigma_\theta$. Therefore, $\mathrm{Sp}(M+\varepsilon) \subseteq [\varepsilon, k+\varepsilon]$ for any $\varepsilon > 0$. By reasoning as in [1, Corollary 3.4] we can show, with the help of Lemma 5.1.4, that $M+\varepsilon$ is a positive element of $C^*_\theta(C,D,I)\otimes M_2$. Since $M+\varepsilon$ is a self-adjoint element of $C^*_\theta(C,D,I) \otimes M_2$, we have [4, Lemma 2.2.9], $\left\|1 - \frac{M+\varepsilon}{\|M+\varepsilon\|}\right\| \leq 1$, so letting $\varepsilon \longrightarrow 0$ we get $\left\|1 - \frac{M}{\|M\|}\right\| \leq 1$, and thus M is positive in $C^*_\theta(C,D,I) \otimes M_2$ [4, Lemma 2.2.9]. (2) and (3) follow similarly. ■

LEMMA 5.1.6. *(1) If* $x_- \in \mathcal{P}^\tau_{(\theta,-)}$, *then* $x_-x_-^*$ *is a positive element of* $C^*_\theta(C,D,I)$, *for* $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ *and* $\frac{3}{4}$.

(2) If $x_\pm \in \mathcal{P}^\zeta_{(\theta,\pm)}$, *then* $x_\pm x_\pm^*$ *are positive elements of* $C^*_\theta(E,F,I)$, *for* $\theta \neq 0, \frac{1}{2}, \frac{1}{3}$ *and* $\frac{2}{3}$.

(3) If $x_- \in \mathcal{P}^\eta_{(\theta,-)}$, *then* $x_-x_-^*$ *is a positive element of* $C^*_\theta(G,H,I)$, *for* θ *irrational.*

PROOF. (1) By Lemma 4.1.2 any $x_- \in \mathcal{P}^\tau_{(\theta,-)}$ can be decomposed as $x_- = y\tau_-(1,0) + z\tau_-(1,1)$ with y and $z \in \mathcal{P}^\tau_\theta$. Therefore,

$$x_-x_-^* = (y \quad z)\begin{pmatrix} \tau_-(1,0)\tau_-(1,0)^* & \tau_-(1,0)\tau_-(1,1)^* \\ \tau_-(1,1)\tau_-(1,0)^* & \tau_-(1,1)\tau_-(1,1)^* \end{pmatrix}\begin{pmatrix} y^* \\ z^* \end{pmatrix} = (y \quad z) M \begin{pmatrix} y^* \\ z^* \end{pmatrix}.$$

Since M is a positive element of $C^*_\theta(C,D,I) \otimes M_2$ by Lemma 5.1.5, it follows that $x_-x_-^*$ is a positive element of $C^*_\theta(C,D,I)$. The cases (2) and (3) follow similarly using Lemmas 4.2.2 and 4.3.2. ■

THEOREM 5.1.7. *(1) For* $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ *and* $\frac{3}{4}$ *the universal* C^**-norm* $|||\ |||$ *on* $\mathcal{P}_\theta(C,D,I)$ *coincides with the* C^**-norm that* $\mathcal{P}_\theta(C,D,I)$ *has as a subalgebra of* $\mathcal{A}_\theta$, *that is* $C^*_\theta(C,D,I) = \mathcal{A}^\tau_\theta$.

(2) For $\theta \neq 0, \frac{1}{2}, \frac{1}{3}$ *and* $\frac{2}{3}$ *the universal* C^**-norm* $|||\ |||$ *on* $\mathcal{P}_\theta(E,F,I)$ *coincides with the* C^**-norm that* $\mathcal{P}_\theta(E,F,I)$ *has as a subalgebra of* $\mathcal{A}_\theta$, *that is* $C^*_\theta(E,F,I) = \mathcal{A}^\zeta_\theta$.

(3) For θ *irrational the universal* C^**-norm* $|||\ |||$ *on* $\mathcal{P}_\theta(G,H,I)$ *coincides with the* C^**-norm that* $\mathcal{P}_\theta(G,H,I)$ *has as a subalgebra of* $\mathcal{A}_\theta$, *that is* $C^*_\theta(G,H,I) = \mathcal{A}^\eta_\theta$.

PROOF. (1) This proof is similar to the one for σ in [1]. We regard $\mathcal{A}^\tau_\theta$ as a C^*-subalgebra of $\mathcal{A}^\sigma_\theta$. Recall from [1] and [2] that $\mathcal{A}^\sigma_\theta$ is the completion of $\mathcal{P}^\sigma_\theta$ in the universal norm. Therefore it will be enough to show that any Hilbert space representation of $\mathcal{P}_\theta(C,D,I) \cong \mathcal{P}^\tau_\theta$ extends to a representation of $\mathcal{P}^\sigma_\theta$, which is equivalent to proving that any state ω on $C^*_\theta(C,D,I)$ extends to a state ψ on $\mathcal{A}^\sigma_\theta$ (it is enough to consider elements in $\mathcal{P}_\theta(C,D,I)$ only). Now each $x \in \mathcal{P}^\sigma_\theta$ has a unique decomposition, $x = x_+ + x_-$, with

$x_+ \in \mathcal{P}_\theta^\tau$ and $x_- \in \mathcal{P}_{(\theta,-)}^\tau$. Therefore if we define, $\psi(x) = \omega(x_+)$, ψ is obviously linear so to finish the proof we only have to show that for any $x \in \mathcal{P}_\theta^\sigma$, $\psi(xx^*) \geq 0$, that is,

$$\omega(x_+x_+^*) + \omega(x_-x_-^*) \geq 0,$$

which is a consequence of Lemma 5.1.6. The other two cases (2) and (3) follow similarly. ∎

COROLLARY 5.1.8. *If θ is irrational then $\mathcal{P}_\theta(C,D,I)$, $\mathcal{P}_\theta(E,F,I)$ and $\mathcal{P}_\theta(G,H,I)$ have a unique normalized C^*-seminorm, namely the one inherited from $\mathcal{A}_\theta$.*

PROOF. We shall only give the proof for $\mathcal{P}_\theta(C,D,I)$. The other two cases can be proved by using an analogous argument. Let $|||\ |||$ be the C^*-norm inherited from $\mathcal{A}_\theta$, and let $|||\ |||_1$ be an arbitrary C^*-seminorm on $\mathcal{P}_\theta(C,D,I)$. By Theorem 5.1.7, $|||x|||_1 \leq |||x|||$, $\forall x \in \mathcal{P}_\theta(C,D,I)$. If $\mathcal{B}_\theta$ is the C^*-algebra obtained by completing $\mathcal{P}_\theta(C,D,I)$ with respect to the norm $|||\ |||_1$, then the identity map of $\mathcal{P}_\theta(C,D,I)$ extends to a morphism $\beta : C_\theta^*(C,D,I) \longrightarrow \mathcal{B}_\theta$. Since $\mathcal{A}_\theta$ is simple, as θ is irrational, and τ is properly outer, the fixed point subalgebra $C_\theta^*(C,D,I)$ is simple [15]. Hence β is a $*$-isomorphism and so it is isometric. ∎

COROLLARY 5.1.9. *If θ, θ' are irrational, $\mathcal{A}_\theta^\gamma$ and $\mathcal{A}_{\theta'}^\gamma$ ($\gamma = \tau, \zeta$ and η) are isomorphic if and only if $\theta' = \theta$ or $1-\theta$.*

PROOF. We shall only give the proof for $\mathcal{A}_\theta^\tau$. The other two cases follow by an analogous argument. First note that the relations (1), (2) and (3) only depend on $(\rho+\rho^{-1})$ so it is clear that $\mathcal{A}_\theta^\tau \cong \mathcal{A}_{(1-\theta)}^\tau$. Conversely, since $\mathcal{A}_\theta^\tau$ is simple for θ irrational (as remarked in the proof of Corollary 5.1.8), $\mathcal{A}_\theta^\tau$ is isomorphic to $e\,(\mathcal{A}_\theta \rtimes_\tau \mathbb{Z}_4)\,e$, where e is a projection of trace $\frac{1}{4}$. Now by reasoning as in [1, Remark 4.6] it follows that $\mathcal{A}_\theta \rtimes_\tau \mathbb{Z}_4$, and hence $\mathcal{A}_\theta^\tau$, have a unique normalized trace and therefore the tracial range is an isomorphism invariant. But the tracial range of $\mathcal{A}_\theta^\tau$ ($\mathcal{A}_{\theta'}^\tau$) is equal to $\mathbb{Z} + \theta\mathbb{Z}$ ($\mathbb{Z} + \theta'\mathbb{Z}$) (*cf.* [3, proof of Theorem 1.2]) which implies $\theta = \theta'$ or $\theta = 1-\theta'$. ∎

6. Proof of Theorem 5.1.2(1) for $\tau(n,m)$.

6.1 *Introduction.* In order to prove Theorem 5.1.2(1) we need the following result, which will be proved in Sections 6.2 through 6.4. Theorem 5.1.2(1) is then proved in Section 6.5.

THEOREM 6.1.1. *Let Π be any representation of $\mathcal{P}_\theta(C,D,I)$. Then there exists a $K > 0$ (independent of θ and Π) such that $\|C\|_{(\theta,\Pi)} \leq K$ for $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ and $\frac{3}{4}$ ($\rho \neq \pm 1, \pm i$). So in particular $|||C||| \leq K$ for $\theta \neq 0, \frac{1}{2}, \frac{1}{4}$ and $\frac{3}{4}$ ($\rho \neq \pm 1, \pm i$).*

DEFINITION 6.1.2. For $n, m \in \mathbb{Z}$ define $\tilde{U} = \rho^{-nm/2}U^mV^{-n}$ and $\tilde{V} = \rho^{nm/2}U^nV^m$ then $\tilde{V}\tilde{U} = \tilde{\rho}\tilde{U}\tilde{V}$ where $\tilde{\rho} = \rho^{(m^2+n^2)}$, $\tilde{C} = \sum_{i=0}^{3} \tau^i(\tilde{U}) = \tau(n,m)$ and $\tilde{D} = \tilde{\rho}^{1/2} \sum_{i=0}^{3} \tau^i(\tilde{U}\tilde{V}) = \tau(m+n, m-n)$.

DEFINITION 6.1.3. Let ω_C be the state on $\overline{\mathcal{P}_\theta(C,D,I)}^{\|\ \|_{(\theta,\Pi)}}$ given by $|\omega_C(C)| = \|C\|_\theta$ (so $\omega_C(C) = \varepsilon_C\|C\|_\theta$, $\varepsilon_C = \pm 1$), where for simplicity we denote by $\|x\|_\theta$ the norm $\|x\|_{(\theta,\Pi)}$ for x in $\mathcal{P}_\theta(C,D,I)$.

6.2 *A nonuniform bound for* $\|C\|_\theta$ *when* $\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$.

DEFINITION 6.2.1. Let ω_C be the state defined in 6.1.3 and define $t_{n,m} = \omega_C\big(\tau(n,m)\big)$. In particular, $t_{1,1} = \omega_C(D)$.

Apply the state ω_C to CDC (by (1), for brevity's sake we call the coefficients λ_i, $i = 1, \ldots, 5$) to obtain,

$$t_{1,1}\big(\|C\|_\theta^2(1-2\lambda_1) - \lambda_4\big) = \lambda_2\omega_C(D^2) - \lambda_5\|C\|_\theta^2 + \lambda_3. \tag{12}$$

We may assume $\|C\|_\theta^2(1-2\lambda_1) - \lambda_4 \neq 0$ since otherwise, $\|C\|_\theta^2 = (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2})$, which implies $\|C\|_\theta^2 \leq 32$.

If we apply the state ω_C to DCD (by (2), call the new coefficients λ_6, λ_7 and λ_8) we obtain,

$$\omega_C(DCD) = 2\lambda_1\varepsilon_C\|C\|_\theta\omega_C(D^2) + \lambda_6\varepsilon_C\|C\|_\theta^3 - 2\lambda_7\varepsilon_C\|C\|_\theta t_{1,1} + \lambda_8\varepsilon_C\|C\|_\theta.$$

Therefore using the inequality $\big|\|C\|_\theta\varepsilon_C\omega_C(DCD)\big| \leq \|C\|_\theta^2\omega_C(D^2)$ we have in particular,

$$-(2\lambda_1+1)\omega_C(D^2) - \lambda_6\|C\|_\theta^2 + 2\lambda_7 t_{1,1} - \lambda_8 \leq 0. \tag{13}$$

By using (12) to substitute for $t_{1,1}$ in (13) we get,

(14)
$$\left\{\frac{-(1+2\lambda_1)(1-2\lambda_1)\|C\|_\theta^2 + \lambda_4(1+2\lambda_1) + 2\lambda_2\lambda_7}{[(1-2\lambda_1)\|C\|_\theta^2 - \lambda_4]}\right\}\omega_C(D^2)$$
$$\leq \frac{\lambda_6(1-2\lambda_1)\|C\|_\theta^4 + \lambda_8(1-2\lambda_1)\|C\|_\theta^2 - \lambda_4\lambda_6\|C\|_\theta^2 - \lambda_4\lambda_8 + 2\lambda_7\lambda_5\|C\|_\theta^2 - 2\lambda_7\lambda_3}{[(1-2\lambda_1)\|C\|_\theta^2 - \lambda_4]}.$$

By substituting the expressions for $\{\lambda_i\}_{i=1,\ldots,8}$ into (14) we obtain, when $\theta \in (\frac{1}{4}, \frac{3}{4})\setminus\{\frac{1}{2}\}$,

$$\left[\frac{\|C\|_\theta^2 - (2\rho^4 + \rho^3 + 5\rho^2 + 3\rho + 10 + 3\rho^{-1} + 5\rho^{-2} + \rho^{-3} + 2\rho^{-4})}{\|C\|_\theta^2 - (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2})}\right]\omega_C(D^2)$$
$$\leq \left[\frac{(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2(\rho + \rho^{-1})\{-\|C\|_\theta^4 - R(\rho)(\rho+\rho^{-1})^{-1}\|C\|_\theta^2 + S(\rho)(\rho+\rho^{-1})^{-1}\}}{\|C\|_\theta^2 - (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2})}\right],$$

where $R(\rho)$ and $S(\rho)$ are polynomials in $\rho + \rho^{-1}$.

If $\|C\|_\theta^2 > 32$, the denominators are positive and the $\omega_C(D^2)$ coefficient is non-negative so we have, as $(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2(\rho + \rho^{-1}) > 0$ for $\theta \in (\frac{1}{4}, \frac{3}{4})$,

$$-\|C\|_\theta^4 - R(\rho)(\rho+\rho^{-1})^{-1}\|C\|_\theta^2 + S(\rho)(\rho+\rho^{-1})^{-1} \geq 0. \tag{15}$$

Thus $\|C\|_\theta^2$ has to be bounded by the roots $C_0^{\pm}$ of the quadratic equation associated to (15). Using the parameter $t = (\rho+\rho^{-1})/2 = \cos\phi$ we explicitly get,

$$C_0^{\pm} = \frac{-1}{2t}\Big\{(8t^4 + 4t^3 - 2t^2 - 9t + 1)$$
$$\mp \sqrt{64t^8 + 64t^7 + 48t^6 + 32t^5 + 68t^4 - 36t^3 + 29t^2 - 2t + 1}\Big\}.$$

Hence $\|C\|_\theta^2 \le \max\{32, C_0^{\mp}\}$ for $\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$.

6.3 *A uniform bound for $\|C\|_\theta$ when $\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$.* Recall, from Definition 6.1.2, if $\tilde U = \rho^{-mn/2} U^m V^{-n}$ and $\tilde V = \rho^{nm/2} U^n V^m$ then $\tilde U$ and $\tilde V$ are unitaries satisfying $\tilde V \tilde U = \tilde\rho \tilde U \tilde V$ with $\tilde\rho = \rho^{m^2+n^2}$, $\tilde C = \tau(n,m)$ and $\tilde D = \tau(m+n, m-n)$. Note that the special case $n = m = 1$ gives $\tilde C = D$ and $\tilde D = \tau(2,0)$. Thus $\|D\|_\theta$ is bounded if $\|\tilde C\|_{\tilde\theta}$ is bounded. This implies, using the results of Section 6.2:

LEMMA 6.3.1. *$\|D\|_\theta$ is uniformly bounded by 6 for $\theta \in (\frac{1}{6}, \frac{1}{3}) \setminus \{\frac{1}{4}\}$.*

LEMMA 6.3.2. *If $\theta \in (\frac{1}{6}, \frac{1}{3}) \setminus \{\frac{1}{4}\}$ and $\|C\|_\theta^2 > 32$ then $\|C\|_\theta \le |t_{3,0}| \le \|\tau(3,0)\|_\theta$.*

PROOF. By applying the state ω_C to the equation giving $\tau(3,0)$ in Section 3.1 we have,

$$\|\tau(3,0)\|_\theta \ge |t_{3,0}| = \|C\|_\theta \left| \|C\|_\theta^2 - 2(\rho + 1 + \rho^{-1})(\rho^{1/2} + \rho^{-1/2})^{-1} t_{1,1} + (\rho - 5 + \rho^{-1}) \right|.$$

Now if $\theta \in (\frac{1}{6}, \frac{1}{3}) \setminus \{\frac{1}{4}\}$ then $|t_{1,1}| \le \|D\|_\theta \le 6$ so the factor multiplying $\|C\|_\theta$ is larger than one. ∎

COROLLARY 6.3.3. *If $\cos(9^i\phi) \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ $\forall i = 0, \ldots, (r-1)$ and $\|C\|_\theta^2 > 32$ then,*

$$\|C\|_\theta \le \|\tau(3,0)\|_\theta \le \cdots \le \|\tau(3^{r-1}, 0)\|_\theta \le \|\tau(3^r, 0)\|_\theta.$$

PROOF. From Lemma 6.3.2 we have $\|C\|_\theta \le \|\tau(3,0)\|_\theta$ if $\cos\phi = \cos(2\pi\theta) \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. Now fix $i \in \{0, \ldots, r-1\}$ and regard $\tau(3^i, 0)$ as the $\tilde C$ corresponding to $\tilde\theta = 9^i\theta$ ($\tilde\phi = 9^i\phi$) (Definition 6.1.2). So, by Lemma 6.3.2, if $\cos\tilde\phi = \cos(9^i\phi) \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ and $\|\tilde C\|_{\tilde\theta}^2 > 32$ then $\|\tau(3^{i+1}, 0)\|_\theta = \|\tilde\tau(3,0)\|_{\tilde\theta} \ge \|\tilde C\|_{\tilde\theta} = \|\tau(3^i, 0)\|_\theta$. Iterating from $i = 0$ to $r - 1$ we obtain the result. ∎

PROPOSITION 6.3.4. *There exists a $K > 0$ such that $\|C\|_\theta \le K$ for $\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$.*

PROOF. From Section 6.2 it is clear that if we bound $\|C\|_\theta$ uniformly for $\theta \in (\frac{1}{4}, \frac{1}{4} + \frac{1}{200\pi})$ (as well as for $\theta \in (\frac{3}{4} - \frac{1}{200\pi}, \frac{3}{4})$), we have that $\|C\|_\theta$ is uniformly bounded for $\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$. Suppose that $\|C\|_\theta^2 \le K$ ($K > 32$) for $\theta \in \left[\frac{1}{4} + \frac{1}{200\pi}, \frac{3}{4} - \frac{1}{200\pi}\right]$, that is $\phi \in \left[\frac{\pi}{2} + \frac{1}{100}, \frac{3\pi}{2} - \frac{1}{100}\right]$. Now let $\theta = \frac{1}{4} + \frac{\varepsilon}{2\pi}$, $\phi = \frac{\pi}{2} + \varepsilon$, $0 < \varepsilon < \frac{1}{100}$. If $\|C\|_\theta^2 \le K$ we are done, so we can suppose $\|C\|_\theta^2 > K > 32$. Let k be the first positive integer such that $9^k\varepsilon > \frac{1}{100}$ (so $9^{(k-1)}\varepsilon < \frac{1}{100}$). Then $9^k\varepsilon = 9(9^{k-1}\varepsilon) < \frac{9}{100}$, hence $\frac{\pi}{2} + \frac{1}{100} < \frac{\pi}{2} + 9^k\varepsilon < \frac{\pi}{2} + \frac{9}{100} < \pi$. Since $\cos(9^i\phi) = \cos(\frac{\pi}{2} + 9^i\varepsilon) \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ for $i = 0, \ldots, k-1$ by Corollary 6.3.3, $\sqrt{K} < \|C\|_\theta \le \|\tau(3^k, 0)\|_\theta$. But $\tau(3^k, 0) = \tilde C$ for $\tilde\rho = \rho^{9^k}$ and $\tilde\phi = 9^k\phi = \frac{\pi}{2} + 9^k\varepsilon (\operatorname{mod} 2\pi)$. The angle $\frac{\pi}{2} + 9^k\varepsilon$ is in the range where we have a uniform bound, so $\|\tilde C\|_{\tilde\theta} = \|\tau(3^k, 0)\|_\theta \le \sqrt{K}$, giving a contradiction. Therefore $\|C\|_\theta^2 \le K$ for $\theta = \frac{1}{4} + \frac{\varepsilon}{2\pi}$. For points of type $\theta = \frac{3}{4} - \frac{\varepsilon}{2\pi}$ we use an analogous argument. ∎

6.4 *A uniform bound for $\|C\|_\theta$ for $\theta \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$.*

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LEMMA 6.4.1. *If* $\theta \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ *and* $\|C\|_\theta > 6$ *then* $|t_{1,1}| \leq 108$.

PROOF. Firstly assume $t_{1,1}(\rho^{1/2} + \rho^{-1/2}) > 0$ and apply the state ω_C to (1) to get,

$$\begin{aligned}\|C\|_\theta^2 \leq (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})\{(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2}) \\ + (\rho + 1 + \rho^{-1})t_{1,1}^{-1}\omega_C(D^2) + 4(\rho + \rho^{-1})^2 t_{1,1}^{-1}\}.\end{aligned} \tag{16}$$

If we also apply the state ω_C to (2) and use the inequality $|\omega_C(DCD)| \leq \omega_C(D^2)\|C\|_\theta$ we have,

$$\omega_C(D^2) \leq (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho + \rho^{-1})\|C\|_\theta^2 + (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho^2 - 5\rho + 4 - 5\rho^{-1} + \rho^{-2}). \tag{17}$$

Now by substituting (16) in (17) we obtain,

$$\begin{aligned}&\omega_C(D^2)(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^{-2}\{1 - (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^3(\rho + \rho^{-1})(\rho + 1 + \rho^{-1})t_{1,1}^{-1}\} \\ &\quad \leq (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho + \rho^{-1})(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2}) \\ &\quad\quad + 4(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})(\rho + \rho^{-1})^3 t_{1,1}^{-1} + (\rho^2 - 5\rho + 4 - 5\rho^{-1} + \rho^{-2}).\end{aligned}$$

This implies $|t_{1,1}| \leq 108$ since otherwise the last inequality yields a contradiction as $t_{1,1}^2 \leq \omega_C(D^2)$.

Now assume $t_{1,1}(\rho^{1/2} + \rho^{-1/2}) < 0$ and $|t_{1,1}| > 108$. If we apply the state ω_C to (1) then we get,

$$\begin{aligned}\|C\|_\theta^2 &\leq \frac{(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2})t_{1,1}}{\{t_{1,1} + (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})(\rho + \rho^{-1})(\rho + 1 + \rho^{-1})\}} \\ &\leq \frac{(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2(\rho^2 + \rho + 4 + \rho^{-1} + \rho^{-2})|t_{1,1}|}{\left(|t_{1,1}| - |(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})(\rho + \rho^{-1})(\rho + 1 + \rho^{-1})|\right)},\end{aligned}$$

since the $\omega_C(D^2)$ and the constant term are negative. This implies $\|C\|_\theta \leq 6$.

Combining these results proves the Lemma. ∎

LEMMA 6.4.2. *If* $\theta \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ *and* $\|C\|_\theta > 16$, *then* $\|C\|_\theta \leq |t_{2,0}| \leq \|\tau(2,0)\|_\theta$.

PROOF. We have, see Section 3.1, $|t_{2,0}| = \left|\|C\|_\theta^2 - (\rho^{1/2} + \rho^{-1/2})t_{1,1} - 4\right|$. Therefore, $\|C\|_\theta \leq \|C\|_\theta^2 - 220 \leq |t_{2,0}| \leq \|\tau(2,0)\|_\theta$. ∎

COROLLARY 6.4.3. *If* $\cos(4^i\phi) > 0\ \forall i = 0, \ldots, (r-1)$ *and* $\|C\|_\theta > 16$ *then,*

$$\|C\|_\theta \leq \|\tau(2,0)\|_\theta \leq \cdots \leq \|\tau(2^{r-1},0)\|_\theta \leq \|\tau(2^r,0)\|_\theta.$$

PROOF. Follows from Lemma 6.4.2 in the same style as Corollary 6.3.3 from Lemma 6.3.2. ∎

LEMMA 6.4.4. *If* $\theta \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ *and* $\|C\|_\theta > 19$, *then* $\|C\|_\theta \leq |t_{3,0}| \leq \|\tau(3,0)\|_\theta$.

PROOF. Follows in the same manner as Lemma 6.3.2. ∎

PROPOSITION 6.4.5. *There exists a $K > 0$ such that $\|C\|_\theta \leq K$ for $\theta \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$.*

PROOF. Since $\|C\|_\theta = \|C\|_{(1-\theta)}$ we need only consider the region $(0, \frac{1}{4})$. Suppose $\|C\|_\theta > K > 19$, where K is chosen as in Proposition 6.3.4. Firstly if $\theta \in (\frac{1}{16}, \frac{3}{16}) \setminus \{\frac{1}{8}\}$, then $4\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$ so $\|C\|_{4\theta} \leq K$ by Proposition 6.3.4. But $\|C\|_{4\theta} = \|\tau(2,0)\|_\theta \geq \|C\|_\theta > K$ (by Corollary 6.4.3) which is a contradiction. Similarly if $\theta \in (\frac{1}{36}, \frac{1}{12}) \setminus \{\frac{1}{18}\}$, then $9\theta \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$ so $\|C\|_{9\theta} \leq K$ and $\|C\|_{9\theta} = \|\tau(3,0)\|_\theta \geq \|C\|_\theta > K$ (By Lemma 6.4.4). Now if $\theta = \frac{1}{18}$ we can apply Corollary 6.4.3 with $r = 3$ to get $\|C\|_{\frac{1}{18}} \leq K$ by contradiction. Also if $\theta = \frac{1}{8}$ we can apply the state ω_C to the expression for $\tau(2,3)$ given in Section 3.1, with $\rho = e^{\pi i/4}$ to obtain,

$$t_{2,3} = \frac{\varepsilon_C \|C\|_{\frac{1}{8}}}{\sqrt{2}} \left\{ \omega_C(D^2) - \sqrt{2}\|C\|^2_{\frac{1}{8}} - \frac{(\sqrt{2}+1)\sqrt{2}}{\pm\sqrt{2+\sqrt{2}}} t_{1,1} + (5\sqrt{2}-4) \right\}.$$

Using equation (12) we can solve for $\omega_C(D^2)$ and substituting this into the above, we get,

$$t_{2,3} = \frac{\varepsilon_C \|C\|_{\frac{1}{8}}}{\sqrt{2}} \left\{ \|C\|^2_{\frac{1}{8}} \left[\frac{t_{1,1}}{\pm\sqrt{2+\sqrt{2}}(\sqrt{2}+1)} \right] - \left[\frac{(4+5\sqrt{2})}{\pm\sqrt{2+\sqrt{2}}} \right] t_{1,1} + (4-3\sqrt{2}) \right\}.$$

Since $|t_{1,1}| \leq 108$ (Lemma 6.4.1), if $t_{1,1}$ is bounded away from 0, say $t_{1,1} \notin [-1,1]$, the last equality gives $\|C\|_{\frac{1}{8}} \leq K$, since $\tau(2,3)$ is the $\tilde{C}$ corresponding to the angle $13\theta = \frac{13}{8} = \frac{5}{8}$ mod(1) and $\|C\|_{\frac{5}{8}} \leq K$. Now, if $t_{1,1} \in [-1,1]$, by considering the expression for $\tau(2,1)$ in Section 3.1 we have,

$$|t_{2,1}| \geq \|C\|_{\frac{1}{8}} \sqrt{2+\sqrt{2}} \left| t_{1,1} \mp \sqrt{2+\sqrt{2}} \right| \geq \|C\|_{\frac{1}{8}}.$$

which implies $\|C\|_{\frac{1}{8}} \leq K$, since $\tau(2,1)$ is the $\tilde{C}$ corresponding to the angle $5\theta = \frac{5}{8}$. Thus $\|C\|_\theta \leq K$ for $\theta \in (\frac{1}{36}, \frac{3}{16})$. For $\theta \in (0, \frac{1}{36}]$, $\phi \in (0, \frac{\pi}{18}]$, let r be the smallest integer such that $4^r\phi > \frac{\pi}{18}$ (hence $4^{r-1}\phi < \frac{\pi}{18}$) so $4^r\phi = 4(4^{r-1}\phi) < \frac{\pi}{4}$. Then $4^i\phi \in (0, \frac{3\pi}{8})$ $\forall i = 0, \ldots, r$, which implies by Corollary 6.4.3, $K < \|C\|_\theta \leq \|\tau(2^r, 0)\|_\theta$. But $\|\tau(2^r,0)\|_\theta = \|C\|_{4^r\theta} \leq K$, as $4^r\theta \in (\frac{1}{36}, \frac{3}{16})$, which gives a contradiction. Thus $\|C\|_\theta \leq K$ for $\theta \in (0, \frac{3}{16})$. If $\theta \in [\frac{2}{9}, \frac{1}{4})$ then $4\theta \in [\frac{8}{9}, 1)$ so $1-4\theta \in (0, \frac{1}{9}]$ and thus $\|C\|_\theta = \|C\|_{1-\theta} \leq K$ for $\theta \in [\frac{2}{9}, \frac{1}{4})$. Therefore the only remaining region is $\theta \in [\frac{3}{16}, \frac{2}{9})$. Firstly, by using (13) and noticing $\lambda_6 = (\rho - \rho^{-1})^2 < 0$ we have,

$$\begin{aligned}\|C\|^2_\theta \leq{} & -(\rho+\rho^{-1})^{-1}[(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^{-2}\omega_C(D^2) + (\rho^2 - 5\rho + 4 - 5\rho^{-1} + \rho^{-2})] \\ & + 2(\rho + 1 + \rho^{-1})(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^{-1} t_{1,1}.\end{aligned}$$

Now since $\|D\|_\theta \leq 6$ (Corollary 6.3.1) then $|t_{1,1}|^2 \leq \omega_C(D^2) \leq \|D\|^2_\theta \leq 36$ and therefore $\|C\|^2_\theta \leq 80$. Hence $\|C\|_\theta \leq K$ for $\theta \in [\frac{3}{16}, \frac{2}{9})$ and we have completed the proof. ∎

Combining Propositions 6.3.4 and 6.4.5 we obtain Theorem 6.1.1.

6.5 *Proof of Theorem 5.1.2(1) for $\tau(n, m)$.*

LEMMA 6.5.1. *If $|||C||| \leq K$ for all θ irrational, then $|||\tau(n,m)||| \leq K$ for all θ irrational.*

PROOF. We have shown $|||C||| \leq K$ for all θ irrational just using the equations (1) and (2) that C and D satisfy, which in turn are derived using only $VU = \rho UV$. It follows that we can derive the same equations for $\tilde{C}, \tilde{D}, \tilde{\rho}$ defined in 6.1.2 and then, $|||\tilde{C}||| = |||\tau(n,m)||| \leq K$. ■

The proof given above in fact holds for all θ, m and n such that $\theta(m^2+n^2)(\mathrm{mod}\ 1) \neq 0$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{4}$. Note that this implies ($n = m = 1$) $\|D\|_\theta \leq K$ for $\theta \neq \frac{k}{8}$, for $k = 0, \ldots, 7$. In the rest of this section we will show that a uniform bound for $\|\tau(n,m)\|_\theta$ does indeed exist for all $n, m \in \mathbb{Z}$ and all rational $\theta \neq 0$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{4}$. This proves Theorem 5.1.2(1) and hence Theorem 5.1.7(1).

LEMMA 6.5.2. *Suppose that* $\max\{\|\tau(n,m)\|_\theta, \|C\|_\theta, \|D\|_\theta, \|\tau(1,2)\|_\theta, \|\tau(2,1)\|_\theta\} \leq$ *K for some $K > 0$ and $n, m \in \mathbb{Z}$ fixed. Then, if any three of $\|\tau(n+1, m+\Delta)\|_\theta$, $\|\tau(n-\Delta, m+1)\|_\theta$, $\|\tau(n-1, m-\Delta)\|_\theta$ and $\|\tau(n+\Delta, m-1)\|_\theta$ are $\leq K$, the fourth one is $\leq K^2+3K$ ($\Delta = 0, 1, \pm 2$).*

PROOF. Straightforward using the product rule identities for $\tau(n,m)C$ and $\tau(n,m)D$ for $\Delta = 0, 1$ and those for $\tau(n,m)\tau(1,2)$ and $\tau(n,m)\tau(2,1)$ for $\Delta = \pm 2$. ■

PROPOSITION 6.5.3. *If $\theta = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $(p,q) = 1$ with $q \neq 1, 2, 4$ then there exists a $K > 0$ (independent of θ) such that, $\|\tau(n,m)\|_\theta \leq K$, $\forall n, m \in \mathbb{Z}$.*

PROOF. Let $B = \{\tau(n,m) : 4p(n^2+m^2)q^{-1} \in \mathbb{Z}\}$. If $\tau(n,m) \notin B$, $\|\tau(n,m)\|_\theta \leq K$, with K chosen as in Lemma 6.5.1. It remains to show that there is a $K' > 0$ such that $\|\tau(n,m)\|_\theta \leq K'$ for all $\tau(n,m) \in B$. We will only give a sketch of the proof since the actual computations are relatively straightforward and repetitive.

(i) q odd: then $\|C\|_\theta$, $\|D\|_\theta \leq K$ and $\tau(n,m) \in B$ only when $q|(n^2+m^2)$. If $q = 3$ this means that n and m are multiples of 3 so a simple application of Lemma 6.5.2 with $\Delta = 0$ or 1 is sufficient. If $q = 5$ we have $5|(n^2+m^2)$ so applying Lemma 6.5.2 with $\Delta = \pm 2$ is sufficient for this case. If $q \neq 3, 5$ and n and m are such that $\tau(n,m) \in B$ the idea is to show that we may assume the eight neighboring points on the $\mathbb{Z}^2$-lattice $\{\tau(n,m) : n, m \in \mathbb{Z}\}$ are not in B (it is sufficient to consider $\tau(n, m+1)$ and $\tau(n+1, m+1)$, for example, show by contradiction together with Lemma 6.5.2 that we may assume these are not in B and then consider the symmetry of the situation). Thus using the product rule identities we have $\|\tau(n,m)C\|_\theta$ and $\|\tau(n,m)D\|_\theta$ are bounded. We can now use relation (1), expressing the identity in terms of C and D, to obtain $\|\tau(n,m)\|_\theta$ is bounded.

(ii) q even: suppose $q = 2^i r$, $i = 1, 2$, r odd, $r \geq 3$, then $\tau(n,m) \in B$ when $r \mid (n^2+m^2)$. Similarly if $q = 8r$, r odd, $r \geq 3$, then $\tau(n,m) \in B$ when $(2r)|(n^2+m^2)$ which implies $r|(n^2+m^2)$. Thus, in either case, we can apply the methods for q odd. If $q = 16r$, $r \geq 1$, then $\tau(n,m) \in B$ when $(4r)|(n^2+m^2)$, hence $n^2+m^2 \equiv 0(\mathrm{mod}\ 4)$ and $n, m \in 2\mathbb{Z}$. This implies the eight neighboring points of a point in B are not in B so we may apply the final argument given for q odd. Finally if $q = 8$, $\tau(n,m) \in B$ when

$n^2 + m^2 \equiv 0(\mathrm{mod}\, 2)$, equivalently $n + m \equiv 0(\mathrm{mod}\, 2)$. Suppose that n and m are both odd. First note that $\|D\|_{\frac{\rho}{8}} \leq \bar{K}$ (apply the state ω_D such that $|\omega_D(D)| = \|D\|_{\frac{\rho}{8}}$ to (1) and note $\omega_D(C^2) \leq \|C\|^2_{\frac{\rho}{8}} \leq K^2$). Now we use the change of coordinates, $\bar{n} = (n-m)/2$, $\bar{m} = (m+n)/2$ so $\tau(n,m)$ corresponds to $\tilde{D}$ with $\tilde{\rho} = \rho^{(\bar{m}^2+\bar{n}^2)}$ (Definition 6.1.2). But $\|\tilde{D}\|_{\tilde{\theta}} = \|D\|_{\frac{\rho}{8}(\bar{m}^2+\bar{n}^2)} \leq \bar{K}$ so $\|\tau(n,m)\|_{\frac{\rho}{8}}$ is bounded for all $n, m \in (\mathbb{Z}\times\mathbb{Z}) \setminus (2\mathbb{Z}\times 2\mathbb{Z})$. This implies the eight neighboring points of a point in B are not in B (with a suitable choice of K) so we may apply the final argument given for q odd. ∎

REMARK 6.5.4. For $\theta = 0, \frac{1}{2}$ $(\rho = \pm 1)$, the relations (1), (2) and (3) become,

$$\pm 2CDC = C^2D + DC^2, \quad \pm 2DCD = CD^2 + D^2C, \quad DCDC = 3CDCD \mp 2C^2D^2,$$

where we choose + if $\rho = 1$ and − if $\rho = -1$. These imply $DC = CD$ or $DC = -CD$ respectively [1]. Hence we clearly cannot have an analog of Theorem 6.1.1 if $\rho = \pm 1$.

For $\theta = \frac{1}{4}, \frac{3}{4}$ $(\rho = \pm i)$, the relations (1), (2) and (3) become,

$$C^2D + DC^2 = 2D(4 \pm \sqrt{2}D), \quad CD^2 + D^2C = 8C, \quad C^2D^2 = D^2C^2,$$

where we choose + if $\rho = i$ and − if $\rho = -i$. Let Π be the 2-dimensional representation given by,

$$\Pi(C) = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad \Pi(D) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

with h an arbitrary real number, $g = \frac{\pm(\sqrt{2}+\sqrt{2+4(h-2)^2})}{2}$, $f = \pm\sqrt{2} - g$ and $e = 4 - h$, where we choose + if $\rho = i$ and − if $\rho = -i$. It is straightforward to check that $\Pi(C)$ and $\Pi(D)$ satisfy the relations above. Since h is arbitrary, we can choose a sequence of 2-dimensional representations, $\{\Pi_n\}_{n\in\mathbb{N}}$, of $\mathcal{P}_\theta(C, D, I)$ with $\|C\|_{(\theta,\Pi_n)} \longrightarrow +\infty$ as $n \longrightarrow +\infty$, thus Theorem 6.1.1 fails.

7. **Proof of Theorem 5.1.2(2) for $\zeta(n,m)$.**

7.1 *A bound for $|||E|||$ and $|||F|||$.*

DEFINITION 7.1.1. Let Π be any representation of $\mathcal{P}_\theta(E, F, I)$ and define ω_E and ω_F to be the states on $\overline{\mathcal{P}_\theta(E,F,I)}^{\|\ \|_{(\theta,\Pi)}}$ given by $|\omega_E(E)| = \|E\|_\theta$ (so $\omega_E(E) = \varepsilon_E\|E\|_\theta$, $\varepsilon_E = \pm 1$) and $|\omega_F(F)| = \|F\|_\theta$ (so $\omega_F(F) = \varepsilon_F\|F\|_\theta$, $\varepsilon_F = \pm 1$), where we denote by $\|x\|_\theta$ the norm $\|x\|_{(\theta,\Pi)}$ for $x \in \mathcal{P}_\theta(E, F, I)$.

THEOREM 7.1.2. *Let Π be any representation of $\mathcal{P}_\theta(E, F, I)$. Then $\|E\|_\theta$ and $\|F\|_\theta \leq 52$ for $\theta \neq 0$ $(\rho \neq 1)$. So in particular $|||E|||$ and $|||F||| \leq 52$ for $\theta \neq 0$ $(\rho \neq 1)$.*

PROOF. The proof below is similar in style to the one for τ. If we apply the state ω_F to (7) we have,

$$\begin{aligned}(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2\omega_F(EFE) &= (\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2\varepsilon_F\|F\|^3_\theta + 2(\rho+\rho^{-1})\varepsilon_F\|F\|_\theta\omega_F(E^2) \\ &\quad + (\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2(\rho^2 + 3\rho + 1 + 3\rho^{-1} + \rho^{-2})\varepsilon_F\|F\|_\theta \\ &\quad + 2(\rho - \rho^{-1})(\rho^{\frac{3}{2}} - \rho^{-\frac{3}{2}})\varepsilon_F\|F\|_\theta\omega_F(E).\end{aligned} \tag{18}$$

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Thus as $-(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})^2\varepsilon_F\omega_F(EFE) \le (\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})^2\|F\|_\theta\omega_F(E^2)$, rearranging slightly and assuming $\|F\|_\theta \ne 0$,

$$\begin{aligned}-(3\rho+2+3\rho^{-1})\omega_F(E^2) \le (\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2\{\|F\|_\theta^2+(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})\\ +2(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})(\rho+1+\rho^{-1})\omega_F(E)\}.\end{aligned}$$

So if $-2 \le \rho+\rho^{-1} \le -2/3$ then $-(3\rho+2+3\rho^{-1})\omega_F(E^2) \ge 0$ and as $(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2 < 0$, we have,

$$\|F\|_\theta^2 \le 2|(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})(\rho+1+\rho^{-1})|\,\|E\|_\theta-(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2}). \tag{19}$$

Now if we apply the state ω_E to (6) we have,

$$\begin{aligned}(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})^2\omega_E(FEF) &= (\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2\varepsilon_E\|E\|_\theta^3+2(\rho+\rho^{-1})\varepsilon_E\|E\|_\theta\omega_E(F^2)\\ &\quad+(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})\varepsilon_E\|E\|_\theta\\ &\quad-(\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})\big(\|E\|_\theta^2-\omega_E(F^2)\big).\end{aligned} \tag{20}$$

Thus as $-(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})^2\varepsilon\omega_E(FEF) \le (\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})^2\|E\|_\theta\omega_E(F^2)$ we get,

$$\begin{aligned}&\{-(3\rho+2+3\rho^{-1})\|E\|_\theta-(\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})\varepsilon_E\}\omega_E(F^2)\\ &\qquad\le(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2\|E\|_\theta\{\|E\|_\theta^2-(\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}})(\rho+1+\rho^{-1})\varepsilon_E\|E\|_\theta\\ &\qquad\quad+(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})\}.\end{aligned}$$

That is, $Z(\|E\|_\theta)\omega_E(F^2) \le (\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2\|E\|_\theta Z'(\|E\|_\theta)$, where $Z(\|E\|_\theta)$ and $Z'(\|E\|_\theta)$ have the obvious definitions. Now, if $-2 \le \rho+\rho^{-1} \le -2/3$ and $Z(\|E\|_\theta) \ge 0$, assuming $\|E\|_\theta \ne 0$ we have $Z'(\|E\|_\theta) \le 0$ which implies that $\|E\|_\theta \le 7$. If we suppose $-2 \le \rho+\rho^{-1} \le -2/3$ and $Z(\|E\|_\theta) < 0$ then, $-\frac{16}{9\sqrt{3}} < -|(\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})| \le Z(\|E\|_\theta) < 0$. Thus if we assume $Z'(\|E\|_\theta) > 0$, since $\|E\|_\theta \le 7$ otherwise, we have, as $\|F\|_\theta^2 \ge \omega_E(F^2)$, $\|F\|_\theta^2 \ge -\frac{9\sqrt{3}}{16}(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2\|E\|_\theta Z'(\|E\|_\theta)$. Combining this with (19) we have

$$\begin{aligned}0 \ge -9\sqrt{3}\|E\|_\theta\{&(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2\|E\|_\theta^2-(\rho-\rho^{-1})(\rho^{\frac{3}{2}}-\rho^{-\frac{3}{2}})\varepsilon_E\|E\|_\theta\\ &+(\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}})^2(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})\}\\ &-|32(\rho^{\frac{1}{2}}+\rho^{1\frac{1}{2}})(\rho+1+\rho^{-1})|\,\|E\|_\theta+16(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2}),\end{aligned}$$

which implies that $\|E\|_\theta \le 3$. Consequently we have shown that $\|E\|_\theta \le 7$ for $-2 \le \rho+\rho^{-1} \le -2/3$ and thus using (19) that $\|F\|_\theta \le 3$.

Now if we apply the state ω_E to (8) and substitute for $\omega_E(FEF)$ from (20) we get,

$$
\begin{aligned}
&(\rho+1+\rho^{-1})\{\omega_E(FE^2F)+\omega_E(F^4)\}\\
&= \omega_E(F^2)\big\{(\rho-3+\rho^{-1})\|E\|_\theta^2 - 2(3\rho^3+6\rho^2+10\rho+10+10\rho^{-1}\\
&\quad +6\rho^{-2}+3\rho^{-3})(\rho^{\frac12}+\rho^{-\frac12})^{-1}\varepsilon_E\|E\|_\theta + (2\rho^4-3\rho^3-7\rho^2-8\rho\\
&\quad -4-8\rho^{-1}-7\rho^{-2}-3\rho^{-3}+2\rho^{-4})\big\} + \big\{(\rho-3+\rho^{-1})\|E\|_\theta^4\\
&\quad -2(\rho^3-2\rho^2-2\rho-2-2\rho^{-1}-2\rho^{-2}+\rho^{-3})(\rho^{\frac12}+\rho^{-\frac12})^{-1}\varepsilon_E\|E\|_\theta^3\\
&\quad -2(\rho^{\frac12}-\rho^{-\frac12})(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})(\rho^{\frac52}-\rho^{-\frac52})(\rho^{\frac12}+\rho^{-\frac12})^{-1}\varepsilon_E\|E\|_\theta\\
&\quad +(2\rho^4-\rho^3-3\rho^2-10\rho+6-10\rho^{-1}-3\rho^{-2}-\rho^{-3}+2\rho^{-4})\|E\|_\theta^2\\
&\quad +3(\rho+1+\rho^{-1})^2(\rho-1+\rho^{-1})\big\}\\
&= \omega_E(F^2)\{X(\|E\|_\theta)\} + \{Y(\|E\|_\theta)\},
\end{aligned}
$$

where $X(\|E\|_\theta)$ and $Y(\|E\|_\theta)$ have the obvious definitions.

Thus if $-1 \le \rho+\rho^{-1} < 2$, the left hand side of this equality is non-negative and hence the right hand side must be non-negative. So if $X(\|E\|_\theta) \ge 0$, then $\|E\|_\theta$ is bounded, by 49, as $X(\|E\|_\theta)$ is a negative quadratic in $\|E\|_\theta$. If $X(\|E\|_\theta) \le 0$ then $Y(\|E\|_\theta) \ge 0$ and since $Y(\|E\|_\theta) \ge 0$ is a negative quartic in $\|E\|_\theta$, $\|E\|_\theta$ is bounded by 11 say. Thus $\|E\|_\theta$ is bounded by 49 for $-1 \le \rho+\rho^{-1} < 2$.

Similarly if we apply the state ω_F to (8) and substitute for $\omega_F(EFE)$ from (18) we get,

$$
\begin{aligned}
&(\rho+1+\rho^{-1})\{\omega_F(EF^2E)+\omega_F(E^4)\}\\
&= \omega_F(E^2)\{(\rho-3+\rho^{-1})\|F\|_\theta^2 - (\rho^{\frac12}+\rho^{-\frac12})^2(3\rho^2+\rho+1+\rho^{-1}+3\rho^{-2})\}\\
&\quad + \big\{(\rho-3+\rho^{-1})\|F\|_\theta^4 - \big((\rho^3+3\rho^2+8\rho-6+8\rho^{-1}+3\rho^{-2}+\rho^{-3})\\
&\quad + 2(\rho^{\frac12}+\rho^{-\frac12})(\rho^2+3\rho+1+3\rho^{-1}+\rho^{-2})\omega_F(E)\big)\|F\|_\theta^2\\
&\quad + \big(3(\rho+1+\rho^{-1})^2(\rho-1+\rho^{-1}) + 2(\rho^{\frac12}+\rho^{-\frac12})(\rho^2+\rho+1+\rho^{-1}+\rho^{-2})\omega_F(E^3)\big)\big\}\\
&= \omega_F(E^2)\{X'(\|E\|_\theta)\} + \{Y'(\|E\|_\theta)\},
\end{aligned}
$$

where $X'(\|E\|_\theta)$ and $Y'(\|E\|_\theta)$ have the obvious definitions.

A similar argument to that for $\|E\|_\theta$, noticing that $\|E\|_\theta$ is bounded so $|\omega(E^n)| \le \|E\|_\theta^n$ for any state ω and n any positive integer, gives that $\|F\|_\theta$ is bounded, by 52, for $-1 \le \rho+\rho^{-1} < 2$. That is $\|E\|_\theta$ and $\|F\|_\theta$ are uniformly bounded, by 52, for $-1 \le \rho+\rho^{-1} < 2$. Combining this with the fact that $\|E\|_\theta$ and $\|F\|_\theta$ are uniformly bounded, by 7, for $-2 \le \rho+\rho^{-1} \le -\frac{2}{3}$, we have $\|E\|_\theta$ and $\|F\|_\theta$ are uniformly bounded, by 52, for $-2 \le \rho+\rho^{-1} < 2$. ∎

COROLLARY 7.1.3. *For $\theta \ne 0$ ($\rho \ne 1$) we have $|||E|||, |||F||| \le 52$. Thus $|||\hat{E}||| = |||\hat{F}||| \le 104$.*

7.2 *Proof of Theorem 5.1.2(2) for $\zeta(n,m)$.*

DEFINITION 7.2.1. For $n, m \in \mathbb{Z}$ define $\tilde{U} = \rho^{-n(n+m)/2}U^{n+m}V^{-n}$ and $\tilde{V} = \rho^{nm/2}U^nV^m$ then $\tilde{V}\tilde{U} = \tilde{\rho}\tilde{U}\tilde{V}$ where $\tilde{\rho} = \rho^{(m^2+nm+n^2)}$, $\tilde{\hat{E}} = \sum_{i=0}^2 \zeta^i(\tilde{V}) = \zeta(n,m)$ and $\tilde{\hat{F}} = \tilde{\rho}^{1/2}\sum_{i=0}^2 \zeta^i(\tilde{U}^{-1}\tilde{V}) = \zeta(n,m)^*$.

LEMMA 7.2.2. *If* $|||\hat{E}||| \leq K$ *for all* θ *irrational, then* $|||\zeta(n,m)||| \leq K$ *for all* θ *irrational.*

PROOF. We have shown $|||\hat{E}||| \leq K$ for all θ irrational just by using equations (4) and (5), which in turn are derived using only $VU = \rho UV$. It follows that we can derive the same equations for $\tilde{\hat{E}}, \tilde{\hat{F}}, \tilde{\rho}$ defined in 7.2.1 and then, $|||\tilde{\hat{E}}||| = |||\zeta(n,m)||| \leq K$. ∎

The proof given above in fact holds for all θ, m and n such that $\theta(m^2 + nm + n^2) \notin \mathbb{Z}$. In the rest of this section we will show that a uniform bound for $||\zeta(n,m)||_\theta$ exists for all $n, m \in \mathbb{Z}$ and all rational $\theta \neq 0, \frac{1}{2}, \frac{1}{3}$ and $\frac{2}{3}$. This proves Theorem 5.1.2(2) and hence Theorem 5.1.7(2).

LEMMA 7.2.3. *Suppose* $\max\{||\zeta(n,m)||_\theta, ||\hat{E}||_\theta\} \leq K$ *for some* $K > 0$ *and* $n, m \in \mathbb{Z}$ *fixed. Then, if any two of* $||\zeta(n,m+\Delta)||_\theta$, $||\zeta(n-\Delta,m)||_\theta$ *and* $||\zeta(n+\Delta,m-\Delta)||_\theta$ *are* $\leq K$, *the third one is* $\leq K^2 + 3K$ $(\Delta = \pm 1)$.

PROOF. Straightforward using the product identities for $\zeta(n,m)E$ and $\zeta(n,m)F$. ∎

PROPOSITION 7.2.4. *If* $\theta = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $(p,q) = 1$ *with* $q \neq 1,2,3$ *then there exists a* $K > 0$ *(independent of* θ*) such that,* $||\zeta(n,m)||_\theta \leq K$, $\forall n, m \in \mathbb{Z}$.

PROOF. The proof is similar to that for $\tau(n,m)$ given in Proposition 6.5.3. Now let $B = \{\zeta(n,m) : p(n^2+nm+m^2)q^{-1} \in \mathbb{Z}\}$. If $\zeta(n,m) \notin B$, $||\zeta(n,m)||_\theta \leq K$ with K chosen as in Lemma 7.2.2. It remains to show that there exists a $K' > 0$ such that $||\zeta(n,m)||_\theta \leq K'$ for all $\zeta(n,m) \in B$. As in Proposition 6.5.3 we shall only give a sketch of the proof.

(i) q odd: then $||\hat{E}||_\theta$, $||\hat{F}||_\theta \leq K$ and $\zeta(n,m) \in B$ when $q|(n^2 + nm + m^2)$. If $q = 5$ we have $n, m \equiv 0 (\mathrm{mod}\, 5)$ and if $q = 9$ we have $n, m \equiv 0 (\mathrm{mod}\, 3)$ so applying Lemma 7.2.3 in either case is sufficient. If $q \neq 5, 9$ and n and m are such that $\zeta(n,m) \in B$ the idea is to show that we may assume that six of the eight neighboring points (exclude $\zeta(n\pm 1, m\mp 1)$) on the $\mathbb{Z}^2$-lattice $\{\zeta(n,m) : n, m \in \mathbb{Z}\}$ are not in B (it is sufficient to consider $\zeta(n, m+1)$ and $\zeta(n+1, m+1)$, for example, show by contradiction together with Lemma 7.2.3 that we may assume these are not in B and then consider the symmetry of the situation). In this case either $\zeta(n-1, m+2)$ or $\zeta(n+1, m-2)$ is not in B and an application of Lemma 7.2.3 completes the proof.

(ii) q even: if $\zeta(n,m) \in B$ $n, m \equiv 0 (\mathrm{mod}\, 2)$. Hence, using Lemma 7.2.3 we are done. ∎

REMARK 7.2.5. When $\theta = 0$ $(\rho = 1)$ the relations (6), (7) and (8) become,

$$F^2E + EF^2 = 2FEF, \quad E^2F + FE^2 = 2EFE, \quad F^2E^2 + E^2F^2 = FEFE + EFEF.$$

Consequently any one-dimensional representation Π with $\Pi(E) = e$, $\Pi(F) = f$ is permissible showing $\|E\|_\theta$ and $\|F\|_\theta$ are not uniformly bounded in this case. When $\theta = \frac{1}{2}$ ($\rho = -1$) as stated in Theorem 3.2.1 the polynomial algebras $\mathcal{P}_\theta(\hat{E}, I)$ and $\mathcal{P}_\theta^\zeta$ are not isomorphic. When $\theta = \frac{1}{3}$ and $\frac{2}{3}$ it is not clear if $\|\zeta(n,m)\|_\theta$ is uniformly bounded in n and m. However, there is a complete description of $\mathcal{A}_\theta^\zeta$ for all θ rational, using different methods, given in [11].

8. **Proof of Theorem 5.1.2(3) for $\eta(n,m)$.**

8.1 *A bound for* $|||G|||$ *and* $|||H|||$.

DEFINITION 8.1.1. Let Π be a representation of $\mathcal{P}_\theta(G,H,I)$ and define ω_G to be the state on $\overline{\mathcal{P}_\theta(G,H,I)}^{\|\ \|_{(\theta,\Pi)}}$ given by $|\omega_G(G)| = \|G\|_\theta$ (so $\omega_G(G) = \varepsilon_G\|C\|_\theta$, $\varepsilon_G = \pm 1$), where we denote by $\|x\|_\theta$ the norm $\|x\|_{(\theta,\Pi)}$ for $x \in \mathcal{P}_\theta(G,H,I)$.

THEOREM 8.1.2. *Let Π be any representation of $\mathcal{P}_\theta(G,H,I)$. Then there exists a $K > 0$ (independent of θ and Π) such that $\|G\|_\theta$ and $\|H\|_\theta \leq K$ for θ irrational. So in particular $|||G|||$ and $|||H||| \leq K$ for θ irrational.*

PROOF. This proof is similar in style to the proofs of Theorems 6.1.1 and 7.1.2, and computationally even more complicated. Therefore instead of giving the full details, we provide only a sketch. Firstly by applying the state ω_G to (9) we obtain $\omega_G(H)$ in terms of $\omega_G(H^2)$ and powers of $\|G\|_\theta$. Now by applying ω_G to (10), and using the formula for $\omega_G(H)$, we obtain an expression for $\omega_G(HGH)$ in terms of $\omega_G(H^2)$ and powers of $\|G\|_\theta$. If we now use $|\omega_G(HGH)| \leq \|G\|_\theta\omega_G(H^2)$, we get that $\|G\|_\theta$ is bounded by a constant for the following values of ε_G and $t = (\rho+\rho^{-1})/2 = \cos\phi$:

$$\varepsilon_G = 1,\ t \in (\cos 4\pi/5, 0)\cup(0, \cos 2\pi/5),\ \varepsilon_G = -1,\ t \in (-1, \cos 4\pi/5)\cup(\cos 2\pi/5, 1).$$

Now consider the third relation (11). If we first replace H^3 in (11) by using (9) $\times H$ and then apply the state ω_G together with the expressions for $\omega_G(H)$ and $\omega_G(HGH)$ we obtain a formula for $\omega_G(HG^2H)$ in terms of $\omega_G(H^2)$ and powers of $\|G\|_\theta$. We now use, $0 \leq \omega_G(HG^2H) \leq \|G\|_\theta^2\omega_G(H^2)$, to get that $\|G\|_\theta$ is uniformly bounded on the following regions of the parameter t:

$$(\alpha,\beta)\setminus\{\cos 4\pi/5\},\quad (\gamma,\delta)\setminus\{\cos 2\pi/5\},$$

where $\alpha \approx -.91$ and $\gamma \approx .10$ are roots of the polynomial $16t^4+8t^3-16t^2-8t+1$ and $\beta \approx -.74$ and $\delta \approx .55$ are roots of the polynomial $16t^4+24t^3-8t-1$.

Recall the transformation $\tilde{U} = \rho^{-n(n+m)/2}U^{n+m}V^{-n}$, $\tilde{V} = \rho^{nm/2}U^nV^m$. Then $\tilde{V}\tilde{U} = \tilde{\rho}\tilde{U}\tilde{V}$ where $\tilde{\rho} = \rho^{(n^2+nm+m^2)}$. Notice that $\tilde{G}$ (the G corresponding to $\tilde{\rho}$)$= \eta(n,m)$ and $\tilde{H} = \eta(2n+m, m-n)$. Therefore if $\|\tilde{G}\|_{\tilde{\theta}} = \|G\|_{(n^2+nm+m^2)\theta} \leq K$ then $\|\eta(n,m)\|_\theta \leq K$. In particular for the special case $n = m = 1$ we have $\tilde{G} = H$ so $\|H\|_\theta$ is bounded if $\|G\|_{3\theta}$ is bounded. Moreover, by careful examination of the inequalities arising from (9) and (10), it can be shown that $\|G\|_\theta \leq K$ if $\|H\|_\theta \leq K$ provided $\theta \notin [-\varepsilon_K, \varepsilon_K] (\mathrm{mod}\ 1)$ for some $\varepsilon_K > 0$. If we also consider the special case $n = 2, m = 0$ and observe that,

$$\eta(2,0) = G^2 - (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})(G+H) + 6I,$$

together with $0 \leq (\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}})^2 \omega_G(H^2)/\|G\|_\theta^3 \leq M$, we obtain $\|G\|_\theta \leq K$ if $\|G\|_{4\theta} \leq K$.

If we now take the intervals in t where $\|G\|_\theta$ is uniformly bounded and consider the corresponding infinite set of intervals in ϕ (equivalently θ) and use the two observations above it is not too difficult to fill out the rest of the real line except for infinitely many rational multiples of π (equivalently rational values of θ) (it is useful to note here that $\alpha = \cos 13\pi/15$ and $\gamma = \cos 7\pi/15$). In other words that $\|G\|_\theta \leq K$ and hence $\|H\|_\theta \leq K$ for some constant K for all θ irrational. ■

Using the same technique as Lemma 7.2.2, also used in the proof above, it is straightforward to prove that $|||\eta(n,m)||| \leq K$ for all θ irrational giving Theorem 5.1.2(3).

REMARK 8.1.3. Due to the complicated nature of the calculations for this case we have no comments regarding the rational situation for η. However, a complete description of $\mathcal{A}_\theta^\eta$ for all θ rational, using different methods, is given in [12].

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