ANZIAM J. 43(2001), 291–320

SOLVABILITY OF SOME ABEL-TYPE INTEGRAL EQUATIONS INVOLVING THE GAUSS HYPERGEOMETRIC FUNCTION AS KERNELS IN THE SPACES OF SUMMABLE FUNCTIONS

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(Received 30 May 1998)

Abstract

This paper is devoted to the study of the solvability of certain one- and multidimensional Abel-type integral equations involving the Gauss hypergeometric function as their kernels in the space of summable functions. The multidimensional equations are considered over certain pyramidal domains and the results obtained are used to present the multidimensional pyramidal analogues of generalized fractional calculus operators and their properties.

1. Introduction

One-dimensional Abel-type integral equations involving the Gauss hypergeometric function $F(a, b; c; z)$ [5, Section 2.1] as kernel have been studied by many authors ([2, 3, 6, 9–14, 20], [26, Section 35.1], [27, 28]; see also [29]). Such equations arise in the boundary value problems for the Euler-Darboux equation with boundary conditions involving generalized fractional integro-differential operators ([7, 16–19, 21–23, 25, 28, 31, 33]). One such multidimensional integral equation of non-convolution type was investigated in [4, Section 4.6.2]. In the above papers, the integral operators of the equations considered were represented as compositions of simpler fractional integral operators with power weights. On the basis of these representations and the known properties of fractional calculus operators, the sufficient conditions for the

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https://doi.org/10.1017/S1446181100013080 Published online by Cambridge University Press

solvability of the integral equations were given and their inversion formulas were obtained in some function spaces.

The investigation of the necessary and sufficient conditions for the solvability of the above equations is more difficult. This problem is closely connected with the characterization of the images of the corresponding integral operators. The classical Tamarkin's statement [26, Section 2.2] on the solvability of the Abel-type integral equation in the space $L_1(a, b)$ of summable functions is known. A similar result for the multidimensional Abel-type integral equations over pyramidal domains was proved in [8]. Multidimensional type fractional calculus operators were also studied in [15] and [32].

The present paper is devoted to the investigation of the aforementioned results for certain one- and multidimensional integral equations with the Gauss hypergeometric function as their kernels. The one-dimensional equation happens to be the equation which was first considered in [20], and the multidimensional one is taken over a pyramidal domain in $\mathbb{R}^n$. Section 2 contains some preliminary information. Sections 3 to 5 deal with the solvability of one-dimensional Abel-type integral equations in the space of summable functions. The criterion for solvability of multidimensional Abel-type integral equations over pyramidal domains in the space of summable functions is given in Section 6. Section 7 is devoted to the discussion of the conditions of solvability of such multidimensional Abel-type integral equations. On the basis of the results in Sections 6 and 7, the generalized fractional integral and differential operators are introduced and their properties are investigated systematically in Section 8.

2. Preliminaries

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. For $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we denote by $(z)_n$ the Pochhammer symbol [5, Section 2.2.1] defined by

$$(z)_0 = 1,\ (z)_n = z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)} \quad (z \in \mathbb{C};\ n \in \mathbb{N}_0), \tag{2.1}$$

where $\Gamma(z)$ is the Gamma function [5, Section 1.1]. In terms of this relation, the Gauss hypergeometric function $F(a, b; c; z)$ is defined by

$$F(a, b; c; z) \equiv {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (a, b, c \in \mathbb{C};\ |z| < 1), \tag{2.2}$$

with the corresponding analytic continuation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \tag{2.3}$$

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for $z \in \mathbb{C}$ ($|\arg(1-z)| < \pi$; $z \neq 1$) and $0 < \mathrm{Re}\,(b) < \mathrm{Re}\,(c)$ (see [5, 2.1 (2) and 2.1 (10)]). When $z = 1$, we have the Gauss summation theorem [5, 2.3 (9)]:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}\,(c-a-b) > 0;\ c \neq 0, -1, -2, \dots). \tag{2.4}$$

The asymptotic behaviour of $F(a,b;c;z)$ at infinity is given by ([12, (5.6)] and [5, 2.3 (9)])

$$F(a,b;c;z) = \lambda_1 z^{-a} + \lambda_2 z^{-b} + O\left(z^{-a-1}\right) + O\left(z^{-b-1}\right) \quad (z \to \infty), \tag{2.5}$$

when $a-b$ is not an integer, with the addition of $\log z$ near z^{-a} or near z^{-b} in the case of integer $a-b$. We shall also use the relations [5, 2.4 (2), 2.4 (3), 2.1 (22)]:

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 s^{\lambda-1}(1-s)^{c-\lambda-1} F(a,b;\lambda;sx)\,ds; \tag{2.6}$$

$$\begin{aligned} F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 & s^{\lambda-1}(1-s)^{c-\lambda-1}(1-sx)^{-a'} F(a-a',b;\lambda;sx) \\ & \times F\left(a', b-\lambda; c-\lambda; \frac{x(1-s)}{1-sx}\right) ds; \end{aligned} \tag{2.7}$$

$$F(a,b;c;x) = (1-x)^{-a} F\left(a, c-b; c; \frac{x}{x-1}\right). \tag{2.8}$$

Let $\Omega = [\alpha, \beta]$ be a finite interval of the real axis $\mathbb{R}$. We denote by $AC(\Omega)$ the space of absolutely continuous functions on Ω. It is known from [26, Section 1.1] that $AC(\Omega)$ coincides with the space of primitives of Lebesgue summable functions on Ω, that is,

$$f(x) \in AC(\Omega) \iff f(x) = c + \int_\alpha^x \varphi(t)\,dt \qquad \left(\int_\alpha^\beta |\varphi(t)|\,dt < \infty\right). \tag{2.9}$$

We denote by $\mathbb{R}^n$ ($n \in \mathbb{N}$) the n-dimensional Euclidean space. Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$. Then we define

$$\boldsymbol{x} \cdot \boldsymbol{t} = \sum_{k=1}^n x_k t_k.$$

In particular, for $\mathbf{1} = (1, \dots, 1)$

$$\boldsymbol{x} \cdot \mathbf{1} = \mathbf{1} \cdot \boldsymbol{x} = \sum_{k=1}^n x_k$$

and

$$\boldsymbol{x} > \boldsymbol{t} \quad \text{means} \quad x_1 > t_1, \dots, x_n > t_n,$$

https://doi.org/10.1017/S1446181100013080 Published online by Cambridge University Press

and similarly for the inequality symbols $<$, $\geqq$ and $\leqq$. We denote by $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ the subsets of $\mathbb{R}^n$ defined by $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{0}\}$ and $\mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} < \boldsymbol{0}\}$, respectively. Let

$$\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n \equiv \mathbb{N}_0 \times \cdots \times \mathbb{N}_0 \quad (k_i \in \mathbb{N}_0 \ (i = 1, \ldots, n))$$

be multi-index with $\boldsymbol{k}! = k_1! \cdots k_n!$ and $|\boldsymbol{k}| = k_1 + \cdots + k_n$. For $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{k} \in \mathbb{N}_0^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$, let

$$\begin{gathered}(\boldsymbol{x})_{\boldsymbol{k}} = (x_1)_{k_1} \cdots (x_n)_{k_n}, \quad \boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \\ D^{\boldsymbol{k}} = \frac{\partial^{|\boldsymbol{k}|}}{(\partial x_1)^{k_1} \cdots (\partial x_n)^{k_n}} \quad \text{and} \quad \Gamma(\boldsymbol{\alpha}) = \Gamma(\alpha_1) \cdots \Gamma(\alpha_n).\end{gathered} \tag{2.10}$$

Let $A = \|a_{jk}\|$ $(a_{jk} \in \mathbb{R})$ be a matrix of order $n \times n$ with its determinant $|A| \equiv \det A = 1$, $\boldsymbol{a}_j = (a_{j1}, \ldots, a_{jn})$ be its line vectors, and let $\tilde{a}_{ij}$ be the elements of the inverse matrix A^{-1}. We shall also use the notation [26, Section 28.4]:

$$A \cdot \boldsymbol{x} = (\boldsymbol{a}_1 \cdot \boldsymbol{x}, \ldots, \boldsymbol{a}_n \cdot \boldsymbol{x}), \quad (A \cdot \boldsymbol{x})^{\boldsymbol{\alpha}} = (\boldsymbol{a}_1 \cdot \boldsymbol{x})^{\alpha_1} \cdots (\boldsymbol{a}_n \cdot \boldsymbol{x})^{\alpha_n}. \tag{2.11}$$

For $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}^1$, we denote by

$$A_{\boldsymbol{c},r}(\boldsymbol{b}) = \{\boldsymbol{t} \in \mathbb{R}^n : A \cdot (\boldsymbol{b} - \boldsymbol{t}) \geqq \boldsymbol{0},\ \boldsymbol{c} \cdot \boldsymbol{t} + r \geqq 0\} \tag{2.12}$$

the n-dimensional bounded pyramid in $\mathbb{R}^n$ with its vertex at the point $\boldsymbol{b}$, with its base on the hyperplane $\boldsymbol{c} \cdot \boldsymbol{t} + r = 0$ and with lateral faces situated on the hyperplanes

$$\boldsymbol{a}_j \cdot (\boldsymbol{b} - \boldsymbol{t}) = 0 \quad (j = 1, \ldots, n).$$

In particular, if $A = E = \|\delta_{jk}\|$ is a unit matrix and $\boldsymbol{c} = \boldsymbol{1} = (1, \ldots, 1)$ and $r = 0$, then (2.12) is the simplest model pyramid

$$E_1(\boldsymbol{b}) = \{\boldsymbol{t} \in \mathbb{R}^n : \boldsymbol{t} \leqq \boldsymbol{b},\ \boldsymbol{1} \cdot \boldsymbol{t} \geqq 0\}. \tag{2.13}$$

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote by $F[\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{\gamma}; \boldsymbol{x}]$ the function

$$F[\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{\gamma}; \boldsymbol{x}] = \prod_{j=1}^{n} F(\alpha_j, \beta_j; \gamma_j; x_j). \tag{2.14}$$

If $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then the Lauricella multiple hypergeometric series is defined by [30, p. 33]

$$\begin{aligned}F_B(\boldsymbol{a}, \boldsymbol{b}; c; \boldsymbol{x}) &\equiv F_B^{(n)}(a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ &= \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(a_1)_{k_1} \cdots (a_n)_{k_n} (b_1)_{k_1} \cdots (b_n)_{k_n}}{(c)_{k_1 + \cdots + k_n}} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_n^{k_n}}{k_n!}\end{aligned} \tag{2.15}$$

$(\max\{|x_1|, \ldots, |x_n|\} < 1)$. When $n = 2$, $F_B^{(2)}(a_1, a_2, b_1, b_2; c; x_1, x_2)$ coincides with the Appell double hypergeometric series $F_3(a_1, a_2, b_1, b_2; c; x_1, x_2)$ (see [30, p. 23]).

3. Solution of the one-dimensional Abel-type hypergeometric integral equation

We consider the integral equation:

$$\left(I_{a+}^{\gamma,\alpha,\beta}\varphi\right)(x) \equiv \frac{(x-a)^{-\alpha}}{\Gamma(\gamma)} \int_a^x (x-t)^{\gamma-1} F\left(\alpha, \beta; \gamma; \frac{x-t}{x-a}\right) \varphi(t)\, dt$$
$$= f(x) \qquad (x > a) \tag{3.1}$$

with $\alpha,\ \beta \in \mathbb{R}^1$ and $\gamma \in \mathbb{R}^1_+$ $(0 < \gamma < 1)$. This equation generalizes the classical Abel equation [26, Section 2], which is obtained from (3.1) in the special case when $\alpha = 0$. The equation (3.1) was first investigated in [20] with α, β and γ being replaced by $\alpha + \beta$, $-\eta$ and α, respectively, and its solution was obtained in the form:

$$\varphi(x) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \left[(x-a)^{\alpha} \int_a^x (x-t)^{-\gamma} \right.$$
$$\left. \times F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) f(t)\, dt \right]. \tag{3.2}$$

Formally, such a solution can be found in the following way. If we suppose that (3.1) is solvable, then replacing x by t and t by τ, and multiplying both sides of the resulting equation by

$$(x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right)$$

and then integrating over (a, x), we have

$$\int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) (t-a)^{-\alpha}\, dt$$
$$\times \int_a^t (t-\tau)^{\gamma-1} F\left(\alpha, \beta; \gamma; \frac{t-\tau}{t-a}\right) \varphi(\tau)\, d\tau$$
$$= \Gamma(\gamma) \int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) f(t)\, dt. \tag{3.3}$$

Interchanging the order of integration in the left-hand side of (3.3) and making the change of variables $t = \tau + (1-s)(x-\tau)$, we rewrite the left-hand side of (3.3) in the form:

$$(x-a)^{-\alpha} \int_a^x \varphi(\tau)\, d\tau \int_0^1 s^{-\gamma}(1-s)^{\gamma-1} \left(1 - s\frac{x-\tau}{x-a}\right)^{-\alpha}$$
$$\times F\left(\alpha, \beta; \gamma; \frac{(1-s)(x-\tau)/(x-a)}{1-s(x-\tau)/(x-a)}\right) F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; s\frac{x-\tau}{x-a}\right) ds,$$

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which, in view of (2.7), yields

$$B(\gamma, 1-\gamma)(x-a)^{-\alpha}\int_a^x F\left(0, 1+\beta-\gamma; 1; \frac{x-\tau}{x-a}\right)\varphi(\tau)\,d\tau$$
$$= \Gamma(\gamma)\Gamma(1-\gamma)(x-a)^{-\alpha}\int_a^x \varphi(\tau)\,d\tau.$$

Therefore, (3.3) is rewritten as follows:

$$\int_a^x \varphi(\tau)\,d\tau = \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)}\int_a^x (x-t)^{-\gamma}F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right)f(t)\,dt. \tag{3.4}$$

From (3.4) we obtain the solution $\varphi(x)$ of (3.1) in the form (3.2).

REMARK 1. It is directly checked that the above result remains valid for the equation obtained from (3.1) by replacing $(x-a)$ by $(x-h)$ with $h \leqq a$, that is, if the integral equation:

$$\left(I_{a+,h}^{\gamma,\alpha,\beta}\varphi\right)(x) \equiv \frac{(x-h)^{-\alpha}}{\Gamma(\gamma)}\int_a^x (x-t)^{\gamma-1}F\left(\alpha, \beta; \gamma; \frac{x-t}{x-h}\right)\varphi(t)\,dt$$
$$= f(x) \qquad (x > a) \tag{3.5}$$

with $\alpha,\ \beta \in \mathbb{R}^1$, $\gamma \in \mathbb{R}_+^1$ $(0 < \gamma < 1)$ and $h \leqq a$ is solvable, then its solution $\varphi(x)$ is given by

$$\varphi(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\left[(x-h)^{\alpha}\right.$$
$$\left.\times \int_a^x (x-t)^{-\gamma}F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-h}\right)f(t)\,dt\right]. \tag{3.6}$$

4. Solvability of the one-dimensional Abel-type hypergeometric integral equation

To obtain the solvability conditions of (3.1), we put

$$f_{a+}^{\gamma,\alpha,\beta}(x) \equiv (I_{a+}^{1-\gamma,-\alpha,1+\beta-\gamma}f)(x)$$
$$= \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)}\int_a^x (x-t)^{-\gamma}F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right)f(t)\,dt. \tag{4.1}$$

The following preliminary assertion holds true.

LEMMA 1. *Let* $f(t) \in L_1(a,b)$, $\alpha, \beta \in \mathbb{R}^1$, $0 < \gamma < 1$. *Let further*

$$f(t) = O\big((t-a)^{\mu}\big) \qquad (t \to a) \tag{4.2}$$

with $\mu > \max\{0, \beta - \alpha\} - 1$, *and*

$$f(t) = O\big((b-t)^{\nu}\big) \qquad (t \to b) \tag{4.3}$$

with $\nu > \gamma - 2$. *Then* $f_{a+}^{\gamma,\alpha,\beta}(x) \in L_1(a,b)$.

PROOF. Using (4.1) and interchanging the order of integration, we have

$$\int_a^b f_{a+}^{\gamma,\alpha,\beta}(x)\,dx = \frac{1}{\Gamma(1-\gamma)} \int_a^b f(t)\,dt \\ \times \int_t^b (x-a)^{\alpha}(x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) dx. \tag{4.4}$$

To evaluate the inner integral, we make the change of variable $\tau = (b-x)/(b-t)$ and apply (2.7) by putting $a = -\alpha$, $b = 2+\beta-\gamma$, $c = 2-\gamma$, $a' = -\alpha$ and $\lambda = 1$. Thus we obtain

$$\int_t^b (x-a)^{\alpha}(x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) dx \\ = \frac{(b-a)^{\alpha}(b-t)^{1-\gamma}}{1-\gamma} F\left(-\alpha, 2+\beta-\gamma; 2-\gamma; \frac{b-t}{b-a}\right). \tag{4.5}$$

Then, according to (2.8), we find that

$$\int_a^b f_{a+}^{\gamma,\alpha,\beta}(x)\,dx = \int_a^b F^{\gamma,\alpha,\beta}(t)\,dt, \tag{4.6}$$

where

$$F^{\gamma,\alpha,\beta}(t) = \frac{1}{\Gamma(2-\gamma)}(t-a)^{\alpha}(b-t)^{1-\gamma} F\left(-\alpha, -\beta; 2-\gamma; -\frac{b-t}{t-a}\right) f(t).$$

From (2.5) and (4.2) we obtain the following asymptotic behaviour of $F_{a+}^{\gamma,\alpha,\beta}(t)$ near $t = a$:

$$F_{a+}^{\gamma,\alpha,\beta}(t) = O\big((t-a)^{\mu}\big) + O\left((t-a)^{\mu+\alpha-\beta}\right) \quad (t \to a)$$

in the case of noninteger $\alpha - \beta$, and with the addition of $\log(t-a)$ for integer $\alpha - \beta$. Equation (4.3) gives the asymptotics near $t = b$:

$$F_{a+}^{\gamma,\alpha,\beta}(t) = O\left((b-t)^{\nu+1-\gamma}\right) \quad (x \to b).$$

So $f_{a+}^{\gamma,\alpha,\beta}(x)$ is integrable on (a, b) and according to (4.6) we have

$$\int_a^b \left|f_{a+}^{\gamma,\alpha,\beta}(x)\right| dx \leqq \frac{1}{\Gamma(2-\gamma)} \int_a^b (t-a)^{\alpha}(b-t)^{1-\gamma} \times \left|F\left(-\alpha, -\beta; 2-\gamma; -\frac{b-t}{t-a}\right)\right| |f(t)|\, dt < \infty. \tag{4.7}$$

Hence $f_{a+}^{\gamma,\alpha,\beta}(x) \in L_1(a, b)$. This completes the proof of Lemma 1.

THEOREM 2. *The Abel-type integral equation* (3.1) *with real* α, β *and* $0 < \gamma < 1$ *is solvable in* $L_1(a, b)$ *if and only if*

$$f_{a+}^{\gamma,\alpha,\beta}(x) \in AC([a, b]) \quad \text{and} \quad f_{a+}^{\gamma,\alpha,\beta}(a) = 0. \tag{4.8}$$

Under these conditions, (3.1) *has a unique solution given by* (3.2).

PROOF. To prove the necessity part, let (3.1) be solvable in $L_1(a, b)$. Then all steps described above are true in which the change of the order of integration in (3.3) is justified by Fubini's theorem. Thus (3.4) is valid. Hence (4.8) follows from (3.4) if we take (2.9) into account. To prove the sufficiency part, let the conditions in (4.8) hold true. Then

$$\left(f_{a+}^{\gamma,\alpha,\beta}(x)\right)' = \frac{d}{dx} f_{a+}^{\gamma,\alpha,\beta}(x) \in L_1(a, b),$$

in view of (4.6) and (4.7). Therefore, the function given by (3.2) exists almost everywhere and belongs to $L_1(a, b)$. We show that it is a solution of (3.1). Substituting $\varphi(x)$ from (3.2) into the left-hand side of (3.1) and denoting the resulting expression by $g(x)$, we have

$$\frac{(x-a)^{-\alpha}}{\Gamma(\gamma)} \int_a^x (x-t)^{\gamma-1} F\left(\alpha, \beta; \gamma; \frac{x-t}{x-a}\right) \left(f_{a+}^{\gamma,\alpha,\beta}(t)\right)' dt = g(x). \tag{4.9}$$

This is an integral equation of the form (3.1) involving the prescribed function $(f_{a+}^{\gamma,\alpha,\beta}(x))'$. It is certainly solvable, and so by (3.2) we have

$$\left(f_{a+}^{\gamma,\alpha,\beta}(x)\right)' = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \left[(x-a)^{\alpha} \times \int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) g(t)\, dt\right] \equiv \left(g_{a+}^{\gamma,\alpha,\beta}(x)\right)', \tag{4.10}$$

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where $g_{a+}^{\gamma,\alpha,\beta}(x)$ is expressed similarly to (4.1). This shows that $f_{a+}^{\gamma,\alpha,\beta}(x)$ and $g_{a+}^{\gamma,\alpha,\beta}(x)$ differ by a constant k, that is, $f_{a+}^{\gamma,\alpha,\beta}(x) - g_{a+}^{\gamma,\alpha,\beta}(x) = k$. But $f_{a+}^{\gamma,\alpha,\beta}(a) = 0$ by hypothesis and $g_{a+}^{\gamma,\alpha,\beta}(a) = 0$, because (4.9) is a solvable equation. Hence $k = 0$ and therefore

$$\frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)} \int_a^x (x-t)^{1-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) [f(t) - g(t)]\, dt = 0.$$

This is an equation of the form (3.1) and the uniqueness of its solution leads to the result $f(t) = g(t)$. The proof of Theorem 2 is thus completed.

REMARK 2. By replacing $x - a$ by $x - h$ $(h \leqq a)$, (4.1) takes the form:

$$f_{a+,h}^{\gamma,\alpha,\beta}(x) \equiv (I_{a+,h}^{1-\gamma,-\alpha,1+\beta-\gamma} f)(x)$$
$$= \frac{(x-h)^{\alpha}}{\Gamma(1-\gamma)} \int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-h}\right) f(t)\, dt. \quad (4.11)$$

Then, in accordance with Remark 1 in Section 3, the following statements are valid, where each can be proved along the lines of our proofs of Lemma 1 and Theorem 2.

LEMMA 3. *Let $f(t) \in L_1(a,b)$, $\alpha, \beta \in \mathbb{R}^1$, $0 < \gamma < 1$. Let further*

$$f(t) = O\big((t-h)^{\mu}\big) \qquad (t \to h) \quad (4.12)$$

with $\mu > -1$ for $h < a$ and $\mu >$ $\max[0, \beta - \alpha] - 1$ *for $h = a$, and*

$$f(t) = O\big((b-t)^{\nu}\big) \qquad (t \to b) \quad (4.13)$$

with $\nu > \gamma - 2$. Then $f_{a+,h}^{\gamma,\alpha,\beta}(x) \in L_1(a,b)$.

THEOREM 4. *The Abel-type integral equation* (3.5) *with real α, β, $0 < \gamma < 1$ and $h \leqq a$ is solvable in $L_1(a,b)$ if and only if*

$$f_{a+,h}^{\gamma,\alpha,\beta}(x) \in AC([a,b]) \quad \text{and} \quad f_{a+,h}^{\gamma,\alpha,\beta}(a) = 0. \quad (4.14)$$

Under these conditions, (3.5) *has a unique solution given by* (3.6).

5. Sufficient conditions for the solvability of the one-dimensional Abel-type hypergeometric integral equation

The criterion for the solvability of the Abel-type hypergeometric integral equation (3.1) is obtained in Theorem 2 in terms of the auxiliary function $f_{a+}^{\gamma,\alpha,\beta}(x)$. The result below gives simple sufficient conditions in terms of the function $f(x)$ itself. To prove such a result, we need the following assertion.

LEMMA 5. *Let* $f(x) \in AC([a,b])$ *and let* $\alpha, \beta, \gamma \in \mathbb{R}^1$ *such that*

$$0 < \gamma < 1, \quad \gamma - \alpha - 1 < \beta < 1 + \alpha \quad \text{and} \quad \gamma < 1 + \alpha. \tag{5.1}$$

Then $f_{a+}^{\gamma,\alpha,\beta}(x) \in AC([a,b])$ *and*

$$\begin{aligned} f_{a+}^{\gamma,\alpha,\beta}(x) &= \frac{\Gamma(1+\alpha-\beta)}{\Gamma(2+\alpha-\gamma)\Gamma(1-\beta)}(x-a)^{\alpha-\gamma+1} f(a) \\ &\quad + \frac{(x-a)^{\alpha}}{\Gamma(2-\gamma)} \int_a^x (x-t)^{1-\gamma} F\left(-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-a}\right) f'(t)\, dt. \end{aligned} \tag{5.2}$$

PROOF. Since, by hypothesis, $f(t) \in AC([a,b])$, and in view of (2.9), $f(t)$ is representable in the form:

$$f(t) = f(a) + \int_a^t f'(\tau)\, d\tau. \tag{5.3}$$

Substituting this relation into (4.1), we have

$$\begin{aligned} f_{a+}^{\gamma,\alpha,\beta}(x) &= \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)} f(a) \int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) dt \\ &\quad + \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)} \int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) dt \int_a^t f'(\tau)\, d\tau \\ &= I_1(x) + I_2(x). \end{aligned} \tag{5.4}$$

According to (2.6) and (2.4), we evaluate $I_1(x)$ by changing the variable $s = (x-t)/(x-a)$ as follows:

$$\begin{aligned} I_1(x) &= \frac{(x-a)^{\alpha-\gamma+1}}{(1-\gamma)\Gamma(1-\gamma)} f(a) F(-\alpha, 1+\beta-\gamma; 2-\gamma; 1) \\ &= \frac{\Gamma(1+\alpha-\beta)}{\Gamma(2+\alpha-\gamma)\Gamma(1-\beta)}(x-a)^{\alpha-\gamma+1} f(a) \end{aligned} \tag{5.5}$$

taking the conditions $0 < \gamma < 1$ and $1 + \alpha - \beta > 0$ in (5.1) into account. As for $I_2(x)$, after interchanging the order of integration and evaluating the inner integral by using (2.6) again, we obtain

$$\begin{aligned} I_2(x) &= \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)} \int_a^x f'(\tau)\, d\tau \\ &\quad \times \int_\tau^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) dt \\ &= \frac{(x-a)^{\alpha}}{\Gamma(2-\gamma)} \int_a^x (x-t)^{1-\gamma} F\left(-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-a}\right) f'(t)\, dt. \end{aligned} \tag{5.6}$$

By (5.5), $I_1(x)$ is an absolutely continuous function because

$$(x-a)^{\alpha-\gamma+1} = (\alpha-\gamma+1)\int_a^x (t-a)^{\alpha-\gamma}\,dt \tag{5.7}$$

and $(t-a)^{\alpha-\gamma} \in L_1(a,b)$, by the condition $\alpha-\gamma+1>0$ in (5.1). To prove that $I_2(x) \in AC([a,b])$, we first note that, in accordance with (2.5), the Gauss function in the last integrand of (5.6) has the following asymptotics near $x=a$:

$$F\left(-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-a}\right) = O\big((x-a)^{-\alpha}\big) + O\big((x-a)^{1+\beta-\gamma}\big) \quad (x\to a) \tag{5.8}$$

for noninteger $\alpha+\beta-\gamma$, and with the addition of $\log(x-a)$ in the case of integer $\alpha+\beta-\gamma$. Therefore, $I_2(a)=0$ by the condition $\alpha+\beta-\gamma+1>0$ in (5.1). So we can represent $I_2(x)$ in the form:

$$I_2(x) = \int_a^x h(t)\,dt \quad \text{or} \quad h(x) = \frac{d}{dx}I_2(x). \tag{5.9}$$

By using (2.2) and term-by-term differentiation, which can be justified under the conditions in (5.1), it is easily verified that

$$\begin{aligned} h(x) &= h_1(x)+h_2(x) \\ &\equiv \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)}\int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) f'(t)\,dt \\ &\quad + \frac{\alpha(x-a)^{\alpha-1}}{\Gamma(2-\gamma)}\int_a^x (x-t)^{1-\gamma} F\left(1-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-a}\right) f'(t)\,dt. \end{aligned} \tag{5.10}$$

It follows on the pattern of proofs of (4.6) and (4.7) that

$$\int_a^b h_1(x)\,dx = \int_a^b g_1(t)\,dt, \qquad \int_a^b h_2(x)\,dx = \int_a^b g_2(t)\,dt,$$

$$g_1(t) = \frac{1}{\Gamma(2-\gamma)}(t-a)^{\alpha}(b-t)^{1-\gamma} F\left(-\alpha, -\beta; 2-\gamma; -\frac{b-t}{t-a}\right) f'(t),$$

$$g_2(t) = \frac{1}{\Gamma(3-\gamma)}(t-a)^{\alpha-1}(b-t)^{2-\gamma} F\left(1-\alpha, 1-\beta; 3-\gamma; -\frac{b-t}{t-a}\right) f'(t)$$

and

$$\int_a^b |h_1(x)|\,dx \leqq \frac{1}{\Gamma(2-\gamma)}\int_a^b (t-a)^{\alpha}(b-t)^{1-\gamma} \times \left|F\left(-\alpha, -\beta; 2-\gamma; -\frac{b-t}{t-a}\right)\right| |f'(t)|\,dt < \infty,$$

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$$\int_a^b |h_2(x)|\,dx \leqq \frac{1}{\Gamma(3-\gamma)} \int_a^b (t-a)^{\alpha-1}(b-t)^{2-\gamma} \times \left| F\left(1-\alpha, 1-\beta; 3-\gamma; -\frac{b-t}{t-a}\right)\right| |f'(t)|\,dt < \infty.$$

Hence $h(x) \in L_1(a,b)$ and $I_2(x)$ is also an absolutely continuous function in accordance with (5.9) and $f_{a+}^{\gamma,\alpha,\beta}(x) \in AC([a,b])$. The representation (5.2) follows from (5.4) to (5.6). This completes the proof of Lemma 5.

COROLLARY 6. *Under the conditions of Lemma* 3, $f_{a+}^{\gamma,\alpha,\beta}(a) = 0$.

The following result gives a new form of the inversion formula of (3.1) applicable to absolutely continuous functions.

THEOREM 7. *Let* $f(x) \in AC([a,b])$ *and let* $\alpha, \beta, \gamma \in \mathbb{R}^1$ *such that the conditions in* (5.1) *are satisfied. Then the Abel-type hypergeometric equation* (3.1) *is solvable in* $L_1(a,b)$ *and its solution* (3.2) *can be expressed in the form*:

$$\begin{aligned}\varphi(x) = {} & \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+\alpha-\gamma)\Gamma(1-\beta)}(x-a)^{\alpha-\gamma} f(a) \\ & + \frac{(x-a)^{\alpha}}{\Gamma(1-\gamma)} \int_a^x (x-t)^{-\gamma} F\left(-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-a}\right) f'(t)\,dt \\ & + \frac{\alpha(x-a)^{\alpha-1}}{\Gamma(2-\gamma)} \int_a^x (x-t)^{1-\gamma} F\left(1-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-a}\right) f'(t)\,dt. \end{aligned} \tag{5.11}$$

PROOF. By Lemma 5 and Corollary 6, $f_{a+}^{\gamma,\alpha,\beta}(x) \in AC([a,b])$ and $f_{a+}^{\gamma,\alpha,\beta}(a) = 0$. So the conditions (4.8) of Theorem 2 are satisfied and (3.1) is solvable in $L_1(a,b)$. Since $\varphi(x) = \left(f_{a+}^{\gamma,\alpha,\beta}(x)\right)'$, (5.11) is obtained by differentiating (5.2) and using (5.4) to (5.7) and (5.9) to (5.10). Theorem 7 is thus proved.

REMARK 3. The results in Lemmas 1, 3 and 5 and Theorems 2, 4 and 7 generalize the corresponding statements for the classical Abel integral equation studied in [26, Section 2.2].

REMARK 4. The results of Sections 4 and 5 given in Theorems 2, 4 and 7 (in particular, a new form (5.11) for the solution of (3.1)), can be used to solve other similar types of integral equations involving the modified and particular forms of the Gauss hypergeometric function $F(a,b;c;z)$ (see [26] and [29]). For example, if we replace α by $\alpha+\beta$, γ by α and β by $-\eta$ in (3.1), we obtain the equation:

$$(I_{a+}^{\alpha,\beta,\eta}\varphi)(x) = f(x) \tag{5.12}$$

with the generalized fractional integral operator $I_{a+}^{\alpha,\beta,\eta}\varphi$ introduced in [20] (see also [26, p. 439]). Theorems 2 and 4 with the above specializations yield new forms of results concerning the solvability of the integral equation (5.12). They can be applied in solving those boundary value problems where such equations arise (see [7, 16–19, 21–23, 25, 28, 31] and [33]).

6. Solution of the multidimensional Abel-type hypergeometric integral equation

In this and the next sections we shall use the notation introduced in Section 2. Let $A = \|a_{jk}\|$ ($a_{jk} \in \mathbb{R}^1$) be a matrix of order $n \times n$ with its determinant $|A| \equiv \det A = 1$. Also let $A_{c,r}(\boldsymbol{b})$ ($\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$; $r \in \mathbb{R}^1$) be a pyramid defined by (2.12). Let $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^n$ and $F[\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{\gamma}; A \cdot (\boldsymbol{x} - \boldsymbol{t})/A \cdot (\boldsymbol{x} - \boldsymbol{h})]$ be defined by (2.14). We define the Abel-type hypergeometric integral equation on $A_{c,r}(\boldsymbol{b})$ by

$$\left(I_{A_{c,r}}^{\gamma,\alpha,\beta}\varphi\right)(\boldsymbol{x}) \equiv \frac{\left(A\cdot(\boldsymbol{x}-\boldsymbol{h})\right)^{-\boldsymbol{\alpha}}}{\Gamma(\boldsymbol{\gamma})}\int_{A_{c,r}(\boldsymbol{x})}\left(A\cdot(\boldsymbol{x}-\boldsymbol{t})\right)^{\boldsymbol{\gamma}-1}F\left[\boldsymbol{\alpha},\boldsymbol{\beta};\boldsymbol{\gamma};\frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]\varphi(\boldsymbol{t})\,d\boldsymbol{t}$$
$$= f(\boldsymbol{x}) \qquad (\boldsymbol{x} \in A_{c,r}(\boldsymbol{b})) \tag{6.1}$$

with $\mathbf{0} < \boldsymbol{\gamma} < \mathbf{1}$. This equation generalizes the multidimensional Abel-type integral equation [26, Section 28.4], which is obtained from (6.1) in the special case when $r = 0$ and $\boldsymbol{\alpha} = \mathbf{0}$ or $\boldsymbol{\beta} = \mathbf{0}$. To solve (6.1), we apply the method used in Section 3. We replace $\boldsymbol{x}$ by $\boldsymbol{t}$ and $\boldsymbol{t}$ by $\boldsymbol{\tau}$ in (6.1), multiply the resulting equation by

$$\left(A\cdot(\boldsymbol{x}-\boldsymbol{t})\right)^{-\boldsymbol{\gamma}}F\left[-\boldsymbol{\alpha},\mathbf{1}+\boldsymbol{\beta}-\boldsymbol{\gamma};\mathbf{1}-\boldsymbol{\gamma};\frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]$$

and integrate over the pyramid $A_{c,r}(\boldsymbol{x})$. Applying a known result [26, Lemma 28.3], we have

$$\frac{1}{\Gamma(\boldsymbol{\gamma})}\int_{A_{c,r}(\boldsymbol{x})}\varphi(\boldsymbol{\tau})\,d\boldsymbol{\tau}\int_{\sigma(\boldsymbol{x},\boldsymbol{\tau})}\left(A\cdot(\boldsymbol{x}-\boldsymbol{t})\right)^{-\boldsymbol{\gamma}}\left(A\cdot(\boldsymbol{t}-\boldsymbol{\tau})\right)^{\boldsymbol{\gamma}-1}\left(A\cdot(\boldsymbol{t}-\boldsymbol{h})\right)^{-\boldsymbol{\alpha}}$$
$$\times F\left[\boldsymbol{\alpha},\boldsymbol{\beta};\boldsymbol{\gamma};\frac{A\cdot(\boldsymbol{t}-\boldsymbol{\tau})}{A\cdot(\boldsymbol{t}-\boldsymbol{h})}\right]F\left[-\boldsymbol{\alpha},\mathbf{1}+\boldsymbol{\beta}-\boldsymbol{\gamma};\mathbf{1}-\boldsymbol{\gamma};\frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]d\boldsymbol{t}$$
$$= \int_{A_{c,r}(\boldsymbol{x})}\left(A\cdot(\boldsymbol{x}-\boldsymbol{t})\right)^{-\boldsymbol{\gamma}}F\left[-\boldsymbol{\alpha},\mathbf{1}+\boldsymbol{\beta}-\boldsymbol{\gamma};\mathbf{1}-\boldsymbol{\gamma};\frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]f(\boldsymbol{t})\,d\boldsymbol{t}, \tag{6.2}$$

where

$$\sigma(\boldsymbol{x},\boldsymbol{\tau}) = \{\boldsymbol{t} \in \mathbb{R}^n : A\cdot\boldsymbol{\tau} \leqq A\cdot\boldsymbol{t} \leqq A\cdot\boldsymbol{x}\}. \tag{6.3}$$

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To evaluate the inner integral in (6.2), we change the variables as follows:

$$s_j = \frac{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{t})}{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})}, \quad \boldsymbol{a}_j = (a_{j1}, \dots, a_{jn}) \quad (j = 1, \dots, n).$$

Then, by taking into account the equality $1 - s_j = [\boldsymbol{a}_j \cdot (\boldsymbol{t} - \boldsymbol{\tau})]/[\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})]$ and (2.7), the inner integral can be evaluated as

$$\begin{aligned}
&\big(A \cdot (\boldsymbol{x} - \boldsymbol{h})\big)^{-\boldsymbol{\alpha}} \prod_{j=1}^{n} \left\{ \int_0^1 s_j^{-\gamma_j} (1 - s_j)^{\gamma_j - 1} \left[1 - s_j \left(\frac{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})}{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{h})} \right) \right]^{-\alpha_j} \right. \\
&\quad \times F\left(\alpha_j, \beta_j; \gamma_j; \frac{(1 - s_j)\{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})/\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{h})\}}{1 - s_j\{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})/\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{h})\}}\right) \\
&\quad \left. \times F\left(-\alpha_j, 1 + \beta_j - \gamma_j; 1 - \gamma_j; s_j \frac{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})}{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{h})}\right) ds_j \right\} \\
&\quad = \Gamma(\boldsymbol{\gamma})\Gamma(\boldsymbol{1} - \boldsymbol{\gamma})\big(A \cdot (\boldsymbol{x} - \boldsymbol{h})\big)^{-\boldsymbol{\alpha}} \boldsymbol{F}\left[\boldsymbol{0}, \boldsymbol{1} + \boldsymbol{\beta} - \boldsymbol{\gamma}; \boldsymbol{1}; \frac{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{\tau})}{\boldsymbol{a}_j \cdot (\boldsymbol{x} - \boldsymbol{h})}\right] \\
&\quad = \Gamma(\boldsymbol{\gamma})\Gamma(\boldsymbol{1} - \boldsymbol{\gamma})\big(A \cdot (\boldsymbol{x} - \boldsymbol{h})\big)^{-\boldsymbol{\alpha}}.
\end{aligned}$$

Hence (6.2) yields

$$\begin{aligned}
\int_{A_{c,r}(\boldsymbol{x})} \varphi(\boldsymbol{t})\, d\boldsymbol{t} &= \frac{\big(A \cdot (\boldsymbol{x} - \boldsymbol{h})\big)^{\boldsymbol{\alpha}}}{\Gamma(\boldsymbol{1} - \boldsymbol{\gamma})} \int_{A_{c,r}(\boldsymbol{x})} \big(A \cdot (\boldsymbol{x} - \boldsymbol{t})\big)^{-\boldsymbol{\gamma}} \\
&\quad \times \boldsymbol{F}\left[-\boldsymbol{\alpha}, \boldsymbol{1} + \boldsymbol{\beta} - \boldsymbol{\gamma}; \boldsymbol{1} - \boldsymbol{\gamma}; \frac{A \cdot (\boldsymbol{x} - \boldsymbol{t})}{A \cdot (\boldsymbol{x} - \boldsymbol{h})}\right] f(\boldsymbol{t})\, d\boldsymbol{t} \\
&\equiv f_{A_{c,r}}^{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}).
\end{aligned} \tag{6.4}$$

Let $\boldsymbol{x} + r/(n\boldsymbol{c}) = (x_1 + r/(nc_1), \dots, x_n + r/(nc_n))$. Making the change of variables ([26, p. 572] and [8, (2.6)]) by means of

$$\boldsymbol{x} + r/(n\boldsymbol{c}) = A^{-1} \cdot (\boldsymbol{y}/\boldsymbol{d}), \quad \boldsymbol{t} + r/(n\boldsymbol{c}) = A^{-1} \cdot (\boldsymbol{\tau}/\boldsymbol{d}), \tag{6.5}$$

where

$$\boldsymbol{y}/\boldsymbol{d} = (y_1/d_1, \dots, y_n/d_n) \in \mathbb{R}^n, \quad \boldsymbol{d} = A^{-1} \cdot \boldsymbol{c},$$

we find that (6.4) becomes

$$\int_{E_1(\boldsymbol{y})} \psi(\boldsymbol{\tau})\, d\boldsymbol{\tau} = g(\boldsymbol{y}), \tag{6.6}$$

where the model pyramid $E_1(\boldsymbol{x})$ is given by (2.13),

$$\psi(\boldsymbol{\tau}) = \varphi\left(A^{-1} \cdot \frac{\boldsymbol{\tau}}{\boldsymbol{d}} - \frac{r}{n\boldsymbol{c}}\right), \quad g(\boldsymbol{y}) = f_{A_{c,r}}^{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}\left(A^{-1} \cdot \frac{\boldsymbol{y}}{\boldsymbol{d}} - \frac{r}{n\boldsymbol{c}}\right) \prod_{j=1}^{n} d_j. \tag{6.7}$$

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To invert (6.6), we follow the steps used in [26, p. 573] (see also [4, p. 274]) and rewrite (6.6) in the form:

$$\int_{-(y_1+\cdots+y_{n-1})}^{y_n} d\tau_n \int_{-(y_1+\cdots+y_{n-2}+\tau_n)}^{y_{n-1}} d\tau_{n-1} \cdots \int_{-(\tau_2+\cdots+\tau_n)}^{y_1} \psi(\tau)\, d\tau_1 = g(y). \tag{6.8}$$

Differentiating successively with respect to $y_n, y_{n-1}, \ldots, y_1$, we obtain

$$\psi(y) = \frac{\partial}{\partial \boldsymbol{y}} g(y) \equiv \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_n} g(y). \tag{6.9}$$

Here we return to the variable $\boldsymbol{x} = A^{-1} \cdot (\boldsymbol{y}/\boldsymbol{d}) - \boldsymbol{r}/(n\boldsymbol{c})$ similarly to (6.5), so that

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \frac{\tilde{a}_{jk}}{d_k} \frac{\partial}{\partial x_j} \quad (k = 1, \ldots, n), \tag{6.10}$$

where $\tilde{a}_{jk}$ $(j, k = 1, \ldots, n)$ represent the elements of the inverse matrix A^{-1}. So, finally, we have from (6.4), (6.6), (6.7), (6.9) and (6.10) the inversion for the solution $\varphi(\boldsymbol{x})$ of (6.1):

$$\varphi(\boldsymbol{x}) = \frac{1}{\Gamma(1-\gamma)} \prod_{k=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j} \right) \left\{ (A \cdot (\boldsymbol{x} - \boldsymbol{h}))^{\alpha} \int_{A_{c,r}(\boldsymbol{x})} (A \cdot (\boldsymbol{x} - \boldsymbol{t}))^{-\gamma} \right.$$
$$\left. \times F \left[-\alpha, 1 + \beta - \gamma; 1 - \gamma; \frac{A \cdot (\boldsymbol{x} - \boldsymbol{t})}{A \cdot (\boldsymbol{x} - \boldsymbol{h})} \right] f(\boldsymbol{t})\, d\boldsymbol{t} \right\}. \tag{6.11}$$

To formulate the conditions of solvability for (6.1) in the space $L_1(A_{c,r}(\boldsymbol{b}))$ of summable functions, we define the space of functions ([26, p. 574] and [8, p. 3])

$$I_{A_{c,r}}(L_1) = \left\{ g : g(\boldsymbol{x}) = \int_{\substack{A_{c,r}(\boldsymbol{x}) \\ A\cdot(\boldsymbol{b}-\boldsymbol{t}) \geqq A\cdot(\boldsymbol{x}-\boldsymbol{t})}} h(\boldsymbol{t})\, d\boldsymbol{t},\ h(\boldsymbol{t}) \in L_1(A_{c,r}(\boldsymbol{b})) \right\}. \tag{6.12}$$

This space plays the same rôle for the multidimensional integral equation (6.1) as the space $AC([a, b])$ does for the one-dimensional integral equation (3.1). It may be noted that, if $g \in I_{A_{c,r}}(L_1)$, then the partial derivatives of $g(\boldsymbol{x})$ up to the order n exist almost everywhere and

$$\prod_{k=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j} \right) g(\boldsymbol{x}) = h(\boldsymbol{x}), \tag{6.13}$$

where $\tilde{a}_{jk}$ $(j, k = 1, \ldots, n)$ are elements of the inverse matrix A^{-1}. In particular, when $A = E$ is the unit matrix, $\boldsymbol{c} = \boldsymbol{1} = (1, \ldots, 1)$ and $\boldsymbol{r} = 0$, (6.12) and (6.13) take the forms:

$$I_{E_1}(L_1) = \left\{ g : g(\boldsymbol{x}) = \int_{\substack{E_1(\boldsymbol{x}) \\ (\boldsymbol{b}-\boldsymbol{t}) \geqq (\boldsymbol{x}-\boldsymbol{t})}} h(\boldsymbol{t}) d\boldsymbol{t},\ h(\boldsymbol{t}) \in L_1(E_1(\boldsymbol{b})) \right\} \tag{6.14}$$

and

$$\frac{\partial}{\partial \boldsymbol{x}} g(\boldsymbol{x}) \equiv \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} g(\boldsymbol{x}) = h(\boldsymbol{x}), \tag{6.15}$$

respectively. The following result, which can be proved just as Theorem 2 on the basis of (6.4) and (6.12), gives the corresponding multidimensional pyramidal analogue of Theorem 2.

THEOREM 8. *The multidimensional Abel-type hypergeometric integral equation* (6.1) *with* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{h}, \boldsymbol{\gamma} \in \mathbb{R}^n$ $(\mathbf{0} < \boldsymbol{\gamma} < \mathbf{1})$ *is solvable in* $L_1(A_{c,r}(\boldsymbol{b}))$ *if and only if*

$$\begin{aligned} f_{A_{c,r}}^{\gamma,\alpha,\beta}(\boldsymbol{x}) &\equiv \left(I_{A_{c,r}}^{1-\gamma,-\alpha,1+\beta-\gamma}\right)(x) \\ &= \frac{\left(A\cdot(\boldsymbol{x}-\boldsymbol{h})\right)^{\alpha}}{\Gamma(1-\gamma)} \int_{A_{c,r}(x)} \left(A\cdot(\boldsymbol{x}-\boldsymbol{t})\right)^{-\gamma} \\ &\quad \times F\left[-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right] f(\boldsymbol{t})\, d\boldsymbol{t} \in I_{A_{c,r}}(L_1) \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} f_{A_{c,r}}^{\gamma,\alpha,\beta}(\boldsymbol{x})\Big|_{c\cdot x+r=0} &= \sum_{j=1}^{n} \tilde{a}_{jn} \frac{\partial}{\partial x_j} f_{A_{c,r}}^{\gamma,\alpha,\beta}(\boldsymbol{x})\Big|_{c\cdot x+r=0} = \cdots \\ &= \prod_{k=2}^{n} \sum_{j=1}^{n} \left(\tilde{a}_{jk} \frac{\partial}{\partial x_j}\right) f_{A_{c,r}}^{\gamma,\alpha,\beta}(\boldsymbol{x})\Big|_{c\cdot x+r=0}. \end{aligned} \tag{6.17}$$

Under these conditions, (6.1) *has a unique solution given by* (6.11).

COROLLARY 9. *The multidimensional 'model' Abel-type hypergeometric integral equation*:

$$\left(I_{E_1}^{\gamma,\alpha,\beta}\varphi\right)(\boldsymbol{x}) \equiv \frac{(\boldsymbol{x}-\boldsymbol{h})^{-\alpha}}{\Gamma(\boldsymbol{\gamma})} \int_{E_1(x)} (\boldsymbol{x}-\boldsymbol{t})^{\gamma-1} F\left[\alpha, \beta; \gamma; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] \varphi(\boldsymbol{t})\, d\boldsymbol{t} = f(\boldsymbol{x}) \tag{6.18}$$

for $\boldsymbol{x} \in E_1(\boldsymbol{b})$ *with* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{h}, \boldsymbol{\gamma} \in \mathbb{R}^n$ $(0 < \boldsymbol{\gamma} < 1)$ *is solvable in* $L_1(E_1(\boldsymbol{b}))$ *if and only if*

$$\begin{aligned} f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x}) &\equiv \left(I_{E_1}^{1-\gamma,-\alpha,1+\beta-\gamma} f\right)(x) \\ &= \frac{(\boldsymbol{x}-\boldsymbol{h})^{\alpha}}{\Gamma(1-\gamma)} \int_{E_1(x)} (\boldsymbol{x}-\boldsymbol{t})^{-\gamma} F\left[-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] f(\boldsymbol{t})\, d\boldsymbol{t} \\ &\in I_{E_1}(L_1) \end{aligned} \tag{6.19}$$

and

$$f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x})\Big|_{1\cdot x=0} = \frac{\partial}{\partial x_n} f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x})\Big|_{1\cdot x=0} = \cdots = \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x})\Big|_{1\cdot x=0}. \tag{6.20}$$

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Under these conditions, (6.18) *has a unique solution given by*

$$\varphi(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{x}} f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x}) \equiv \frac{1}{\Gamma(\mathbf{1}-\boldsymbol{\gamma})} \frac{\partial}{\partial \boldsymbol{x}} \left\{ (\boldsymbol{x}-\boldsymbol{h})^{\boldsymbol{\alpha}} \int_{E_1(\boldsymbol{x})} (\boldsymbol{x}-\boldsymbol{t})^{-\boldsymbol{\gamma}} \times F\left[-\boldsymbol{\alpha}, \mathbf{1}+\boldsymbol{\beta}-\boldsymbol{\gamma}; \mathbf{1}-\boldsymbol{\gamma}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] f(\boldsymbol{t}) d\boldsymbol{t} \right\} \tag{6.21}$$

with $\partial/\partial \boldsymbol{x}$ *as in* (6.9).

REMARK 5. When $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) = \mathbf{0}$, Theorem 8 and Corollary 9 are reduced to the results obtained in [8, Theorem 2 and Corollary] (see also the case $r = 0$ in [26, Theorem 28.7 and Corollary]).

7. Sufficient conditions for the solvability of the multidimensional model Abel-type hypergeometric integral equation

To discuss the results concerning the solvability of the model Abel-type multidimensional hypergeometric integral equation (6.18), we need some preliminary assertions. For $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$, we define a multivariable function $S(\boldsymbol{x})$ by

$$S(\boldsymbol{x}) = \sum_{\boldsymbol{k}=\mathbf{0}}^{\infty} c(\boldsymbol{k}) \boldsymbol{x}^{\boldsymbol{k}} \equiv \sum_{k_1,\dots,k_n=0}^{\infty} c(k_1, \dots, k_n) x_1^{k_1} \cdots x_n^{k_n}, \tag{7.1}$$

where $c(\boldsymbol{k})$ is a bounded multiple sequence and $|x_i| < R_i$ ($R_i > 0$; $i = 1, \dots, n$).

LEMMA 10. *For* $\boldsymbol{\gamma} \in \mathbb{R}_+^n$ *and* $\boldsymbol{p} \in \mathbb{R}^n$,

$$\int_{E_1(\boldsymbol{x})} (\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{\gamma}-\mathbf{1}} S(\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{t}))\, d\boldsymbol{t} = \frac{\Gamma(\boldsymbol{\gamma})}{\Gamma(1+|\boldsymbol{\gamma}|)} (\mathbf{1}\cdot\boldsymbol{x})^{|\boldsymbol{\gamma}|} \sum_{\boldsymbol{k}=\mathbf{0}}^{\infty} \frac{(\boldsymbol{\gamma})_{\boldsymbol{k}}}{(1+|\boldsymbol{\gamma}|)_{|\boldsymbol{k}|}} c(\boldsymbol{k}) \boldsymbol{p}^{\boldsymbol{k}} (\mathbf{1}\cdot\boldsymbol{x})^{|\boldsymbol{k}|}, \tag{7.2}$$

where $S(\boldsymbol{x})$ *is defined by* (7.1).

PROOF. Using (7.1) and (2.13) and substituting

$$\begin{aligned} x_1 - t_1 &= s_1(x_1 + t_2 + \cdots + t_n), \\ x_2 - t_2 &= s_2(x_1 + x_2 + t_3 + \cdots + t_n), \end{aligned} \tag{7.3}$$

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successively, we have

$$\int_{E_1(x)} (x-t)^{\gamma-1} S(p(x-t))\,dt$$
$$= \sum_{k=0}^{\infty} c(k) \int_{-(x_1+\cdots+x_{n-1})}^{x_n} p_n^{k_n}(x_n-t_n)^{\gamma_n+k_n-1}\,dt_n$$
$$\times \int_{-(x_1+\cdots+x_{n-2}+t_n)}^{x_{n-1}} p_{n-1}^{k_{n-1}}(x_{n-1}-t_{n-1})^{\gamma_{n-1}+k_{n-1}-1}\,dt_{n-1} \times \cdots$$
$$\times \int_{-(x_1+t_3+\cdots+t_n)}^{x_2} p_2^{k_2}(x_2-t_2)^{\gamma_2+k_2-1}\,dt_2 \int_{-(t_2+\cdots+t_n)}^{x_1} p_1^{k_1}(x_1-t_1)^{\gamma_1+k_1-1}\,dt_1$$
$$= \sum_{k=0}^{\infty} c(k) p^k \frac{1}{\gamma_1+k_1} \int_{-(x_1+\cdots+x_{n-1})}^{x_n} (x_n-t_n)^{\gamma_n+k_n-1}\,dt_n \times \cdots$$
$$\times \int_{-(x_1+t_3+\cdots+t_n)}^{x_2} (x_2-t_2)^{\gamma_2+k_2-1}(x_1+t_2+\cdots+t_n)^{\gamma_1+k_1}\,dt_2$$
$$= \sum_{k=0}^{\infty} c(k) p^k \frac{\Gamma(\gamma_2+k_2)\Gamma(\gamma_1+k_1)}{\Gamma(1+\gamma_1+\gamma_2+k_1+k_2)} \int_{-(x_1+\cdots+x_{n-1})}^{x_n} (x_n-t_n)^{\gamma_n+k_n-1}\,dt_n \times \cdots$$
$$\times \int_{-(x_1+x_2+t_4\cdots+t_n)}^{x_3} (x_3-t_3)^{\gamma_3+k_3-1}(x_1+x_2+t_3+\cdots+t_n)^{\gamma_1+\gamma_2+k_1+k_2}\,dt_3.$$

Continuing this process, we obtain

$$\int_{E_1(x)} (x-t)^{\gamma-1} S(p(x-t))\,dt = \sum_{k=0}^{\infty} c(k) p^k \frac{\Gamma(\gamma+k)}{\Gamma(1+|\gamma|+|k|)} (1\cdot x)^{|\gamma|+|k|},$$

which gives (7.2), in view of (2.1) and (2.10). This completes the proof of Lemma 10.

The following assertion proved along the lines of our proof of Lemma 10 by applying the successive substitutions (7.3), *etc.*, is an analogue of the relation in which the Beta function is expressed *via* Gamma functions [5].

LEMMA 11. *For* $\gamma \in \mathbb{R}_+^n$ *and* $r > 0$,

$$\int_{E_1(x)} (x-t)^{\gamma-1}(1\cdot t)^{r-1}\,dt = \frac{\Gamma(\gamma)\Gamma(r)}{\Gamma(r+|\gamma|)} (1\cdot x)^{r+|\gamma|-1}. \tag{7.4}$$

COROLLARY 12. *For* $\alpha \in \mathbb{R}^n$ *with* $\alpha < 1$,

$$\int_{E_1(x)} (x-t)^{-\alpha}\,dt = \frac{\Gamma(1-\alpha)}{\Gamma(1+n-|\alpha|)} (1\cdot x)^{n-|\alpha|}. \tag{7.5}$$

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REMARK 6. For $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$, the result in (7.5) was proved by Kilbas *et al.* [8, p. 5, (3.4)].

The next statement, which can be proved by applying (2.15) and Lemma 11, is a generalization of Lemma 11 for the multidimensional model Abel-type hypergeometric integral operator defined in (6.18).

LEMMA 13. *If* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{h}, \boldsymbol{\gamma} \in \mathbb{R}^n$ $(\mathbf{0} < \boldsymbol{\gamma} < \mathbf{1})$, $r > 0$ *and*

$$\max\left[\left|\frac{\mathbf{1}\cdot\boldsymbol{x}}{x_1 - h_1}\right|, \dots, \left|\frac{\mathbf{1}\cdot\boldsymbol{x}}{x_n - h_n}\right|\right] < 1, \tag{7.6}$$

then

$$\left(I_{E_1}^{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{1}\cdot\boldsymbol{t})^{r-1}\right)(\boldsymbol{x}) = \frac{(\boldsymbol{x}-\boldsymbol{h})^{-\boldsymbol{\alpha}}\Gamma(r)}{\Gamma(r+|\boldsymbol{\gamma}|)}(\mathbf{1}\cdot\boldsymbol{x})^{r+|\boldsymbol{\gamma}|-1} F_B\left(\boldsymbol{\alpha}, \boldsymbol{\beta}; r + |\boldsymbol{\gamma}|; \frac{\mathbf{1}\cdot\boldsymbol{x}}{\boldsymbol{x}-\boldsymbol{h}}\right), \tag{7.7}$$

where

$$F_B\left(\boldsymbol{\alpha}, \boldsymbol{\beta}; r + |\boldsymbol{\gamma}|; \frac{\mathbf{1}\cdot\boldsymbol{x}}{\boldsymbol{x}-\boldsymbol{h}}\right) \equiv F_B^{(n)}\left(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n; r + |\boldsymbol{\gamma}|; \frac{\mathbf{1}\cdot\boldsymbol{x}}{x_1 - h_1}, \dots, \frac{\mathbf{1}\cdot\boldsymbol{x}}{x_n - h_n}\right) \tag{7.8}$$

is given by (2.15).

By an analogy with (2.9), we can obtain the subspace $I^1_{E_1,0}(L_1)$ of the space (6.14), introduced in [24] and defined by

$$I^1_{E_1,0}(L_1) = \left\{ f(\boldsymbol{x}) \in I^1_{E_1}(L_1) : f(\boldsymbol{x})\big|_{\mathbf{1}\cdot\boldsymbol{x}=0} = c_0 \in \mathbb{R}^1, \right.$$
$$\left. \frac{\partial^k}{\partial x_{n-k+1}\cdots\partial x_n} f(\boldsymbol{x})\bigg|_{\mathbf{1}\cdot\boldsymbol{x}=0} = c_k \in \mathbb{R}^1 \ (1 \leqq k \leqq n-1) \right\}. \tag{7.9}$$

This space is characterized by the following assertion [24, Corollary of Lemma 1].

LEMMA 14. *The function* $f(\boldsymbol{x}) \in I^1_{E_1,0}(L_1)$ *if and only if*

$$f(\boldsymbol{x}) = \int_{E_1(\boldsymbol{x})} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial t}}\right) f(\boldsymbol{t}) d\boldsymbol{t} + \sum_{k=1}^{n} \frac{c_k}{k!}(\mathbf{1}\cdot\boldsymbol{x})^k, \tag{7.10}$$

where c_k $(k = 1, \dots, n)$ *are given in* (7.9).

The following assertion is a multidimensional analogue of Lemma 5.

LEMMA 15. *Let* $x, h \in \mathbb{R}^n$ *and* $\alpha, \beta, \gamma \in \mathbb{R}^n$ *be such that*

$$0 < \gamma < 1, \quad |1 \cdot x| \equiv |x_1 + \cdots + x_n| < \min\left[|x_1 - h_1|, \ldots, |x_n - h_n|\right]. \tag{7.11}$$

If $f(x) \in I_{E_1,0}(L_1)$, *then* $f_{E_1}^{\gamma,\alpha,\beta}(x)$ *defined by* (6.19) *can be represented in the form:*

$$\begin{aligned} f_{E_1}^{\gamma,\alpha,\beta}(x) &= (x-h)^\alpha \sum_{k=0}^{n-1} \frac{(1\cdot x)^{n+k-|\gamma|} c_k}{\Gamma(n+k-|\gamma|+1)} F_B\left(-\alpha, 1+\beta-\gamma; n+k-|\gamma|+1; \frac{1\cdot x}{x-h}\right) \\ &\quad + \frac{(x-h)^\alpha}{\Gamma(2-\gamma)} \int_{E_1(x)} (x-t)^{1-\gamma} \\ &\quad \times F\left[-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-h}\right] \frac{\partial}{\partial t} f(t)\, dt, \end{aligned} \tag{7.12}$$

where c_k *are given in* (7.9).

PROOF. By using the relations (7.10) and (6.19), we have

$$\begin{aligned} f_{E_1}^{\gamma,\alpha,\beta}(x) &= \sum_{k=1}^{n} \frac{c_k}{(n-k)!} \left(I_{E_1}^{1-\gamma,-\alpha,1+\beta-\gamma}(1\cdot t)^{n-k}\right)(x) + \frac{(x-h)^\alpha}{\Gamma(1-\gamma)} \int_{E_1(x)} (x-t)^{-\gamma} \\ &\quad \times F\left[-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-h}\right] \int_{E_1(t)} \frac{\partial}{\partial \tau} f(\tau)\, d\tau \\ &= I_1(x) + I_2(x). \end{aligned} \tag{7.13}$$

Applying Lemma 13, we find for the first term that

$$\begin{aligned} I_1(x) = (x-h)^\alpha \sum_{k=0}^{n-1} &\frac{(1\cdot x)^{n+k-|\gamma|} c_k}{\Gamma(n+k-|\gamma|+1)} \\ &\times F_B\left(-\alpha, 1+\beta-\gamma; n+k-|\gamma|+1; \frac{1\cdot x}{x-h}\right). \end{aligned} \tag{7.14}$$

Changing the order of integration in the second term $I_2(x)$ and using (2.6) and (2.14), we obtain

$$\begin{aligned} I_2(x) &= \frac{(x-h)^\alpha}{\Gamma(1-\gamma)} \int_{E_1(x)} \frac{\partial}{\partial \tau} f(\tau)\, d\tau \\ &\quad \times \int_{\tau \leqq t \leqq x} (x-t)^{-\gamma} F\left[-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-h}\right] dt \\ &= \frac{(x-h)^\alpha}{\Gamma(2-\gamma)} \int_{E_1(x)} (x-t)^{1-\gamma} F\left[-\alpha, 1+\beta-\gamma; 2-\gamma; \frac{x-t}{x-h}\right] \frac{\partial f(t)}{\partial t}\, dt, \end{aligned} \tag{7.15}$$

and (7.12) follows from (7.13) to (7.15).

Applying Lemma 15 and (6.21), after differentiation of (7.12), we obtain the following result similar to Theorem 7 and concerning the solvability of (6.18).

THEOREM 16. *Let* $\boldsymbol{x}, \boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^n$ *be such that the conditions in* (7.11) *are satisfied. If* $f(\boldsymbol{x}) \in I_{E_1,0}(L_1)$, $f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x})$ *given by* (7.12) *belongs to* $I_{E_1}(L_1)$ *and satisfies the conditions* (6.20), *then the Abel-type model multidimensional integral equation* (6.18) *is solvable in* $L_1(E_1(\boldsymbol{b}))$ *and its solution* (6.21) *can be represented in the form*:

$$\varphi(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{h})^{\boldsymbol{\alpha}} \sum_{k=0}^{n-1}\sum_{m=0}^{n} \left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(m)} \frac{(\boldsymbol{1}\cdot\boldsymbol{x})^{k+m-|\boldsymbol{\gamma}|}}{\Gamma(k+m-|\boldsymbol{\gamma}|+1)} \times F_B\left(-\boldsymbol{\alpha}, \boldsymbol{1}+\boldsymbol{\beta}-\boldsymbol{\gamma}; k+m-|\boldsymbol{\gamma}|+1; \frac{\boldsymbol{1}\cdot\boldsymbol{x}}{\boldsymbol{x}-\boldsymbol{h}}\right), \tag{7.16}$$

where

$$\left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(0)} = 1, \quad \left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(m)} = \sum_{\substack{j_1,j_2,\dots,j_m=1\\ j_k\neq j_l\ (k,l=1,\dots,m)}}^{n} \frac{(\alpha_{j_1}-i_{j_1})\cdots(\alpha_{j_m}-i_{j_m})}{(x_{j_1}-h_{j_1})\cdots(x_{j_m}-h_{j_m})} \tag{7.17}$$

for $m = 1, 2, \dots, n$. *In particular*

$$\begin{aligned}
\left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(1)} &= \sum_{j=1}^{n} \frac{\alpha_j - i_j}{x_j - h_j} = \frac{\alpha_1-i_1}{x_1-h_1} + \cdots + \frac{\alpha_n - i_n}{x_n-h_n},\\
\left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(2)} &= \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n} \frac{(\alpha_{j_1}-i_{j_1})(\alpha_{j_2}-i_{j_2})}{(x_{j_1}-h_{j_1})(x_{j_2}-h_{j_2})}\\
&= \frac{(\alpha_1-i_1)(\alpha_2-i_2)}{(x_1-h_1)(x_2-h_2)} + \cdots + \frac{(\alpha_1-i_1)(\alpha_n-i_n)}{(x_1-h_1)(x_n-h_n)}\\
&\quad + \frac{(\alpha_2-i_2)(\alpha_3-i_3)}{(x_2-h_2)(x_3-h_3)} + \cdots + \frac{(\alpha_2-i_2)(\alpha_n-i_n)}{(x_2-h_2)(x_n-h_n)}\\
&\quad + \cdots + \frac{(\alpha_{n-1}-i_{n-1})(\alpha_n-i_n)}{(x_{n-1}-h_{n-1})(x_n-h_n)},\\
&\dots,\\
\left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(n-1)} &= \frac{(\alpha_1-i_1)\cdots(\alpha_{n-1}-i_{n-1})}{(x_1-h_1)\cdots(x_{n-1}-h_{n-1})} + \cdots + \frac{(\alpha_2-i_2)\cdots(\alpha_n-i_n)}{(x_2-h_2)\cdots(x_n-h_n)},\\
\left(\frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}}\right)_{(n)} &= \frac{\boldsymbol{\alpha}-\boldsymbol{i}}{\boldsymbol{x}-\boldsymbol{h}} = \frac{(\alpha_1-i_1)\cdots(\alpha_n-i_n)}{(x_1-h_1)\cdots(x_n-h_n)}.
\end{aligned} \tag{7.18}$$

REMARK 7. In Theorem 16 we stated sufficient conditions for the solvability of (6.18) in $L_1(E_1)$ *via* the auxiliary function $f_{E_1}^{\gamma,\alpha,\beta}(\boldsymbol{x})$ given by (6.19) and found another

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representation (7.16) for the solution of (6.18). It is an open problem to find the sufficient conditions in terms of the function $f(x)$ itself.

8. Generalized pyramidal fractional calculus operators

The results in Section 6 lead to the definitions of generalized pyramidal fractional integral and differential operators. The former is introduced by (6.1).

DEFINITION 17. Let $A_{c,r}(\boldsymbol{b})$ be a nonempty pyramid (2.12) in $\mathbb{R}^n$ and let $\boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ and $\boldsymbol{\gamma} \in \mathbb{R}^n_+$. For $\boldsymbol{x} \in A_{c,r}(\boldsymbol{b})$ the generalized pyramidal fractional integral operator is defined by (6.1):

$$\left(I^{\gamma,\alpha,\beta}_{A_{c,r}}\varphi\right)(x) = \frac{(A\cdot(x-h))^{-\alpha}}{\Gamma(\gamma)} \times \int_{A_{c,r}(x)} (A\cdot(x-t)r)^{\gamma-1} F\left[\alpha,\beta;\gamma;\frac{A\cdot(x-t)}{A\cdot(x-h)}\right]\varphi(t)\,dt. \tag{8.1}$$

In particular, if $E_1(\boldsymbol{b})$ is a model pyramid (2.13), for $\boldsymbol{x} \in E_1(\boldsymbol{b})$ the model generalized pyramidal fractional integral operator is defined by

$$\left(I^{\gamma,\alpha,\beta}_{E_1}\varphi\right)(x) = \frac{(x-h)^{-\alpha}}{\Gamma(\gamma)}\int_{E_1(x)}(x-t)^{\gamma-1}F\left[\alpha,\beta;\gamma;\frac{x-t}{x-h}\right]\varphi(t)\,dt. \tag{8.2}$$

According to (6.11) for $\boldsymbol{\gamma} \in \mathbb{R}^n_+$ $(\boldsymbol{0} < \boldsymbol{\gamma} < \boldsymbol{1})$, the corresponding generalized fractional differential operator, inverse to (8.1), is defined by

$$\begin{aligned}\left(D^{\gamma,\alpha,\beta}_{A_{c,r}}f\right)(x) &= \prod_{k=1}^{n}\left(\sum_{j=1}^{n}\tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)\left(I^{1-\gamma,-\alpha,1+\beta-\gamma}_{A_{c,r}}f\right)(x)\\ &\equiv \frac{1}{\Gamma(1-\gamma)}\prod_{k=1}^{n}\left(\sum_{j=1}^{n}\tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)\left\{(A\cdot(x-h))^{\alpha}\int_{A_{c,r}(x)}(A\cdot(x-t))^{-\gamma}\right.\\ &\quad\times\left. F\left[-\alpha,1+\beta-\gamma;1-\gamma;\frac{A\cdot(x-t)}{A\cdot(x-h)}\right]f(t)\,dt\right\},\end{aligned} \tag{8.3}$$

where $\tilde{a}_{jk}$ are elements of the inverse matrix A^{-1}. In particular, if $E_1(\boldsymbol{b})$ is a model pyramid (2.13), for $x \in E_1(\boldsymbol{b})$ the model generalized pyramidal fractional differential operator is defined by (6.21):

$$\begin{aligned}\left(D^{\gamma,\alpha,\beta}_{E_1}f\right)(x) &= \frac{\partial}{\partial x}\left(I^{1-\gamma,-\alpha,1+\beta-\gamma}_{E_1}f\right)(x)\\ &\equiv \frac{1}{\Gamma(1-\gamma)}\frac{\partial}{\partial x}\left((x-h)^{\alpha}\int_{E_1(x)}(x-t)^{-\gamma}\right.\end{aligned}$$

$$\times F\left[-\alpha, 1+\beta-\gamma; 1-\gamma; \frac{x-t}{x-h}\right] f(t)\, dt\bigg). \tag{8.4}$$

In order to extend this definition to any $\gamma \in \mathbb{R}^n_+$, we prove the following statement, which is an analogue of a semigroup property for the pyramidal analogues of the Riemann-Liouville mixed fractional integrals (see [26, (28.91)]).

THEOREM 18. *Let $A_{c,r}(b)$ be a nonempty pyramid (2.12) in $\mathbb{R}^n$ and let $h, \alpha, \beta, \eta \in \mathbb{R}^n$ and $\gamma, \xi \in \mathbb{R}^n_+$. Then, for $\varphi(x) \in L_1(A_{c,r}(b))$,*

$$I^{\gamma,\alpha,\beta}_{A_{c,r}} I^{\xi,\eta,\beta-\gamma}_{A_{c,r}} \varphi = I^{\gamma+\xi,\alpha+\eta,\beta}_{A_{c,r}} \varphi. \tag{8.5}$$

In particular, if $E_1(b)$ is a model pyramid (2.13), then

$$I^{\gamma,\alpha,\beta}_{E_1} I^{\xi,\eta,\beta-\gamma}_{E_1} \varphi = I^{\gamma+\xi,\alpha+\eta,\beta}_{E_1} \varphi \tag{8.6}$$

for $x \in E_1(b)$.

PROOF. We apply arguments similar to those in Section 6. By (8.1), after changing the order of integration, we have

$$\begin{aligned}
&\left(I^{\gamma,\alpha,\beta}_{A_{c,r}} I^{\xi,\eta,\beta-\gamma}_{A_{c,r}} \varphi\right)(x) \\
&\quad = \frac{(A\cdot(x-h))^{-\alpha}}{\Gamma(\gamma)\Gamma(\xi)} \int_{A_{c,r}(x)} \varphi(\tau)\, d\tau \int_{\sigma(x,\tau)} (A\cdot(x-t))^{\gamma-1} \\
&\qquad \times (A\cdot(t-\tau))^{\xi-1} \cdot (A\cdot(t-h))^{-\eta} \\
&\qquad \times F\left[\alpha, \beta; \gamma; \frac{A\cdot(x-t)}{A\cdot(x-h)}\right] F\left[\eta, \beta-\gamma; \xi; \frac{A\cdot(t-\tau)}{A\cdot(t-h)}\right] dt.
\end{aligned} \tag{8.7}$$

Making the following change of variables:

$$s_j = \frac{a_j\cdot(x-t)}{a_j\cdot(x-\tau)}, \quad a_j = (a_{j1}, \dots, a_{jn}) \qquad (j = 1, \dots, n)$$

in the inner integral and using (2.7) and (2.14), we obtain

$$\begin{aligned}
&\left(I^{\gamma,\alpha,\beta}_{A_{c,r}} I^{\xi,\eta,\beta-\gamma}_{A_{c,r}} \varphi\right)(x) \\
&\quad = \frac{(A\cdot(x-h))^{-\alpha}}{\Gamma(\gamma)\Gamma(\xi)} \int_{A_{c,r}(x)} \prod_{j=1}^{n} (a_j\cdot(x-h))^{-\eta}(a_j\cdot(x-\tau))^{\gamma+\xi-1}\varphi(\tau)\, d\tau \\
&\qquad \times \prod_{j=1}^{n} \int_0^1 \left\{ s_j^{\gamma_j-1}(1-s_j)^{\xi_j-1}\left[1 - s_j\frac{a_j\cdot(x-\tau)}{a_j\cdot(x-h)}\right]^{-\eta_j}\right.
\end{aligned}$$

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$$\times F\left(\alpha_j, \beta_j; \gamma_j; s_j \frac{a_j \cdot (x-\tau)}{a_j \cdot (x-h)}\right)$$
$$\times F\left(\eta_j, \beta_j - \gamma_j; \xi_j; \frac{(1-s_j)\{a_j \cdot (x-\tau)/a_j \cdot (x-h)\}}{1 - s_j\{a_j \cdot (x-\tau)/a_j \cdot (x-h)\}}\right) ds_j\Bigg\}$$
$$= \frac{(A \cdot (x-h))^{-\alpha-\eta}}{\Gamma(\gamma+\xi)}$$
$$\times \int_{A_{c,r}(x)} (A \cdot (x-\tau))^{\gamma+\xi-1} F\left[\alpha+\eta, \beta; \gamma+\xi; \frac{A \cdot (x-t)}{A \cdot (x-h)}\right] \varphi(\tau)\, d\tau,$$

and (8.5) is proved.

The following statement gives the solution of the Abel-type hypergeometric integral equation (6.1) with any $\gamma > 0$.

THEOREM 19. *Let* $\alpha, \beta, h \in \mathbb{R}^n$, $\gamma \in \mathbb{R}^n_+$ *and* $m = [\gamma] + 1 = (m_1, \dots, m_n)$. *If the Abel-type hypergeometric integral equation* (6.1) *is solvable in* $L_1(A_{c,r}(b))$, *then its solution* $\varphi(x)$ *is given by*

$$\varphi(x) = \left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} \tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)^{m_k}\right]\left(I_{A_{c,r}}^{m-\gamma,-\alpha,\beta-\gamma+m} f\right)(x)$$
$$\equiv \frac{1}{\Gamma(m-\gamma)}\left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} \tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)^{m_k}\right]\Bigg\{(A \cdot (x-h))^{\alpha}\int_{A_{c,r}(x)} (A \cdot (x-t))^{m-\gamma-1}$$
$$\times F\left[-\alpha, m+\beta-\gamma; m-\gamma; \frac{A \cdot (x-t)}{A \cdot (x-h)}\right] f(t)\, dt\Bigg\}. \tag{8.8}$$

PROOF. Applying the operator $I_{A_{c,r}}^{m-\gamma,-\alpha,m+\beta-\gamma}$ to both sides of the relation $I_{A_{c,r}}^{\gamma,\alpha,\beta}\varphi = f$ and using (8.5), we have

$$\frac{1}{\Gamma(m)}\int_{A_{c,r}} (A \cdot (x-t))^{m-1}\varphi(t)\, dt = \left(I_{A_{c,r}}^{m-\gamma,-\alpha,m+\beta-\gamma} f\right)(x).$$

Making the change of variables (6.5), we rewrite this relation in the form:

$$\frac{1}{\Gamma(m)}\int_{E_1(y)} (y-\tau)^{m-1}\psi(\tau)\, d\tau = g(y),$$

where

$$\psi(\tau) = \varphi\left(A^{-1} \cdot \frac{\tau}{d} - \frac{r}{nc}\right),$$
$$g(y) = \left(I_{A_{c,r}}^{m-\gamma,-\alpha,m+\beta-\gamma} f\right)\left(A^{-1} \cdot \frac{y}{d} - \frac{r}{nc}\right)\prod_{k=1}^{n} d_k, \tag{8.9}$$

with $\boldsymbol{d} = A^{-1} \cdot \boldsymbol{c}$. Now, just as in (6.8), we have

$$\frac{1}{\Gamma(\boldsymbol{m})} \int_{-(y_1+\cdots+y_{n-1})}^{y_n} (y_n - \tau_n)^{m_n-1}\, d\tau_n \int_{-(y_1+\cdots+y_{n-2}+\tau_n)}^{y_{n-1}} (y_{n-1} - \tau_{n-1})^{m_{n-1}-1}\, d\tau_{n-1}$$
$$\times \cdots \times \int_{-(\tau_2+\cdots+\tau_n)}^{y_1} \psi(\boldsymbol{\tau})(y_1 - \tau_1)^{m_1-1}\, d\tau_1 = g(\boldsymbol{y}).$$

By successively differentiating with respect to $y_n, y_{n-1}, \ldots, y_1$, we obtain

$$\psi(\boldsymbol{y}) = \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{y}^{\boldsymbol{m}}} g(\boldsymbol{y}) \equiv \frac{\partial^{m_1}}{\partial y_1^{m_1}} \cdots \frac{\partial^{m_n}}{\partial y_n^{m_n}} g(\boldsymbol{y}), \tag{8.10}$$

which is equivalent to (8.8) in view of (8.9) and (6.10). This completes the proof of Theorem 19.

COROLLARY 20. *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{h} \in \mathbb{R}^n$, $\boldsymbol{\gamma} \in \mathbb{R}^n_+$ *and* $\boldsymbol{m} = [\boldsymbol{\gamma}] + \boldsymbol{1} = (m_1, \ldots, m_n)$. *If the model Abel-type hypergeometric integral equation* (6.18) *is solvable in* $L_1(E_1(\boldsymbol{b}))$, *then its solution* $\varphi(\boldsymbol{x})$ *is given by*

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} \left(I_{E_1}^{\boldsymbol{m}-\boldsymbol{\gamma}, -\boldsymbol{\alpha}, \boldsymbol{m}+\boldsymbol{\beta}-\boldsymbol{\gamma}} f \right)(\boldsymbol{x}) \\ &\equiv \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\gamma})} \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} \Bigg\{ (\boldsymbol{x}-\boldsymbol{h})^{\boldsymbol{\alpha}} \int_{E_1(\boldsymbol{x})} (\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{m}-\boldsymbol{\gamma}-\boldsymbol{1}} \\ &\qquad \times F\left[-\boldsymbol{\alpha}, \boldsymbol{m}+\boldsymbol{\beta}-\boldsymbol{\gamma}; \boldsymbol{m}-\boldsymbol{\gamma}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] f(\boldsymbol{t}) d\boldsymbol{t} \Bigg\}.\end{aligned} \tag{8.11}$$

Theorem 19 leads to the following definition of the generalized fractional differential operators, inverse to (8.1), for any $\boldsymbol{\gamma} \in \mathbb{R}^n_+$.

DEFINITION 21. Let $A_{c,r}(\boldsymbol{b})$ be a nonempty pyramid (2.12) in $\mathbb{R}^n$ and let $\boldsymbol{h}, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\gamma} \in \mathbb{R}^n_+$ and $\boldsymbol{m} = [\boldsymbol{\alpha}] + \boldsymbol{1} = (m_1, \cdots, m_n)$. For $\boldsymbol{x} \in A_{c,r}(\boldsymbol{b})$ the generalized pyramidal fractional differential operator is defined by (6.11):

$$\begin{aligned}\left(D_{A_{c,r}}^{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{x}) &= \left[\prod_{k=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j}\right)^{m_k}\right] \left(I_{A_{c,r}}^{\boldsymbol{m}-\boldsymbol{\gamma}, -\boldsymbol{\alpha}, \boldsymbol{m}+\boldsymbol{\beta}-\boldsymbol{\gamma}} f \right)(\boldsymbol{x}) \\ &\equiv \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\gamma})} \left[\prod_{k=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j}\right)^{m_k}\right] \\ &\qquad \times \Bigg\{ (A\cdot(\boldsymbol{x}-\boldsymbol{h}))^{\boldsymbol{\alpha}} \int_{A_{c,r}(\boldsymbol{x})} (A\cdot(\boldsymbol{x}-\boldsymbol{t}))^{\boldsymbol{m}-\boldsymbol{\gamma}-\boldsymbol{1}} \\ &\qquad \times F\left[-\boldsymbol{\alpha}, \boldsymbol{m}+\boldsymbol{\beta}-\boldsymbol{\gamma}; \boldsymbol{m}-\boldsymbol{\gamma}; \frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right] f(\boldsymbol{t})\, d\boldsymbol{t} \Bigg\},\end{aligned} \tag{8.12}$$

where $\tilde{a}_{jk}$ are elements of the inverse matrix A^{-1}.

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In particular, if $E_1(\boldsymbol{b})$ is a model pyramid (2.13), for $\boldsymbol{x} \in E_1(\boldsymbol{b})$ the model generalized pyramidal fractional differential operator is defined by (6.21):

$$\left(D_{E_1}^{\gamma,\alpha,\beta} f\right)(\boldsymbol{x}) = \frac{\partial^m}{\partial \boldsymbol{x}^m}\left(I_{E_1}^{m-\gamma,-\alpha,m+\beta-\gamma} f\right)(\boldsymbol{x})$$
$$\equiv \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\gamma})}\frac{\partial^m}{\partial \boldsymbol{x}^m}\left\{(\boldsymbol{x}-\boldsymbol{h})^{\alpha}\int_{E_1(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{t})^{m-\gamma-1}\right.$$
$$\left. \times F\left[-\boldsymbol{\alpha}, \boldsymbol{m}+\boldsymbol{\beta}-\boldsymbol{\gamma}; \boldsymbol{m}-\boldsymbol{\gamma}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] f(\boldsymbol{t})\, d\boldsymbol{t}\right\}. \tag{8.13}$$

The following statement is proved just as Theorem 18 was.

THEOREM 22. *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{h} \in \mathbb{R}^n$, $\boldsymbol{\gamma} \in \mathbb{R}_+^n$ *and* $\boldsymbol{m} = [\boldsymbol{\gamma}] + \mathbf{1} = (m_1, \dots, m_n)$. *If* $\varphi \in L_1(A_{c,r}(\boldsymbol{b}))$, *then*

$$D_{A_{c,r}}^{\gamma,\alpha,\beta} I_{A_{c,r}}^{\gamma,\alpha,\beta} f = f. \tag{8.14}$$

In particular, for $\varphi \in L_1(E_1(\boldsymbol{b}))$,

$$D_{E_1}^{\gamma,\alpha,\beta} I_{E_1}^{\gamma,\alpha,\beta} f = f. \tag{8.15}$$

It was noted in Remark 4 at the end of Section 5 that the one-dimensional generalized fractional integral and differential operators, given in (3.1) and (3.6) with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ being replaced by $\boldsymbol{\alpha}+\boldsymbol{\beta}$, $-\boldsymbol{\eta}$ and $\boldsymbol{\alpha}$, respectively, arise in solving certain boundary value problems for differential equations ([7, 16–19, 21–23, 25, 28, 31, 33]). Similarly, we define the generalized pyramidal fractional integral and differential operators.

DEFINITION 23. Let $A_{c,r}(\boldsymbol{b})$ be a nonempty pyramid (2.12) in $\mathbb{R}^n$ and let $\boldsymbol{h}, \boldsymbol{\beta}, \boldsymbol{\eta} \in \mathbb{R}^n$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$. For $\boldsymbol{x} \in A_{c,r}(\boldsymbol{b})$, the generalized pyramidal fractional integral operator is defined by

$$\left(J_{A_{c,r}}^{\alpha,\beta,\eta}\varphi\right)(\boldsymbol{x}) \equiv \left(I_{A_{c,r}}^{\alpha,\alpha+\beta,-\eta}\varphi\right)(\boldsymbol{x})$$
$$= \frac{(A\cdot(\boldsymbol{x}-\boldsymbol{h}))^{-\alpha-\beta}}{\Gamma(\boldsymbol{\alpha})}\int_{A_{c,r}(\boldsymbol{x})}(A\cdot(\boldsymbol{x}-\boldsymbol{t}))^{\alpha-1}$$
$$\times F\left[\boldsymbol{\alpha}+\boldsymbol{\beta}, -\boldsymbol{\eta}; \boldsymbol{\alpha}; \frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]\varphi(\boldsymbol{t})\, d\boldsymbol{t}. \tag{8.16}$$

In particular, for $x \in E_1(\boldsymbol{b})$ the model generalized pyramidal fractional integral operator has the form:

$$\left(J_{E_1}^{\alpha,\beta,\eta}\varphi\right)(\boldsymbol{x}) \equiv \left(I_{E_1}^{\alpha,\alpha+\beta,-\eta}\varphi\right)(\boldsymbol{x})$$
$$= \frac{(\boldsymbol{x}-\boldsymbol{h})^{-\alpha-\beta}}{\Gamma(\boldsymbol{\alpha})}\int_{E_1(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{t})^{\alpha-1} F\left[\boldsymbol{\alpha}+\boldsymbol{\beta}, -\boldsymbol{\eta}; \boldsymbol{\alpha}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right]\varphi(\boldsymbol{t})\, d\boldsymbol{t}. \tag{8.17}$$

DEFINITION 24. Let $A_{c,r}(\boldsymbol{b})$ be a nonempty pyramid (2.12) in $\mathbb{R}^n$ and let $\boldsymbol{h}, \boldsymbol{\beta}, \eta \in \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n_-$ and $\boldsymbol{m} = [-\boldsymbol{\alpha}] + \mathbf{1} = (m_1, \ldots, m_n)$. For $\boldsymbol{x} \in A_{c,r}(\boldsymbol{b})$, the generalized pyramidal fractional differential operator is defined by

$$\left(J^{\alpha,\beta,\eta}_{A_{c,r}} f\right)(\boldsymbol{x}) \equiv \left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} \tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)^{m_k}\right]\left(I^{m-\alpha,\beta-m,\eta-m}_{A_{c,r}} f\right)(\boldsymbol{x}), \tag{8.18}$$

where $\tilde{a}_{jk}$ are elements of the inverse matrix A^{-1}.

In particular, for $\boldsymbol{x} \in E_1(\boldsymbol{b})$ the model generalized pyramidal fractional differential operator has the form:

$$\left(J^{\alpha,\beta,\eta}_{E_1} f\right)(\boldsymbol{x}) = \frac{\partial^m}{\partial \boldsymbol{x}^m}\left(I^{m-\alpha,\beta-m,\eta-m}_{E_1} f\right)(\boldsymbol{x}). \tag{8.19}$$

The proofs of the following statements are similar to those of Theorems 18 and 19.

THEOREM 25. *Let $A_{c,r}(\boldsymbol{b})$ be a nonempty pyramid* (2.12) *in $\mathbb{R}^n$ and let $\boldsymbol{h}, \boldsymbol{\beta}, \eta, \boldsymbol{\delta} \in \mathbb{R}^n$ and $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbb{R}^n_+$. Then, for $\varphi(x) \in L_1(A_{c,r}(\boldsymbol{b}))$,*

$$J^{\alpha,\beta,\eta}_{A_{c,r}} J^{\gamma,\delta,\alpha+\eta}_{A_{c,r}} \varphi = J^{\alpha+\gamma,\beta+\delta,\eta}_{A_{c,r}} \varphi. \tag{8.20}$$

In particular, if $E_1(\boldsymbol{b})$ is a model pyramid (2.13), *then*

$$J^{\alpha,\beta,\eta}_{E_1} J^{\gamma,\delta,\alpha+\eta}_{E_1} \varphi = J^{\alpha+\gamma,\beta+\delta,\eta}_{E_1} \varphi, \tag{8.21}$$

for $x \in E_1(\boldsymbol{b})$.

THEOREM 26. *Let $\boldsymbol{h}, \boldsymbol{\beta}, \eta \in \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ and $\boldsymbol{m} = [\boldsymbol{\alpha}] + \mathbf{1}$. If the Abel-type hypergeometric integral equation*:

$$\frac{(A\cdot(\boldsymbol{x}-\boldsymbol{h}))^{-\alpha-\beta}}{\Gamma(\alpha)}\int_{A_{c,r}(\boldsymbol{x})}(A\cdot(\boldsymbol{x}-\boldsymbol{t}))^{\alpha-1} \times F\left[\alpha+\beta, -\eta; \alpha; \frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]\varphi(\boldsymbol{t})\,d\boldsymbol{t} = f(\boldsymbol{x}) \qquad \left(\boldsymbol{x} \in A_{c,r}(\boldsymbol{b})\right) \tag{8.22}$$

is solvable in $L_1\left(A_{c,r}(\boldsymbol{b})\right)$, then its solution $\varphi(\boldsymbol{x})$ is given by

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} \tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)^{m_k}\right]\left(J^{m-\alpha,-m-\beta,\alpha+\eta-m}_{A_{c,r}} f\right)(\boldsymbol{x}) \\ &\equiv \frac{1}{\Gamma(m-\alpha)}\left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} \tilde{a}_{jk}\frac{\partial}{\partial x_j}\right)^{m_k}\right]\Bigg\{(A\cdot(\boldsymbol{x}-\boldsymbol{h}))^{\alpha+\beta}\int_{A_{c,r}(\boldsymbol{x})}(A\cdot(\boldsymbol{x}-\boldsymbol{t}))^{m-\alpha-1} \\ &\quad \times F\left[-\alpha-\beta, m-\alpha-\eta; m-\alpha; \frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right]d\boldsymbol{t}\Bigg\}.\end{aligned} \tag{8.23}$$

https://doi.org/10.1017/S1446181100013080 Published online by Cambridge University Press

COROLLARY 27. *Let $\boldsymbol{h}, \boldsymbol{\beta}, \eta \in \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ and $\boldsymbol{m} = [\boldsymbol{\alpha}] + \mathbf{1}$. If the Abel-type hypergeometric integral equation*:

$$\frac{(\boldsymbol{x}-\boldsymbol{h})^{-\boldsymbol{\alpha}-\boldsymbol{\beta}}}{\Gamma(\boldsymbol{\alpha})} \int_{E_1(\boldsymbol{x})} (\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{\alpha}-1} F\left[\boldsymbol{\alpha}+\boldsymbol{\beta}, -\eta; \boldsymbol{\alpha}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] \varphi(\boldsymbol{t})\, d\boldsymbol{t} = f(\boldsymbol{x}), \tag{8.24}$$

where $\boldsymbol{x} \in E_1(\boldsymbol{b})$, is solvable in $L_1(E_1(\boldsymbol{b}))$, then its solution $\varphi(\boldsymbol{x})$ is given by

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\alpha})} \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} \left(J_{E_1}^{\boldsymbol{m}-\boldsymbol{\alpha}, -\boldsymbol{m}-\boldsymbol{\beta}, \boldsymbol{\alpha}+\eta-\boldsymbol{m}} f\right)(\boldsymbol{x}) \\ &\equiv \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\alpha})} \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} \Bigg\{(\boldsymbol{x}-\boldsymbol{h})^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \int_{E_1(\boldsymbol{x})} (\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{m}-\boldsymbol{\alpha}-1} \\ &\quad \times F\left[-\boldsymbol{\alpha}-\boldsymbol{\beta}, \boldsymbol{m}-\boldsymbol{\alpha}-\eta; \boldsymbol{m}-\boldsymbol{\alpha}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] f(\boldsymbol{t})\, d\boldsymbol{t}\Bigg\}. \end{aligned} \tag{8.25}$$

On the basis of Theorem 26, we introduce the generalized pyramidal fractional derivative as an inversion of the corresponding generalized pyramidal fractional integral.

DEFINITION 28. Let $A_{c,r}(\boldsymbol{b})$ be a nonempty pyramid (2.12) in $\mathbb{R}^n$ and let $\boldsymbol{h}, \boldsymbol{\beta}, \eta \in \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ and $\boldsymbol{m} = [\boldsymbol{\alpha}]+\mathbf{1} = (m_1, \dots, m_n)$. For $x \in A_{c,r}(\boldsymbol{b})$ the generalized pyramidal fractional differential operator is defined by

$$\begin{aligned}\left(\mathscr{D}_{A_{c,r}}^{\boldsymbol{\alpha},\boldsymbol{\beta},\eta} f\right)(\boldsymbol{x}) &= \left[\prod_{k=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j}\right)^{m_k}\right] \left(J_{A_{c,r}}^{\boldsymbol{m}-\boldsymbol{\alpha}, -\boldsymbol{m}-\boldsymbol{\beta}, \boldsymbol{\alpha}+\eta-\boldsymbol{m}} f\right)(\boldsymbol{x}) \\ &\equiv \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\alpha})} \left[\prod_{k=1}^{n} \left(\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j}\right)^{m_k}\right] \\ &\quad \times \Bigg\{(A\cdot(\boldsymbol{x}-\boldsymbol{h}))^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \int_{A_{c,r}(\boldsymbol{x})} (A\cdot(\boldsymbol{x}-\boldsymbol{t}))^{\boldsymbol{m}-\boldsymbol{\alpha}-1} \\ &\quad \times F\left[-\boldsymbol{\alpha}-\boldsymbol{\beta}, \boldsymbol{m}-\boldsymbol{\alpha}-\eta; \boldsymbol{m}-\boldsymbol{\alpha}; \frac{A\cdot(\boldsymbol{x}-\boldsymbol{t})}{A\cdot(\boldsymbol{x}-\boldsymbol{h})}\right] f(\boldsymbol{t})\, d\boldsymbol{t}\Bigg\}. \end{aligned} \tag{8.26}$$

In particular, for $\boldsymbol{x} \in E_1(\boldsymbol{b})$ the model generalized pyramidal fractional differential operator has the form:

$$\begin{aligned}\left(\mathscr{D}_{E_1}^{\boldsymbol{\alpha},\boldsymbol{\beta},\eta} f\right)(\boldsymbol{x}) &= \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} \left(J_{E_1}^{\boldsymbol{m}-\boldsymbol{\alpha}, -\boldsymbol{m}-\boldsymbol{\beta}, \boldsymbol{\alpha}+\eta-\boldsymbol{m}} f\right)(\boldsymbol{x}) \\ &\equiv \frac{1}{\Gamma(\boldsymbol{m}-\boldsymbol{\alpha})} \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} \Bigg\{(\boldsymbol{x}-\boldsymbol{h})^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \int_{E_1(\boldsymbol{x})} (\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{m}-\boldsymbol{\alpha}-1} \\ &\quad \times F\left[-\boldsymbol{\alpha}-\boldsymbol{\beta}, \boldsymbol{m}-\boldsymbol{\alpha}-\eta; \boldsymbol{m}-\boldsymbol{\alpha}; \frac{\boldsymbol{x}-\boldsymbol{t}}{\boldsymbol{x}-\boldsymbol{h}}\right] f(\boldsymbol{t})\, d\boldsymbol{t}\Bigg\}. \end{aligned} \tag{8.27}$$

The proof of the following result is similar to that of Theorem 22.

THEOREM 29. *Let* $\boldsymbol{h}, \boldsymbol{\beta}, \boldsymbol{\eta} \in \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ *and* $\boldsymbol{m} = [\boldsymbol{\alpha}] + \mathbf{1}$. *If* $\varphi \in L_1(A_{c,r}(\boldsymbol{b}))$, *then*

$$\mathscr{D}^{\alpha,\beta,\eta}_{A_{c,r}} J^{\alpha,\beta,\eta}_{A_{c,r}} f = f. \tag{8.28}$$

In particular, for $\varphi \in L_1\,(E_1(\boldsymbol{b}))$,

$$\mathscr{D}^{\alpha,\beta,\eta}_{E_1} J^{\alpha,\beta,\eta}_{E_1} f = f. \tag{8.29}$$

Acknowledgements

The present investigation was initiated during the second-named author's visit to Fukuoka University in October 1997. This work was supported, in part, by the Department of Science and Technology (Government of India) under Grant DST/MS/PM-001/93, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353, by the Belarusian Fundamental Scientific Research Fund, and by Grant-in-Aid for Scientific Research (Ministry of Education of Japan) under Project 09640238.

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