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On Gibbs Measures and Spectra of Ruelle Transfer Operators

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Abstract. We prove a comprehensive version of the Ruelle–Perron–Frobenius Theorem with explicit estimates of the spectral radius of the Ruelle transfer operator and various other quantities related to spectral properties of this operator. The novelty here is that the Hölder constant of the function generating the operator appears only polynomially, not exponentially as in previously known estimates.

1 Introduction

We consider a one-sided shift space

$$\Sigma_A^+ = \{\xi = (\xi_0, \xi_1, \dots, \xi_m, \dots) : 1 \le \xi_i \le q, A(\xi_i, \xi_{i+1}) = 1 \text{ for all } \ i \ge 0\},$$

where A is a $q \times q$ matrix of 0's and 1's ($q \ge 2$). We assume that A is *aperiodic*, *i.e.*, there exists an integer $M > 0$ such that $A^M(i, j) > 0$ for all i, j (see [5, Chapter 1]). The *shift map* $\sigma: \Sigma_A^+ \to \Sigma_A^+$ is defined by $\sigma(\xi) = \xi'$, where $\xi_i' = \xi_{i+1}$ for all $i \ge 0$.

In this paper we consider *Ruelle transfer operators* $L_f: C(\Sigma_A^+) \to C(\Sigma_A^+)$ defined by real-valued functions $f: \Sigma_A^+ \to \mathbb{R}$ by $L_f g(x) = \sum_{\sigma(y)=x} e^{f(y)} g(y)$. Here $C(\Sigma_A^+)$ denotes the space of all continuous functions $g: \Sigma_A^+ \to \mathbb{R}$ with the product topology. Given $\theta \in (0,1)$, consider the metric d_θ on Σ_A^+ defined by $d_\theta(\xi, \eta) = 0$ if $\xi = \eta$ and $d_\theta(\xi, \eta) = \theta^k$ if $\xi \ne \eta$ and $k \ge 0$ is the maximal integer with $\xi_i = \eta_i$ for $0 \le i \le k$. For any function $g: \Sigma_A^+ \to \mathbb{R}$ set

$$\operatorname{var}_k g = \sup\{|g(\xi) - g(\eta)| : \xi_i = \eta_i,\ 0 \le i \le k\}, \quad |g|_\theta = \sup\Big\{\frac{\operatorname{var}_k g}{\theta^k} : k \ge 0\Big\},$$
$$|g|_\infty = \sup\{|g(\xi)| : \xi \in \Sigma_A^+\}, \quad \|g\|_\theta = |g|_\theta + |g|_\infty.$$

Denote by $\mathcal{F}_\theta(\Sigma_A^+)$ the *space of all functions* g on Σ_A^+ with $\|g\|_\theta < \infty$, and by $\operatorname{spec}_\theta(L_g)$ the *spectrum* of $L_g: \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$.

The Ruelle–Perron–Frobenius Theorem concerns spectral properties of the transfer operator $L_f: \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$. Assuming A is aperiodic and $f \in \mathcal{F}_\theta(\Sigma_A^+)$ is real-valued, it asserts that L_f has a simple maximal positive eigenvalue λ, a corresponding strictly positive eigenfunction h, and a probability measure ν on Σ_A^+ such that $\operatorname{spec}_\theta(L_f) \setminus \{\lambda\}$ is contained in a disk of radius $\rho\lambda$ for some $\rho \in (0,1)$, $L_f^* \nu = \lambda\nu$, and assuming h is normalized by $\int h\,d\nu = 1$, we also have

$$\lim_{n\to\infty} \frac{1}{\lambda^n} L_f^n g = h \int g\,d\nu \tag{1.1}$$

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for all $g \in \mathcal{F}_\theta(\Sigma_A^+)$. This was proved by Ruelle [7] (see also [8]). In the case of a complex-valued function f, similar results were established by Pollicott [6].

In this paper a comprehensive version of the Ruelle–Perron–Frobenius Theorem is considered which provides explicit estimates for the various constants and functions involved, *e.g.*, the function h and the constant ρ mentioned above, as well as the speed of convergence in (1.1). Estimates of this kind were derived in [9], however the constants that appeared there, including the estimate ρ for the spectral radius of the operator L_f, involved terms of the form $e^{C|f|_\theta}$ for various constants $C > 0$. The same applies to the estimates that appear in [2,5,7,8] and also to the estimate of the spectral radius of L_f obtained in [4].

From our personal experience, when estimates for families of Ruelle transfer operators L_f are considered for a class of functions f, usually the norms $|f|_\infty$ are uniformly bounded. However, the Hölder constants $|f|_\theta$ can vary widely and in some cases can be arbitrarily large. That is why estimates involving terms of the form $e^{C|f|_\theta}$ are particularly unpleasant.

All estimates obtained in this paper involve only powers of $|f|_\theta$, and, in this sense, they are significantly sharper than the existing ones.

The motivation for [9] came from investigations in scattering theory on distribution of scattering resonances, in particular in dealing with the so-called modified Lax–Phillips conjecture for obstacles K in $\mathbb{R}^n$ that are finite disjoint unions of strictly convex bodies with smooth boundaries [10]. The present work stems from studies on decay of correlations for Axiom A flows and spectra of Ruelle transfer operators in the spirit of [3,11].

Section 2 contains the statement of the Ruelle–Perron–Frobenius Theorem with comprehensive estimates of the constants involved, while Section 3 is devoted to a proof of the theorem. As in [9], we follow the main frame of the proof in [2] with necessary modifications.

2 The Ruelle–Perron–Frobenius Theorem

In what follows A will be a $q \times q$ matrix ($q \ge 2$) such that $A^M > 0$ for some integer $M \ge 1$, $\theta \in (0,1)$ will be a fixed number, and $f \in \mathcal{F}_\theta(\Sigma_A^+)$ will be a fixed real-valued function. Set $b = b_f = \max\{1, |f|_\theta\}$.

Theorem 2.1 (Ruelle–Perron–Frobenius Theorem) (i) *There exist a unique $\lambda = \lambda_f > 0$, a probability measure $\nu = \nu_f$ on Σ_A^+, and a positive function $h = h_f \in \mathcal{F}_\theta(\Sigma_A^+)$ such that $L_f h = \lambda h$ and $\int h \, d\nu = 1$. The spectral radius of L_f as an operator on $\mathcal{F}_\theta(\Sigma_A^+)$ is λ, and its essential spectral radius is $\theta\lambda$. The eigenfunction h satisfies $\frac{1}{K} \le h \le K$ and $|h|_\theta \le BbK$, where*

$$K = Bb^{r_0}, \tag{2.1}$$

and the constants B and r_0 can be chosen as

$$B = \frac{e^{\frac{2\theta}{1-\theta}} q^{M+1} e^{2(M+1)|f|_\infty}}{1-\theta}, \qquad r_0 = \frac{\log q + 2|f|_\infty}{|\log \theta|}. \tag{2.2}$$

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(ii) *The probability measure $\widehat{\nu} = h\nu$ (this is the so called Gibbs measure generated by f) is σ-invariant.*

(iii) *We have* $\mathrm{spec}_\theta(L_f) \cap \{z \in \mathbb{C} : |z| = \lambda\} = \{\lambda\}$. *Moreover λ is a simple eigenvalue for L_f and every $z \in \mathrm{spec}_\theta(L_f)$ with $|z| < \lambda$ satisfies $|z| \le \rho\lambda$, where*

$$\rho = 1 - \frac{1-\theta}{8K^3} \in (0,1). \tag{2.3}$$

(iv) *For every $g \in \mathcal{F}_\theta(\Sigma_A^+)$ and every integer $n \ge 0$ we have*

$$\left\| L_f^n g - h \int g \, d\nu \right\|_\theta \le D_f \lambda^n \rho^n \|g\|_\theta, \tag{2.4}$$

where $D_f = \frac{100K^5 b^3}{1-\theta}$.

The constants K, ρ, D_f, etc., are not optimal; slightly better estimates are possible as one can see from the proof in Section 3.

3 Proof of Theorem 2.1

We will use the notation and assumptions from Section 2. Set $L = L_f$. Given $x = (x_0, x_1, \dots) \in \Sigma_A^+$ and $m \ge 0$, consider the *cylinder of length m*

$$\mathcal{C}_m[x] = \{y = (y_0, y_1, \dots) \in \Sigma_A^+ : y_j = x_j \text{ for all } j = 0, 1, \dots, m\}$$

determined by x. Set $g_m(x) = g(x) + g(\sigma x) + \cdots + g(\sigma^{m-1} x)$.

As in [2], it follows from the Schauder–Tychonoff Theorem that there exist a Borel probability measure ν on Σ_A^+ and a number $\lambda > 0$ such that $L_f^* \nu = \lambda\nu$, that is, $\lambda \int g \, d\nu = \int L_f g \, d\nu$ for every $g \in C(\Sigma_A^+)$. With $g = 1$, this gives $\lambda = \int L_f 1 \, d\nu$. Clearly, $(L_f 1)(x) = \sum_{\sigma\xi = x} e^{f(\xi)} \le q e^{|f|_\infty}$, and also $(L_f 1)(x) \ge e^{-|f|_\infty}$ for all $x \in \Sigma_A^+$. Thus,

$$e^{-|f|_\infty} \le \lambda \le q e^{|f|_\infty}. \tag{3.1}$$

Let $m_0 = m_0(f, \theta) \ge 1$ be *the integer such that*

$$\theta^{m_0} < \frac{1}{b} \le \theta^{m_0 - 1}. \tag{3.2}$$

Then $m_0 \log\theta < -\log b \le (m_0 - 1)\log\theta$, so

$$m_0 - 1 \le \frac{\log b}{|\log\theta|} < m_0.$$

The first significant difference between our argument and the one in [2] is in the definitions of the constants B_m and the space Λ below. In our argument they depend on m_0, *i.e.*, on b.

For $m \ge m_0$, set $B_m = e^{2\sum_{j=m-m_0+1}^{\infty} \theta^j}$ and define

$$\Lambda = \Big\{ g \in C(\Sigma_A^+) : g \ge 0, \\ \int g \, d\nu = 1, g(y) \le B_m\, g(x) \text{whenever } y \in \mathcal{C}_m[x], m \ge m_0 \Big\}.$$

Then $B_{m_0} = e^{2\sum_{j=1}^{\infty} \theta^j} = e^{\frac{2\theta}{1-\theta}}$. Notice that in the above definitions *we only consider integers m with $m \ge m_0$.* This will be significant later on.

Lemma 3.1 *Λ is non-empty, convex, and closed in a $C(\Sigma_A^+)$ equicontinuous family of functions, and the operator $T = \frac{1}{\lambda}L$ maps Λ into Λ.*

Proof We use a modification of the proof of [2, Lemma 1.8]. It is clear that Λ is convex and closed in $C(\Sigma_A^+)$, and also $\Lambda \neq \varnothing$ since $1 \in \Lambda$.

Consider arbitrary $x = (x_0, x_1, \dots), z = (z_0, z_1, \dots) \in \Sigma_A^+$, and $g \in \Lambda$. Since $A^M > 0$, there exists a sequence $(z'_{m_0+1}, z'_{m_0+2}, \dots, z'_{m_0+M-1}, z'_{m_0+M} = x_0)$ such that

$$y = (z_0, z_1, \dots, z_{m_0}, z'_{m_0+1}, z'_{m_0+2}, \dots, z'_{m_0+M-1}, z'_{m_0+M} = x_0, x_1, x_2, \dots) \in \Sigma_A^+.$$

Then $d_\theta(y, z) \le \theta^{m_0}$, so $g \in \Lambda$ implies $g(z) \le B_{m_0} g(y)$. Moreover, $\sigma^{m_0+M} y = x$, so

$$(L^{m_0+M} g)(x) = \sum_{\sigma^{m_0+M}(\xi)=x} e^{f_{m_0+M}(\xi)} g(\xi) \ge e^{f_{m_0+M}(y)} g(y) \ge \frac{e^{-(m_0+M)|f|_\infty}}{B_{m_0}} g(z). \tag{3.3}$$

Keeping z fixed and integrating (3.3) with respect to x, gives

$$1 = \int g\, d\nu = \frac{1}{\lambda^{m_0+M}} \int L^{m_0+M} g\, d\nu \ge \frac{e^{-(m_0+M)|f|_\infty}}{\lambda^{m_0+M} B_{m_0}} g(z).$$

Setting $K' = B_{m_0} \lambda^{m_0+M} e^{(m_0+M)|f|_\infty}$, the above implies $g(z) \le K'$. This is true for all $z \in \Sigma_A^+$, so $|g|_\infty \le K'$ for all $g \in \Lambda$. Using (3.1), (3.2), and the definition of B_{m_0}, we get

$$K' \le e^{\frac{2\theta}{1-\theta}} (q e^{|f|_\infty})^{m_0+M} e^{(m_0+M)|f|_\infty} \le B e^{\frac{\log b}{|\log\theta|}(\log q + 2|f|_\infty)} < B b^{r_0} = K,$$

where K is as in (2.1), while B and r_0 are defined by (2.2). (For later convenience we take slightly larger B and r_0 than necessary here.) Thus,

$$|g|_\infty \le K, \quad g \in \Lambda. \tag{3.4}$$

Next, integrating (3.3) with respect to z yields

$$(T^{m_0+M} g)(x) = \frac{1}{\lambda^{m_0+M}} (L^{m_0+M} g)(x) \ge \frac{e^{-(m_0+M)|f|_\infty}}{\lambda^{m_0+M} B_{m_0}} = \frac{1}{K'} \ge \frac{1}{K}.$$

Thus,

$$\frac{1}{K} \le \min(T^{m_0+M} g), \quad g \in \Lambda. \tag{3.5}$$

Let is now prove that Λ is an equicontinuous family of functions. Given $\epsilon > 0$, take $m \ge m_0$ so that $e^{2\theta^{m-m_0+1}/(1-\theta)} - 1 < \epsilon/K$. Let $x, y \in \Sigma_A^+$ be such that $d_\theta(x, y) \le \theta^m$. Then for any $g \in \Lambda$, we have $g(x) \le B_m g(y)$, so $g(x) - g(y) \le (B_m - 1) g(y) \le (B_m - 1)K$. Similarly, $g(y) - g(x) \le (B_m - 1)K$, so

$$|g(x) - g(y)| \le (B_m - 1)K = \left(e^{2\theta^{m-m_0+1}/(1-\theta)} - 1\right) K < \epsilon.$$

Hence Λ is equicontinuous.

It remains to show that $T(\Lambda) \subset \Lambda$. Let $g \in \Lambda$. Then $Tg \ge 0$ and $\int Tg\, d\nu = 1$. Let $m \ge m_0$ and let $y \in \mathcal{C}_m[x]$. Given $\xi = (\xi_0, \xi_1, \dots) \in \Sigma_A^+$ with $\sigma\xi = x$, we have $\xi_1 = x_0 = y_0, \xi_2 = x_1 = y_1, \dots, \xi_{m+1} = x_m = y_m$. Set

$$\eta = \eta(\xi) = (\xi_0, \xi_1, \dots, \xi_m, \xi_{m+1} = y_m, y_{m+1}, y_{m+2}, \dots) \in \Sigma_A^+. \tag{3.6}$$

Then $\sigma\eta = y$ and $d(\xi,\eta(\xi)) \le \theta^{m+1}$, so by (3.2),

$$|f(\xi) - f(\eta(\xi))| \le |f|_\theta d_\theta(\xi,\eta(\xi)) \le b\,\theta^{m+1} \le \theta^{m-m_0+1}.$$

This and $g \in \Lambda$ imply $g(\xi) \le B_{m+1} g(\eta(\xi))$ and

$$\begin{aligned}(Tg)(x) &= \frac{1}{\lambda}\sum_{\sigma\xi=x} e^{f(\xi)} g(\xi) \le \frac{1}{\lambda}\sum_{\sigma\xi=x} e^{f(\eta(\xi))+\theta^{m-m_0+1}} B_{m+1} g(\eta(\xi))\\ &= \frac{e^{\theta^{m-m_0+1}} B_{m+1}}{\lambda}\sum_{\sigma\eta=y} e^{f(\eta)} g(\eta) = e^{\theta^{m-m_0+1}} e^{2\sum_{j=m-m_0+2}^{\infty}\theta^j}(Tg)(y)\\ &\le B_m (Tg)(y).\end{aligned}$$

Thus, $Tg \in \Lambda$. ∎

Using the above lemma and the Schauder–Tychonoff Theorem we derive the following corollary.

Corollary 3.2 *There exists $h \in \Lambda$ with $Th = h$,* i.e., *with $Lh = \lambda h$. Moreover we have $\frac{1}{K} \le h \le K$, where K is given by (2.1).*

The latter follows from (3.4) and (3.5), since $T^{m_0+M} h = h$.

Lemma 3.3 *There exists a constant $\mu \in (0,1)$ such that for every $g \in \Lambda$ there exists $\widetilde{g} \in \Lambda$ with $T^{m_0+M} g = \mu h + (1-\mu)\widetilde{g}$. More precisely, we can take*

$$\mu = \frac{1-\theta}{4K^2 e^{2\theta/(1-\theta)}} < \frac{1}{4K^2}. \tag{3.7}$$

Proof We use a modification of the proof of [2, Lemma 1.9]. Define μ by (3.7). Given $g \in \Lambda$, set $g_1 = T^{m_0+M} g - \mu h$ and $\widetilde{g} = \frac{g_1}{1-\mu}$. Then (3.7) and $T^{m_0+M} g \in \Lambda$ imply $\mu h \le \mu K < \frac{1}{K} \le \min(T^{m_0+M} g)$, so $g_1 > 0$. Moreover $\int g_1\, d\nu = 1-\mu$, so $\int \widetilde{g}\, d\nu = 1$.

Next, let $m \ge m_0$ and let $x \in \Sigma_A^+$, $y \in \mathcal{C}_m[x]$. We will show that $\widetilde{g}(x) \le B_m \widetilde{g}(y)$, which is equivalent to $g_1(x) \le B_m g_1(y)$, *i.e.*, to

$$(T^{m_0+M} g)(x) - \mu h(x) \le B_m\big((T^{m_0+M} g)(y) - \mu h(y)\big),$$

that is, to

$$\mu(B_m h(y) - h(x)) \le B_m (T^{m_0+M} g)(y) - (T^{m_0+M} g)(x). \tag{3.8}$$

Given $\xi \in \Sigma_A^+$ with $\sigma\xi = x$ define $\eta = \eta(\xi)$ by (3.6); then $\sigma\eta = y$ and $\eta \in \mathcal{C}_{m+1}[\xi]$. For any $G \in \Lambda$, as in the proof of Lemma 3.1, we have

$$(LG)(x) = \sum_{\sigma\xi=x} e^{f(\xi)} G(\xi) \le \sum_{\sigma\xi=x} e^{f(\eta)+\theta^{m-m_0+1}} B_{m+1} G(\eta) \le e^{\theta^{m-m_0+1}} B_{m+1} (LG)(y).$$

Using this with $G = T^{m_0+M-1} g = \frac{1}{\lambda^{m_0+M-1}} L^{m_0+M-1} g \in \Lambda$ gives

$$(T^{m_0+M} g)(x) \le e^{\theta^{m-m_0+1}} B_{m+1} (T^{m_0+M} g)(y).$$

This and $h(x) \ge \frac{h(y)}{B_m}$ show that to prove (3.8) it is enough to establish

$$\mu\Big(B_m - \frac{1}{B_m}\Big) h(y) \le \Big(B_m - e^{\theta^{m-m_0+1}} B_{m+1}\Big)(T^{m_0+M} g)(y).$$

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Next, the definition of B_m, $h(y) \le K$, and $(T^{m_0+M}g)(y) \ge 1/K$ show that the latter will be true if we prove

$$\mu\Big(e^{\frac{2\theta^{m-m_0+1}}{1-\theta}} - e^{-\frac{2\theta^{m-m_0+1}}{1-\theta}}\Big) \le \Big(e^{2\theta^{m-m_0+1}}B_{m+1} - e^{\theta^{m-m_0+1}}B_{m+1}\Big)\frac{1}{K^2},$$

which is equivalent to

$$\mu\Big(e^{\frac{2\theta^{m-m_0+1}}{1-\theta}} - e^{-\frac{2\theta^{m-m_0+1}}{1-\theta}}\Big) \le e^{\theta^{m-m_0+1}+\frac{2\theta^{m-m_0+2}}{1-\theta}} \cdot \frac{e^{\theta^{m-m_0+1}}-1}{K^2}. \tag{3.9}$$

For the left-hand-side of (3.9) there exists some z with $|z| < 2\theta^{m-m_0+1}/(1-\theta)$ such that

$$\mu\Big(e^{\frac{2\theta^{m-m_0+1}}{1-\theta}} - e^{-\frac{2\theta^{m-m_0+1}}{1-\theta}}\Big) = \mu e^z \frac{4e^{\theta^{m-m_0+1}}}{1-\theta} \le \mu\frac{4e^{\frac{2\theta}{1-\theta}}}{1-\theta}\theta^{m-m_0+1}.$$

For the right-hand-side of (3.9) we have

$$e^{\theta^{m-m_0+1}+\frac{2\theta^{m-m_0+2}}{1-\theta}} \cdot \frac{e^{\theta^{m-m_0+1}}-1}{K^2} > \frac{e^{\theta^{m-m_0+1}}-1}{K^2} \ge \frac{\theta^{m-m_0+1}}{K^2}.$$

Thus, (3.9) would follow from $\mu\frac{4e^{\frac{2\theta}{1-\theta}}}{1-\theta}\theta^{m-m_0+1} \le \frac{\theta^{m-m_0+1}}{K^2}$. The latter is clearly true by (3.7). This proves (3.9) which, as we observed, implies (3.8). Hence $\widetilde{g}(x) \le B_m\widetilde{g}(y)$ which shows that $\widetilde{g} \in \Lambda$. ■

Lemma 3.4 *There exist constants $A > 0$ and $\beta \in (0,1)$ such that*

$$|T^n g - h|_\infty \le A\beta^n \tag{3.10}$$

for every $g \in \Lambda$ and every integer $n \ge 0$. More precisely we can take

$$A = 4K^2, \quad \beta = 1 - \frac{1-\theta}{4K^3} \in (\theta, 1). \tag{3.11}$$

Proof We use a modification of the proof of [2, Lemma1.10].

Let $g \in \Lambda$. Given an integer $n \ge 0$ write $n = p(m_0+M) + r$ for some integers $p \ge 0$ and $r = 0, 1, \ldots, m_0+M-1$. By Lemma 3.3, $T^{m_0+M}g = \mu h + (1-\mu)g_1$ for some $g_1 \in \Lambda$. Similarly, $T^{m_0+M}g_1 = \mu h + (1-\mu)g_2$ for some $g_2 \in \Lambda$, so

$$T^{2(m_0+M)}g = \mu h + (1-\mu)(\mu h + (1-\mu)g_2) = \mu h(1 + (1-\mu)) + (1-\mu)^2 g_2.$$

Continuing in this way, we prove by induction

$$T^{p(m_0+M)}g = \mu h(1 + (1-\mu) + \cdots + (1-\mu)^{p-1}) + (1-\mu)^p g_p$$

for some $g_p \in \Lambda$. Thus,

$$T^{p(m_0+M)}g = \mu h\frac{1-(1-\mu)^p}{1-(1-\mu)} + (1-\mu)^p g_p = h(1-(1-\mu)^p) + (1-\mu)^p g_p,$$

and therefore, using (3.4),

$$|T^{p(m_0+M)}g - h|_\infty \le (1-\mu)^p|h - g_p|_\infty \le 2K(1-\mu)^p.$$

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Next, notice that by (3.1) for every bounded function G on Σ_A^+ we have

$$|(TG)(x)| = \frac{1}{\lambda}\Big|\sum_{\sigma\xi=x} e^{f(\xi)}G(\xi)\Big| \le \frac{qe^{|f|_\infty}}{\lambda}|G|_\infty \le qe^{2|f|_\infty}|G|_\infty,$$

so $|TG|_\infty \le qe^{2|f|_\infty}|G|_\infty$. Using this r times and setting $\beta' = (1-\mu)^{\frac{1}{m_0+M}}$, yields

$$\begin{aligned}|T^n g - h|_\infty &= |T^r(T^{p(m_0+M)}g - h)|_\infty \le (qe^{2|f|_\infty})^r |T^{p(m_0+M)}g - h|_\infty\\ &\le 2K(qe^{2|f|_\infty})^{m_0+M}\frac{(\beta')^n}{(\beta')^r} \le \frac{2K}{1-\mu}(qe^{2|f|_\infty})^{m_0+M}(\beta')^n.\end{aligned}$$

As in previous estimates, using (3.2) and (3.4), we get

$$\begin{aligned}(qe^{2|f|_\infty})^{m_0+M} &= q^M e^{2M|f|_\infty} e^{m_0(\log q + 2|f|_\infty)} \le q^M e^{2M|f|_\infty} e^{(\frac{\log b}{|\log\theta|}+1)(\log q+2|f|_\infty)}\\ &\le q^{M+1}e^{(2M+1)|f|_\infty} b^{r_0} < K.\end{aligned}$$

We have $1-\mu \ge 1/2$ by (3.7), so the above and (3.11) imply $|T^n g - h|_\infty \le A(\beta')^n$.

It remains to show that $\beta' \le \beta$. We will use the elementary inequality $(1-x)^a \le 1 - ax$ for $0 \le x < 1$ and $0 < a < 1$. It implies

$$\beta' = (1-\mu)^{\frac{1}{m_0+M}} \le 1 - \frac{\mu}{m_0+M} < 1 - \frac{\mu}{e^{m_0+M}} = 1 - \frac{1-\theta}{4K^2 e^{m_0+M} e^{2\theta/(1-\theta)}} < 1 - \frac{1-\theta}{4K^3} = \beta.$$

This proves the lemma. ∎

Lemma 3.5 *For every $g \in \Lambda$ we have $|g|_\theta < BbK$, and so $\|g\|_\theta < 2BbK$.*

Proof Let $g \in \Lambda$ and let $x, y \in \Sigma_A^+$ be such that $d_\theta(x,y) = \theta^m$. If $m \le m_0 - 1$, then by (3.4),

$$|g(x) - g(y)| \le 2K = 2K\frac{d_\theta(x,y)}{\theta^m} \le \frac{2K}{\theta^{m_0-1}}d_\theta(x,y) \le 2bKd_\theta(x,y) \le BbKd_\theta(x,y).$$

Next, assume that $m \ge m_0$. Then using again (3.2) and (3.4), we get

$$B_m - 1 = e^{\frac{2\theta^{m-m_0+1}}{1-\theta}} - 1 \le e^{2\theta/(1-\theta)}\frac{2\theta^{m-m_0+1}}{1-\theta} = \frac{2e^{2\theta/(1-\theta)}}{(1-\theta)\theta^{m_0-1}}\theta^m < Bb\theta^m.$$

Since $g(x) \le B_m g(y)$, we have

$$g(x) - g(y) \le (B_m - 1)g(y) \le (Bb\theta^m)K = BbKd_\theta(x,y).$$

Similarly, $g(y) - g(x) < BbKd_\theta(x,y)$, so $|g(x) - g(y)| < BbKd_\theta(x,y)$. ∎

In particular, $\Lambda \subset \mathcal{F}_\theta(\Sigma_A^+)$, so λ is an eigenvalue of the transfer operator

$$L_f : \mathcal{F}_\theta(\Sigma_A^+) \to \mathcal{F}_\theta(\Sigma_A^+)$$

and $h > 0$ is a corresponding eigenfunction. Moreover, following arguments from the proof of [5, Theorem 2.2], one proves that λ is a simple eigenvalue and $\mathrm{spec}_\theta(L_f) \subset \{z : |z| \le \lambda\}$. Also, following the argument from the proof of [1, Theorem 1.5], one shows that the essential spectral radius of L_f as an operator on $\mathcal{F}_\theta(\Sigma_A^+)$ is $\theta\lambda$.

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Lemma 3.6 *For every $g \in \mathcal{F}_\theta(\Sigma_A^+)$, we have*

$$\Big|\frac{1}{\lambda^n}L^n g - h\int g\,d\nu\Big|_\infty \le A_1\beta^n\|g\|_\theta, \quad n \ge 0, \tag{3.12}$$

where $A_1 = 2Ab = 8K^2 b$.

Proof We will proceed as in [9] with some modifications. Let $g \in \mathcal{F}_\theta(\Sigma_A^+)$. First assume that $g \ge 0$. The case $|g|_\theta = 0$ follows trivially from Lemma 3.4, so assume $|g|_\theta > 0$ and set $\widetilde{g} = C\,g + 1$, where $C = \frac{2}{(1-\theta)b|g|_\theta}$. Then $\omega = \int \widetilde{g}\,d\nu \ge 1$.

We will check that $\widetilde{g}/\omega \in \Lambda$. Let $m \ge m_0$, and let $x, y \in \Sigma_A^+$ be such that $y \in \mathcal{C}_m[x]$. Assume for example that $\widetilde{g}(x) \ge \widetilde{g}(y)$. We have

$$\widetilde{g}(x) - \widetilde{g}(y) = C(g(x) - g(y)) \le C|g|_\theta d_\theta(x,y) = C|g|_\theta\theta^m.$$

Hence, using $\widetilde{g}(y) \ge 1$ and (3.2), it follows that

$$\begin{aligned}\widetilde{g}(x) &\le \widetilde{g}(y) + \frac{C|g|_\theta\theta^{m-m_0+1}}{\theta^{m_0-1}} \le \widetilde{g}(y)\Big(1 + \frac{2\theta^{m-m_0+1}}{(1-\theta)b\,\theta^{m_0-1}}\Big)\\ &\le \widetilde{g}(y)\Big(1 + \frac{2\theta^{m-m_0+1}}{(1-\theta)}\Big) \le \widetilde{g}(y)e^{\frac{2\theta^{m-m_0+1}}{(1-\theta)}} = \widetilde{g}(y)B_m.\end{aligned}$$

This shows that $\widetilde{g}/\omega \in \Lambda$, and by (3.10), $|T^n\widetilde{g} - \omega h|_\infty \le A\omega\beta^n$. Thus,

$$\Big|T^n(Cg+1) - h\Big(C\int g\,d\nu + 1\Big)\Big|_\infty \le A\omega\beta^n.$$

Using this and (3.10) with $g = 1$ yields

$$C\Big|T^n g - h\int g\,d\nu\Big|_\infty \le \Big|T^n 1 - h\int 1\,d\nu\Big|_\infty + A\omega\beta^n \le A(\omega+1)\beta^n,$$

so $|T^n g - h\int g\,d\nu|_\infty \le A\frac{\omega+1}{C}\beta^n$. Finally,

$$\frac{1}{C}(\omega+1) = \int g\,d\nu + \frac{2}{C} \le |g|_\infty + (1-\theta)b|g|_\theta \le b\|g\|_\theta.$$

Hence

$$\Big|T^n g - h\int g\,d\nu\Big|_\infty \le Ab\|g\|_\theta\beta^n. \tag{3.13}$$

For general $g \in \mathcal{F}_\theta(\Sigma_A^+)$, write $g = g_+ - g_-$, where $g_+ = \max\{g, 0\} \ge 0$ and $g_- = g_+ - g \ge 0$. Then $g_+, g_- \in \mathcal{F}_\theta(\Sigma_A^+)$, $\|g_+\|_\theta \le \|g\|_\theta$, $\|g_-\|_\theta \le \|g\|_\theta$, $g_+ \le |g|_\infty$, and $g_- \le |g|_\infty$, so $\|g_+\|_\theta \le \|g\|_\theta$ and $\|g_-\|_\theta \le \|g\|_\theta$. Using (3.13) for g_+ and g_- implies $|T^n g - h\int g\,d\nu|_\infty \le 2Ab\|g\|_\theta\beta^n$. ∎

We will now sketch the proofs of the Basic Inequalities (see [5, Proposition 2.1] or [2, Lemma 1.2]) keeping track on the constants involved. We continue to use the notation from Section 2 and also the one introduced above for the function f and the operator $L = L_f$.

Lemma 3.7 (Basic Inequalities) *We have*

$$|L^n g|_\infty \le K^2\lambda^n|g|_\infty, \quad g \in C(\Sigma_A^+), n \ge 0, \tag{3.14}$$

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and

$$|L^n g|_\theta \le K^2 \lambda^n \Big[\frac{2|f|_\theta}{1-\theta} |g|_\infty + \theta^n |g|_\theta \Big], \quad g \in \mathcal{F}_\theta(\Sigma_A^+), n \ge 0. \tag{3.15}$$

Consequently,

$$\|L^n g\|_\theta \le \frac{4bK^2}{1-\theta} \lambda^n \|g\|_\theta, \quad g \in \mathcal{F}_\theta(\Sigma_A^+), n \ge 0. \tag{3.16}$$

Proof We will just follow the standard arguments to derive the above estimates. It follows from Corollary 3.2 that $L^n 1 = K(L^n 1/K) \le K L^n h = K\lambda^n h \le K^2 \lambda^n$, so $L^n 1 \le K^2 \lambda^n$ for all $n \ge 0$.

Given $g \in C(\Sigma_A^+)$, for any $x \in \Sigma_A^+$ and any $n \ge 1$, we have

$$|(L^n g)(x)| \le \sum_{\sigma^n \xi = x} e^{f_n(\xi)} |g(\xi)| \le |g|_\infty (L^n 1)(x) \le K^2 \lambda^n |g|_\infty.$$

This proves (3.14).

Next, let $g \in \mathcal{F}_\theta(\Sigma_A^+)$, and let $n \ge 1$. Given $x \in \Sigma_A^+$ and $y \in \mathcal{C}_n[x]$, for any $\xi \in \Sigma_A^+$ with $\sigma^n \xi = x$, denote by $\eta = \eta(\xi)$ the unique element of Σ_A^+ such that $\sigma^n \eta = y$ and $d_\theta(\xi, \eta) = \theta^n d_\theta(x, y)$. Then

$$|f_n(\xi) - f_n(\eta(\xi))| \le \sum_{j=0}^{n-1} |f(\sigma^j \xi) - f(\sigma^j \eta)| \le \sum_{j=0}^{n-1} |f|_\theta \theta^{n-j} d_\theta(x, y) \le \frac{|f|_\theta}{1-\theta} d_\theta(x, y),$$

and therefore

$$|e^{f_n(\xi)} - e^{f_n(\eta)}| \le |f_n(\xi) - f_n(\eta)| e^{\max\{f_n(\xi), f_n(\eta)\}} \le \frac{|f|_\theta}{1-\theta} d_\theta(x, y) \big[e^{f_n(\xi)} + e^{f_n(\eta)} \big].$$

The above yields

$$\begin{aligned}
|(L^n g)(x) - (L^n g)(y)| &\le \sum_{\sigma^n \xi = x} |e^{f_n(\xi)} g(\xi) - e^{f_n(\eta(\xi))} g(\eta(\xi))| \\
&\le \sum_{\sigma^n \xi = x} \big[|e^{f_n(\xi)} - e^{f_n(\eta)}| \, |g(\xi)| + e^{f_n(\eta)} |g(\xi) - g(\eta)| \big] \\
&\le \frac{|g|_\infty |f|_\theta}{1-\theta} d_\theta(x, y) \sum_{\sigma^n \xi = x} [e^{f_n(\xi)} + e^{f_n(\eta)}] + |g|_\theta \theta^n d_\theta(x, y) \sum_{\sigma^n \xi = x} e^{f_n(\eta)} \\
&\le \frac{|g|_\infty |f|_\theta}{1-\theta} d_\theta(x, y) [(L^n 1)(x) + (L^n 1)(y)] + |g|_\theta \theta^n d_\theta(x, y) (L^n 1)(y) \\
&\le K^2 \lambda^n \Big[\frac{2|f|_\theta}{1-\theta} |g|_\infty + \theta^n |g|_\theta \Big] d_\theta(x, y),
\end{aligned}$$

which proves (3.15). The latter obviously implies (3.16). ∎

To derive Theorem 2.1(iii), just notice that (2.3) implies $\rho > \beta$, where β is given by (3.11). If $z \in \operatorname{spec}_\theta(L_f)$ with $\rho\lambda < |z|$ and $z \ne \lambda$, then z is an eigenvalue of L. If g is a corresponding eigenfunction, then $\int g \, d\nu = 0$ by (3.12), and using (3.12) again, gives $|z| \le \beta\lambda < \rho\lambda$, a contradiction. This shows that $\operatorname{spec}_\theta(L_f) \cap \{z : |z| > \rho\,\lambda\} = \{\lambda\}$.

We will now use (3.12) to prove the following lemma.

Lemma 3.8 *For every $g \in \mathcal{F}_\theta(\Sigma_A^+)$ we have*

$$\Big\| \frac{1}{\lambda^n} L_f^n g - h \int g\, d\nu \Big\|_\theta \le A_2\, \rho^n \|g\|_\theta, \quad n \ge 0, \tag{3.17}$$

where ρ is given by (2.3) *and $A_2 = \frac{100K^5 b^3}{(1-\theta)}$.*

Proof We will again use a corresponding argument in [9] with some modifications. Let $g \in \mathcal{F}_\theta(\Sigma_A^+)$ and let $n \ge 1$.

Case 1: $\int g\, d\nu = 0$. Set $C = \frac{2|f|_\theta}{1-\theta}$, $\ell = [n/2]$, and $k = n - \ell$. First notice that in the present case (3.12) gives $|L^\ell g|_\infty \le A_1 \lambda^\ell \beta^\ell \|g\|_\theta$. Using this, (3.15), (3.12), and $\theta \le \beta$, yields

$$\begin{aligned}|L^n g|_\theta &= |L^k(L^\ell g)|_\theta \le K^2 \lambda^k (C|L^\ell g|_\infty + \theta^k |L^\ell g|_\theta)\\ &\le K^2\lambda^k\big[\, CA_1\, \lambda^\ell \beta^\ell \|g\|_\theta + \theta^k K^2 \lambda^\ell (C|g|_\infty + \theta^\ell |g|_\theta)\big] \le A' \lambda^n \beta^{n/2} \|g\|_\theta,\end{aligned}$$

where $A' = \frac{40K^4 b^2}{1-\theta}$. This proves (3.17) in the case considered.

Case 2: General case: let $g \in \mathcal{F}_\theta(\Sigma_A^+)$ and let $n \ge 1$. Set $\widetilde{g} = g - \alpha h$, where $\alpha = \int g\, d\nu$. Then $\int \widetilde{g}\, d\nu = 0$, so by Case 1, we have $|L^n \widetilde{g}|_\theta \le A' \lambda^n \beta^{n/2} \|\widetilde{g}\|_\theta$. By Corollary 3.2 we have $|\widetilde{g}|_\infty \le |g|_\infty + K|g|_\infty \le (1+K)\|g\|_\theta$, while Lemma 3.5 implies

$$|\widetilde{g}|_\theta \le |g|_\theta + |g|_\infty |h|_\theta \le BbK\|g\|_\theta.$$

Thus, $\|\widetilde{g}\|_\theta \le 2BbK\|g\|_\theta$. This and the above estimate imply

$$\begin{aligned}\Big| \frac{1}{\lambda^n} L^n g - h \int g\, d\nu \Big|_\theta &= \frac{1}{\lambda^n} |L^n(g - \alpha h)|_\theta = \frac{1}{\lambda^n} |L^n \widetilde{g}|_\theta\\ &\le A' \beta^{n/2} \|\widetilde{g}\|_\theta \le A'\, 2BbK \beta^{n/2} \|g\|_\theta.\end{aligned}$$

Combining with (3.12), gives

$$\Big\| \frac{1}{\lambda^n} L^n g - h \int g\, d\nu \Big\|_\theta \le \frac{100BK^5 b^3}{1-\theta} \beta^{n/2} \|g\|_\theta.$$

Finally it follows from $\sqrt{1-x} < 1 - x/2$ for $0 < x < 1$ and (2.3) that $\sqrt{\beta} = \sqrt{1 - \frac{1-\theta}{4K^3}} \le 1 - \frac{1-\theta}{8K^3} = \rho$. This proves (2.4). ∎

References

[1] V. Baladi, *Positive transfer operators and decay of correlations*. Advanced Series in Nonlinear Dynamics, 16. World Scientific, River Edge, NJ, 2000. http://dx.doi.org/10.1142/9789812813633

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, 470. Springer-Verlag, Berlin, 1975.

[3] D. Dolgopyat, *On decay of correlations in Anosov flows*. Ann. of Math. **147**(1998), no. 2, 357–390. http://dx.doi.org/10.2307/121012

[4] F. Naud, *Dynamics on Cantor sets and analytic properties of zeta functions*. Ph.D. thesis, University of Bordeaux I, 2003.

[5] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*. Astérisque 187–188, 1990.

[6] M. Pollicott, *A complex Ruelle operator theorem and two counterexamples*. Ergodic Theory Dynam. Systems **4**(1984), 135–146. http://dx.doi.org/10.1017/S0143385700002327

[7] D. Ruelle, *Statistical mechanics of a one-dimensional lattice gas*. Commun. Math. Phys. **9**(1968), 267–278. http://dx.doi.org/10.1007/BF01654281

[8] D. Ruelle, *A measure associated with Axiom A attractors*. Amer. J. Math. **98**(1976), 619–654. http://dx.doi.org/10.2307/2373810

[9] L. Stoyanov, *On the Ruelle–Perron–Frobenius Theorem*. Asymptot Anal. **43**(2005), 131–150.

[10] ———, *Scattering resonances for several small convex bodies and the Lax–Phillips conjecture*. Mem. Amer. Math. Soc. **199**(2009).

[11] ———, *Spectra of Ruelle transfer operators for Axiom A flows*. Nonlinearity **24**(2011), 1089–1120. http://dx.doi.org/10.1088/0951-7715/24/4/005

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