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A NOTE ON BARIC ALGEBRAS

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In this paper we present a characterization of baric algebras. In particular we study those in which the identity $x^3 = w(x)x^2$ holds. Moreover, for every field K, we prove that this identity guarantees that the annihilator of $\operatorname{Ker}(w)$ is an ideal in A and we give example of a subspace of $\operatorname{Ker}(w)$ whose annihilator is not an ideal.

1. INTRODUCTION

In what follows K is an infinite field and A is a finite dimensional, commutative, not necessarily associative algebra over K.

If $w : A \to K$ is a non zero algebra homomorphism, then the ordered pair (A, w) is called a baric algebra and w the weight function of A. A Bernstein algebra is a baric algebra (A, w) such that $(x^2)^2 = w(x)^2 x^2$ for every $x \in A$.

In a Bernstein algebra there exists a decomposition, $A = Ke \oplus N$ where e is an idempotent element of A, $N = \operatorname{Ker}(w)$ and $w(e) = 1$. If $\operatorname{char} K \neq 2$ the relation $2e(ey) - ey = 0$ holds for every $y \in N$. Moreover the map $L_e : N \to N$ defined by $L_e(y) = ey$ for every $y \in N$ satisfies $2L_e^2 = L_e$. Hence, $N = U \oplus V$ where $U = L_e(N)$ and $V = \operatorname{Ker}(L_e)$. The subspaces U and V satisfy the relations $U^2 \subseteq V$, $UV \subseteq U$, $V^2 \subseteq U$, $UV^2 = \{0\}$. Also the following identities are satisfied: $u_1^3 = 0$, $u_1(u_2u_3) + u_2(u_3u_1) + u_3(u_1u_2) = 0$ and $u_1(u_1v) = 0$ for every $u_i \in U$, $i = 1, 2, 3$ and $v \in V$. For references see [**1**, **4**, and **8**].

2. BARIC ALGEBRAS

It is known, see [**8**], that a finite dimensional commutative real algebra is a baric algebra if and only if A has a basis $\{e_1, \cdots, e_n\}$ such that the constants defined by

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \qquad i, j = 1, \cdots, n$$

satisfy the relation

$$\sum_{k=1}^{n} \gamma_{ijk} = 1 \qquad i, j = 1, \cdots, n.$$

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We can easily prove that this result holds whatever the field K under consideration is.

In the case of a field of characteristic not 2, we have the following characterization of baric algebras:

PROPOSITION 2.1. *Let A be a n-dimensional, commutative, not necessarely associative algebra over K. Then the following conditions are equivalent:*

1. *A is a baric algebra.*
2. *There exist an ideal I, of codimension 1 such that $A^2 \not\subseteq I$.*

PROOF: Let (A,w) be a baric algebra. Then w is onto and $A = Ka \oplus \mathrm{Ker}(w)$, as vector spaces, with $w(a) = 1$. Note also, that $\mathrm{Ker}(w) = N$ is an ideal in A of codimension 1. Moreover, $w(a^2) = w(a)w(a)$ and $A^2 \not\subseteq N$. Conversely, let I be an ideal in A such that $A^2 \not\subseteq I$ and $\{b_2, \cdots, b_n\}$ a basis of I, as a vector space. First we prove the existence of an element $b_1 \in A - I$ such that $b_1^2 \notin I$ and $\{b_1, \cdots, b_n\}$ is a basis of A. If $x^2 \in A$ for every $x \in A$, then $2ab \in I$ for each $a, b \in A$, since $(a+b)^2 = a^2 + 2ab + b^2$. But this contradicts the assumption that $A^2 \not\subseteq I$. Therefore, there exists an element $b_1 \in A - I$ such that $b_1^2 \notin I$. Moreover $\{b_1, \cdots, b_n\}$ is a basis of A, and $b_1^2 = \lambda b_1 + y$ with $\lambda \neq 0$ and $y \in I$. If we define $w : A \to K$ by $w(b_1) = \lambda$ and $w(b_i) = 0$ for $i = 2, \cdots, n$ then by straigtforward computations we can see that w is a non zero algebra homomorphism and (A,w) is a baric algebra. ∎

We note here that the above result had been shown by Wörz-Busekros in [8] for a real algebra.

In the following we prove some properties of baric algebras in which the following identity holds

$$x^3 = w(x)x^2. \tag{1}$$

It is well known that w is the only non zero algebra homomorphism from A to the ground field K. Moreover, for a field of characteristic different from 2 these algebras are Bernstein, see [3].

PROPOSITION 2.3. *Suppose (A,w) satifies $x^3 = w(x)x^2$ and $\mathrm{char}\, K \neq 2$. Then for every $x, y, z \in A$ we have:*

$$(xy)z + (yz)x + (zx)y = w(x)(yz) + w(y)(zx) + w(z)(xy). \tag{2}$$

The proof consists of linearising the identity $x^3 = w(x)x^2$ and cancelling out the factor 2.

If the characteristic of K is also different from 3, the relation (2) is equivalent to the identity $x^3 = w(x)x^2$.

Proposition 2.3. *Suppose (A, w) is a baric algebra over K, $\operatorname{char} K = 2$. Then the identity $x^3 = w(x)x^2$ holds in A if and only if for every $x, y \in A$:*

$$x^2 y = w(y)x^2. \tag{3}$$

Proof: Obviously relation (3) implies the identity $x^3 = w(x)x^2$. Conversely, linearising the identity $x^3 = w(x)x^2$ we have relation (3). ∎

Definition: A Jordan algebra is a commutative algebra in which the identity $x^2(yx) = (x^2y)x$ holds.

Theorem 2.4. *Suppose (A, w) satifies $x^3 = w(x)x^2$ and $\operatorname{char} K \neq 2$. Then*

1. *A is a Jordan algebra.*
2. *N is nilpotent.*

Proof: By setting $x = y$ in relation (2) we obtain for every $x, z \in A$

$$x^2 z + 2(xz)x = 2w(x)xz + w(z)x^2. \tag{4}$$

Now replacing z by xz in (4) we have

$$x^2(xz) + 2(x(xz))x = 2w(x)x(xz) + w(x)w(z)x^2. \tag{5}$$

On the other hand, relation (4) implies

$$x(x^2 z) + 2x((xz)x) = 2w(x)x(xz) + w(z)x^3. \tag{6}$$

Moreover, relations (1), (5), (6) and commutativity imply $x^2(xz) = x(x^2z)$ for every $x, z \in A$. Therefore A is a Jordan algebra.

Finally, since A is a Jordan algebra, then N is also Jordan. Moreover $x^3 = 0$ for every $x \in N$. Then N is nil and by a Theorem of Albert [6], N is nilpotent. ∎

Remark 2.5. We observe that none of the conditions 1 or 2, by itself, in the above Theorem implies $x^3 = w(x)x^2$ for every $x \in A$, as we can see in the following examples.

Example 1. Let K be a field, $\operatorname{char} K \neq 2$, V a finite dimensional vector space over K, $T : V \to V$ a linear operator and $w : V \to K$ a non zero linear form over V such that $w \circ T = w$. If we define $xy = (1/2)(w(x)T(y) + w(y)T(x))$ for every $x, y \in V$ then V is a baric algebra denoted by $A_{T,w}$ see [5]. By straightforward computations we can prove that if $T^2 = 2T$, $A_{T,w}$ is an associative algebra and then a Jordan algebra and $x^3 \neq w(x)x^2$.

EXAMPLE 2. Let $A = \langle c_0, c_1, c_2\rangle_K$ be a K-algebra, $\operatorname{char} K \neq 2$ with multiplication given by $c_0^2 = c_0$, $c_0c_1 = c_1c_0 = c_1$, $c_0c_2 = c_2c_0 = c_1$, $c_1^2 = c_2$, $c_1c_2 = c_2c_1 = c_2^2 = 0$ and let I be the ideal in A generated by c_1 and c_2. Then $A^2 \not\subseteq I$. Hence, by Proposition 2.1, A is a baric algebra with weight function $w : A \to K$ defined by $w(c_0) = 1$, $w(c_1) = w(c_2) = 0$ and $I = \operatorname{Ker}(w)$. Moreover $I^2 = \langle c_2\rangle$, $I^3 = \langle 0\rangle$. Therefore $\operatorname{Ker}(w)$ is nilpotent but $(c_0 + c_1)^3 \neq w(c_0 + c_1)(c_0 + c_1)^2$.

REMARK 2.6. Walcher [7] and Ouattara [5] have shown that a baric algebra (A, w) over K, $\operatorname{char} K \neq 2$, is a Bernstein and a Jordan algebra if and only if the identity $x^3 = w(x)x^2$ holds in A. In the case of a field K, $\operatorname{char} K = 2$, this statement is not true, for instance if we take the algebra $A = Ke \oplus N$ where $N = \langle y\rangle$ and with multiplication table $e^2 = e$, $ey = y$, $y^2 = 0$, then A is a Bernstein and a Jordan algebra and $(e + y)^3 \neq w(e + y)(e + y)^2$.

THEOREM 2.7. *Suppose (A, w) satifies $x^3 = w(x)x^2$ and $\operatorname{char} K = 2$. Then*

1. *A is a Bernstein algebra.*
2. *A is a Jordan algebra.*
3. *$eN = \{0\}$.*

PROOF: By Proposition 2.3 relation (1) is equivalent to relation (3) and this identity implies that A is a Bernstein algebra.

Replacing y by xy in relation (3) and using relation (1) we have $x^2(yx) = (x^2y)x$. Therefore A is a Jordan algebra.

Finally, relation (3) implies $ey = 0$ for every $y \in N$. Then $eN = \{0\}$. ∎

THEOREM 2.8. *Let (A, w) be a baric algebra over K, $\operatorname{char} K = 2$. Then the following conditions are equivalent:*

1. *The identity $x^3 = w(x)x^2$ holds in A.*
2. *A is a Bernstein algebra such that $eN = \{0\}$.*

PROOF: Since the identity $x^3 = w(x)x^2$ holds in A, then Theorem 2.7 implies that A is a Bernstein algebra such that $eN = \{0\}$. Conversely if A is a Bernstein algebra over K and $\operatorname{char} K = 2$, then linearising the identity $(x^2)^2 = w(x)^2x^2$, we have $w(x)^2y^2 + w(y)^2x^2 = 0$. Thus, for every $y \in A$

$$y^2 = w(y)^2 e. \tag{7}$$

Let $x = \alpha e + y$ be an element in A. Then by using $eN = \{0\}$ together with relation (7) we have $x^3 = (\alpha e + y)^3 = \alpha^2 e(\alpha e + y) = \alpha^3 e = \alpha(\alpha^2 e) = w(x)x^2$. ∎

3. ANNIHILATORS

Let A be a commutative not necessarily associative algebra and $S \subseteq A$. The annihilator of S in A is the subspace $\operatorname{Ann}(S) = \{x \in A \mid xS = \{0\}\}$.

For associative algebras, this subspace is an ideal, but it is not true in the non associative case. For instance, if A has the following multiplication table: $e^2 = e$, $eu = (1/2)u$, $ev = u^2 = v^2 = 0$, $uv = u$. Then the annihilator of the subspace S generated by u and v is not an ideal, because $(-2e+v)S = 0$ but $(e(-2e+v))u \neq 0$.

Now we prove that the identity $x^3 = w(x)x^2$ guarantees that the annihilator of N is an ideal in A, whatever the field K under consideration is.

THEOREM 3.1. *Suppose (A, w) satisfies $x^3 = w(x)x^2$, $\operatorname{char} K \neq 2$ and let $A = Ke \oplus U \oplus V$ be its decomposition relative to the idempotent e. Then:*

1. *The annihilator of N is an ideal in A.*
2. *If $U = \{0\}$, then $\operatorname{Ann}(N) = A$.*
3. *If $U \neq \{0\}$, then $\operatorname{Ann}(N) \subseteq N$.*

PROOF: Since (A, w) satisfies $x^3 = w(x)x^2$ and $\operatorname{char} K \neq 2$, A is a Bernstein and a Jordan algebra. Then $V^2 = \{0\}$ and $(Uv)v = \{0\}$ for every $v \in V$, see [**2**].

1. If $x = \alpha e + u_0 + v_0 \in \operatorname{Ann}(N)$, then for every $u \in U$, $v \in V$, $xu = 0$, $xv = 0$ and xu, $xv \in \operatorname{Ann}(N)$. It remains only to prove that $ex \in \operatorname{Ann}(N)$. Since $xu = 0$, $xv = 0$ for every $u \in U$, $v \in V$, we have the followings relations:

$$\frac{\alpha}{2}u + uu_0 + uv_0 = 0, \tag{8}$$

$$vu_0 + vv_0 = 0. \tag{9}$$

Relation (8) implies that $uu_0 = 0$ and $(\alpha/2)u + uv_0 = 0$ for every $u \in U$. As $V^2 = \{0\}$, by relation (9) we have that $vu_0 = 0$ for every $v \in V$. Moreover $ex = \alpha e + (1/2)u_0$. Then using the previous relations one has $(ex)(u+v) = (\alpha/2)u$ for every $u \in U$, $v \in V$.

If $\alpha = 0$, then $(ex)(u+v) = 0$ for every $u \in U$, $v \in V$ and $ex \in \operatorname{Ann}(N)$.

If $\alpha \neq 0$, then $((\alpha/2)u + uv_0)v_0 = 0$ for every $u \in U$. Since A is a Jordan and a Bernstein algebra, we have $(uv_0)v_0 = 0$ for every $u \in U$. Therefore $(\alpha/2)uv_0 = 0$ and $uv_0 = 0$. But $(\alpha/2)u + uv_0 = 0$, so that $(ex)(u+v) = (\alpha/2)u = 0$ for every $u \in U$, $v \in V$, and $ex \in \operatorname{Ann}(N)$. Thus $\operatorname{Ann}(N)$ is an ideal in A.

2. If $U = \{0\}$, then $A = Ke \oplus V$ and $\operatorname{Ann}(N) = \{x \in A \mid xv = 0 \text{ for every } v \in V\} = \{\alpha e + v_1 \mid (\alpha e + v_1)v = 0 \text{ for every } v \in V\} = \{\alpha e + v_1 \mid \alpha \in K,\ v_1 \in V\} = A$.

3. Let $U \neq \{0\}$ and $x = \beta e + u_0 + v_0 \in \operatorname{Ann}(N)$. By a similar argument to that used in 1 we have $(\beta/2)u = 0$ for every $u \in U$. Since $U \neq \{0\}$ there exists $u_1 \neq 0$, $u_1 \in U$ such that $(\beta/2)u_1 = 0$ and then $\beta = 0$ and $x = u_0 + v_0 \in N$. ∎

REMARK 3.2. Since N is nilpotent, the above Theorem implies that in the case $U \neq \{0\}$, $\operatorname{Ann}(N)$ is nilpotent.

Theorem 3.3. *Suppose (A, w) satifies $x^3 = w(x)x^2$ and $\operatorname{char} K = 2$. Then the annihilator of N is an ideal in A.*

Proof: Since $x^3 = w(x)x^2$ holds in A, by Theorem 2.7 we have that A is a Bernstein algebra such that $eN = \{0\}$. Let $A = Ke \oplus N$ be its decomposition relative to the idempotent e and $x = \alpha e + y$ an element in $\operatorname{Ann}(N)$. Then $tx = ty$ for every $t \in N$.

Now for every $a = \beta e + t$ in A, we have $ax = (\beta e + t)(\alpha e + y) = \beta\alpha e + ty = \beta\alpha e + tx = \beta\alpha e$. Then since $eN = \{0\}$ we have $axN = \{0\}$ for every $a \in A$. Therefore the annihilator of N is an ideal in A. ∎

Remark 3.4. It is not true that for a subspace S of N, $\operatorname{Ann}(S)$ is an ideal in A, as we can see in the following examples.

Example 3. Let A be a commutative real algebra with basis $\{x_1, \cdots, x_4\}$ and with multiplication given by $x_1^2 = x_1$, $x_1x_2 = (1/2)x_2$, $x_1x_3 = (1/2)x_3$, $x_3x_4 = -(1/2)x_2$ and the other products being zero. If $N = \langle x_2, x_3, x_4 \rangle$, then $A = \mathbb{R}x_1 \oplus N$ is a baric algebra with weight function $w : A \to \mathbb{R}$ defined by $w(\lambda x_1 + n) = \lambda$ for every $\lambda \in \mathbb{R}$, $n \in N$. Moreover the identity $x^3 = w(x)x^2$ holds in A. If we take $S = \langle x_2 + x_4 \rangle$, then $x_1 + x_3 \in \operatorname{Ann}(S)$ but $x_1(x_1 + x_3) \notin \operatorname{Ann}(S)$. Thus $\operatorname{Ann}(S)$ is not an ideal in A.

Example 4. In the case of a field K, $\operatorname{char} K = 2$, let A be a commutative algebra with basis $\{x_1, x_2, x_3\}$ and with multiplication given by $x_1^2 = x_1$, $x_2x_3 = x_2$ and the other products being zero. If $N = \langle x_2, x_3 \rangle$, then $A = Kx_1 \oplus N$ is a baric algebra with weight function $w : A \to K$ defined by $w(\lambda x_1 + n) = \lambda$ for every $\lambda \in K$, $n \in N$, and the identity $x^3 = w(x)x^2$ holds in A. Moreover the annihilator of the subspace S generated by $x_2 + x_3$ is not an ideal in A, because $(x_1 + (x_2 + x_3))S = \{0\}$ but $(x_2(x_1 + (x_2 + x_3)))(x_2 + x_3) \neq 0$.

References

[1] M.T. Alcalde, C. Burgueño, A. Labra and A. Micali, 'Sur les algèbres de Bernstein', *Proc. London Math. Soc. (3)* **58** (1989), 51–68.

[2] M.T. Alcalde, R. Baeza and C.Burgueño, 'Autour des algèbres de Bernstein', *Arch. Math.* **53** (1989), 134–140.

[3] I.M.H. Etherington, 'Commutative train algebras of ranks 2 and 3', *J. London Math. Soc.* **15** (1940), 136–149.

[4] P. Holgate, 'Genetic algebras satisfying Bernstein's stationarity principle', *J. London Math. Soc. (2)* **9** (1975), 613–623.

[5] M. Ouattara, *Algèbres de Jordan et algèbres génétiques*, Thèse de Doctorat (Université de Montpellier II, France, 1988).

[6] R.D. Shafer, *Introduction to nonassociative algebras* (Academic Press, New York, 1966).

https://doi.org/10.1017/S0004972700015823 Published online by Cambridge University Press

[7] S. Walcher, 'Bernstein algebras which are Jordan algebras', *Arch. Math.* **50** (1988), 218–222.

[8] A. Wörz-Busekros, *Algebras in genetics*, Lecture Notes in Biomathematics **36** (Springer-Verlag, Berlin, Heidelberg, New York, 1980).

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