Canad. Math. Bull.Vol. **34** (3), 1991 pp. 295–300

ON RINGS WITH ENGEL CYCLES

H. E. BELL AND A. A. KLEIN

ABSTRACT. A ring R is called an EC-ring if for each $x, y \in R$, there exist distinct positive integers m, n such that the extended commutators $[x, y]_m$ and $[x, y]_n$ are equal. We show that in certain EC-rings, the commutator ideal is periodic; we establish commutativity of arbitrary EC-domains; we prove that a ring R is commutative if for each $x, y \in R$, there exists $n > 1$ for which $[x, y] = [x, y]_n$.

Let R denote an arbitrary ring. For each $x, y \in R$ define extended commutators $[x, y]_k$ as follows: let $[x, y]_1$ be the ordinary commutator $xy - yx$, and for $k > 1$ extend the notion inductively by taking $[x, y]_k = \big[[x, y]_{k-1}, y\big]$. We say that R satisfies an Engel condition (or alternatively, R is an E-ring) if for each $x, y \in R$ there exists a positive integer r, depending on x and y, such that $[x, y]_r = 0$. We call R an Engel-cycle ring (EC-ring) if for each $x, y \in R$ there exist distinct positive integers r and s for which $[x, y]_r = [x, y]_s$. In the event that we can choose r (resp. r and s) independent of x and y, we call R an E^*-ring or EC^*-ring respectively.

Prompted by questions from Luise-Charlotte Kappe and Rolf Brandl, we explore commutativity in EC-rings and EC^*-rings, of which E-rings and E^*-rings are special cases. It has been known for some time that E^*-rings have nil commutator ideal [8]; however, it is apparently still an open question as to whether general E-rings have the same property—a situation which is an impediment in our study of EC-rings. Moreover, all finite rings are EC-rings, so the commutativity theory of EC-rings cannot in general be better than that of finite rings. As we shall see, the class of periodic rings—a class which includes all finite rings—plays a central role in our study.

Throughout the paper, the center of the ring R will be denoted by Z or $Z(R)$, and the set of nilpotent elements by N or $N(R)$. The symbols $\mathcal{C}(R)$, $\mathcal{N}(R)$ and $\mathcal{J}(R)$ will denote respectively the commutator ideal, the nil radical, and the Jacobson radical. The symbols $\mathbb{Z}$ and $\mathbb{Z}_p$ will stand for the ring of integers and the ring of integers mod p.

1. **Remarks on periodic and algebraic ideals.** Define a ring R to be periodic if for each $x \in R$ there exist distinct positive integers m and n such that $x^m = x^n$. Periodic rings entered the arena of commutativity theorems early—with Wedderburn's theorem on finite division rings; and various authors have investigated their special commutativity properties. One of the most useful results on periodic rings is one due to Chacron ([6], [2, Theorem 1]):

The first author was supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. A 3961.

Received by the editors October 18, 1989.

AMS subject classification: 16A70, 16A15, 16A38.

LEMMA 1. *Suppose that for each x in the ring R, there exists a positive integer* $n = n(x)$ *and a polynomial* $p(X) = p_x(X) \in \mathbb{Z}[X]$ *such that* $x^n = x^{n+1}p(X)$. *Then R is periodic.*

As an immediate application, we establish the existence of a maximal periodic ideal.

LEMMA 2. *Let R be any ring. Then R contains a maximal periodic ideal* $\mathcal{P}(R)$, *and* $\frac{R}{\mathcal{P}(R)}$ *has no nontrivial periodic ideals.*

PROOF. Let $\mathcal{P}(R)$ be the sum of all periodic ideals of R, which is obviously an ideal. To show it is periodic, we need only show that the sum of two periodic ideals I_1 and I_2 is again periodic. Since $\frac{I_1+I_2}{I_1} \cong \frac{I_2}{I_1 \cap I_2}$, we see that $\frac{I_1+I_2}{I_1}$ is periodic; hence for each $x \in I_1+I_2$, there exist distinct n, m such that $x^n - x^m \in I_1$. Thus, there exist distinct k and j for which $(x^n - x^m)^j = (x^n - x^m)^k$; and $I_1 + I_2$ is periodic by Lemma 1. Another easy application of Lemma 1 shows that $\frac{R}{\mathcal{P}(R)}$ has no nonzero periodic ideals.

In [4] Bergen and Herstein discuss the related notion of algebraic ideals. They assume that R is an algebra over a field F, with *algebraic* having its usual meaning. They define the algebraic hypercenter $A(R)$ to be the set of all $a \in R$ such that for each $x \in R$, there exists $p(X) \in F[X]$, of positive degree and depending on a and x, for which $ap(x) = p(x)a$. One of their principal results is

LEMMA 3 [4, THEOREM 1.6]. *If R is an algebra over a field and has no nonzero algebraic ideals, then* $A(R) = Z(R)$.

This lemma is of interest to us because any ring of prime characteristic p may be regarded as an algebra over $\mathbb{Z}_p$; and in this case, a simple application of Lemma 1 shows that an ideal is periodic if and only if it is algebraic.

2. A basic result on *EC*-rings.

The standard measure of near-commutativity is that $\mathcal{C}(R)$ is nil. In the case of *EC*-rings, we cannot hope to prove this, since it does not hold for all finite rings. However, for a significant class of *EC*-rings, we can establish that $\mathcal{C}(R)$ is periodic.

THEOREM 1. *If R is any EC-ring for which* $(R, +)$ *is a torsion group, then* $\mathcal{C}(R)$ *is periodic.*

Before beginning the proof, we single out some computational details in a lemma. Part (a) is well-known; part (b) is clear.

LEMMA 4. *(a) Let R be any ring of prime charcteristic p. Then if* $m = p^k$, $[x, y]_m = [x, y^m]$ *for all* $x, y \in R$.

(b) If $[x, y]_r = [x, y]_{r+d}$ *for* $r, d > 0$, *then* $[x, y]_m = [x, y]_n$ *for all* m, n *with* $r \leq m < n$ *and* $n \equiv m \pmod{d}$. *In particular, if R is an EC-ring, then for any* x_1, x_2, y_1, $y_2 \in R$, *there is a single pair* m, n *of positive integers for which* $[x_1, y_1]_m = [x_1, y_1]_n$ *and* $[x_2, y_2]_m = [x_2, y_2]_n$.

To avoid further interruption, we state an additional lemma, which will be used in this section and in subsequent sections.

LEMMA 5 [7,8]. *(a) If R is a (Jacobson) semisimple E-ring, then R is commutative.*

(b) If R is any E^*-ring, then $C(R)$ is nil.

PROOF OF THEOREM 1. Consider $\bar{R} = \frac{R}{\mathcal{N}(R)}$, and write it as the direct sum of its primary components $\overline{R_i}$. Since $\bar{R}$ has no nontrivial nil ideals, we have $p\overline{R_i} = \{0\}$, where p is the prime associated with $\overline{R_i}$; hence $\overline{R_i}$ is an algebra over $\mathbb{Z}_p$. Let $R_i^* = \frac{\overline{R_i}}{\mathcal{P}(\overline{R_i})}$, which has no nontrivial periodic ideals, hence no nontrivial algebraic ideals.

Now consider $x, y \in R_i^*$, and choose $r, d > 0$ such that $[x,y]_r = [x,y]_{r+d}$. Since there are only finitely many congruence classes mod d, there must be two distinct powers of p, say p^α and p^β, both at least r and congruent mod d. By Lemma 4(a), we have $[x, y^{p^\alpha}] = [x, y^{p^\beta}]$—i.e. $[x, y^{p^\beta} - y^{p^\alpha}] = 0$. Thus $x \in A(R_i^*)$ for each $x \in R_i^*$ and by Lemma 3, R_i^* is commutative. Thus $C(\overline{R_i}) \subseteq \mathcal{P}(\overline{R_i})$, so that $C(\overline{R_i})$ is periodic. Since each element of $C(\bar{R})$ has nonzero components in only finitely many of the $\overline{R_i}$, it follows that $C(\bar{R})$ is periodic. We now have $C(R)$ periodic mod $\mathcal{N}(R)$, and an application of Lemma 1 shows that $C(R)$ is periodic.

One consequence of this result is

THEOREM 2. *If R is any EC^*-ring, then $C(R)$ is periodic.*

PROOF. Let R satisfy the identity

$$[x,y]_r = [x,y]_s, \quad s > r. \tag{1}$$

Replacing y by $2y$, we obtain the identity

$$(2^s - 2^r)[x,y]_r = 0. \tag{2}$$

Suppose temporarily that R has no nonzero nil ideals. Then there exists a family $\{ P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$ and R is a subdirect product of the factor rings $R_\alpha = \frac{R}{P_\alpha}$, each of which is prime with no nonzero nil ideals and satisfies (1) and (2). If char R_α is 0 or a prime not dividing $2^s - 2^r$, then R_α satisfies the identity $[x,y]_r = 0$—i.e. R_α is an E^*-ring; and R_α is therefore commutative by Lemma 5(b). Note that there are only finitely many primes dividing $2^s - 2^r$, which we call exceptional.

Let $\Lambda_1 = \{\alpha \in \Lambda \mid \text{char}\, R_\alpha \text{ is not exceptional}\}$, and $\Lambda_2 = \Lambda \setminus \Lambda_1$. Define $P_1 = \bigcap_{\alpha\in\Lambda_1} P_\alpha$ and $P_2 = \bigcap_{\alpha\in\Lambda_2} P_\alpha$. Then $P_1 \cap P_2 = \{0\}$, so R is a subdirect product of $R_1 = \frac{R}{P_1}$ and $R_2 = \frac{R}{P_2}$. Now the argument already given shows that R_1 is commutative; and since there are only finitely many exceptional primes, $(R_2, +)$ is a torsion group. Since R_2 clearly satisfies (1), Theorem 1 shows that $C(R_2)$ is periodic; and it follows at once that $C(R)$ is periodic.

Returning to the case of a general ring R satisfying (1), we have $C\left(\frac{R}{\mathcal{N}(R)}\right)$ periodic, so that $C(R)$ is periodic mod $\mathcal{N}(R)$. Applying Lemma 1 again, we conclude that $C(R)$ is periodic.

It is interesting to note that while EC-rings have seldom been studied in the past, groups with Engel cycles have been studied by various authors for some time. The literature contains theorems asserting that EC-groups with some additional finiteness condition have a particular structure—for example, a recent theorem of Brandl [5] asserts that

https://doi.org/10.4153/CMB-1991-048-x Published online by Cambridge University Press

if G is a finitely-generated soluble EC-group, then G is finite-by-nilpotent. Our Theorems 1 and 2 have a similar character; in each case there is a sort of finiteness hypothesis in addition to the basic assumption that R is an EC-ring, and the conclusion is that (in group-theory terminology) R is periodic-by-commutative.

3. ***EC*-domains and related rings.** Our major goal in this section is to prove the following theorem.

THEOREM 3. *If R is any EC-domain, then R is commutative.*

We dispose at once of the characteristic p case. Indeed, if we assume $\mathcal{C}(R) \neq \{0\}$, then by Theorem 1 $\mathcal{C}(R)$ is a periodic domain, which must be commutative by Jacobson's $a^n = a$ theorem; and this contradicts the fact that a domain with a nonzero commutative ideal must itself be commutative.

If R has characteristic 0, then for $x, y \in R$ choose r and s such that $[x,y]_r = [x,y]_s$ and $[x,2y]_r = [x,2y]_s$, this being possible by Lemma 4(b). It follows easily that $(2^s - 2^r)[x,y]_r = 0$, so that R is an E-ring. Thus, Theorem 3 will be proved once we prove the following theorem.

THEOREM 4. *Let R be any E-domain of characteristic 0. Then R is commutative.*

PROOF. If R does not have 1, we can embed it in an E-domain with 1. (If $Z \neq \{0\}$ localize at $Z \backslash \{0\}$; otherwise, use the Dorroh embedding.) Thus we assume that R has 1. Since semi-simple E-rings are commutative by Lemma 5, we have $[x,y] \in \mathcal{J}(R)$ for each $x, y \in R$; hence $1 + [x,y]$ is invertible for all $x, y \in R$.

Assume R is not commutative. Then by Lemma 5, R is not an E^*-domain; and we can find $x, y \in R$ and an integer $n \geq 3$ such that $[x,y]_n = 0 \neq [x,y]_{n-1}$. Taking $z = [x,y]_{n-2}$, we see that $[z,y]_2 = 0 \neq [z,y]$. Now since $n \geq 3$, z is a commutator, so $u = 1 + z$ is invertible; and we clearly have $[u,y]_2 = 0 \neq [u,y]$. Defining d to be the inner derivation $x \longrightarrow xy - yx$, we thus have $d^2(u) = 0 \neq d(u)$.

Now $0 = d(uu^{-1}) = ud(u^{-1}) + d(u)u^{-1}$, hence $d(u^{-1}) = -u^{-1}d(u)u^{-1}$. Using the fact that $d^2(u) = 0$, we can show in a straightforward way that $d^2(u^{-1}) = 2\big(u^{-1}d(u)\big)^2 u^{-1}$; and proceeding by induction, we get $d^n(u^{-1}) = (-1)^n n! \big(u^{-1}d(u)\big)^n u^{-1}$ for all positive integers n. Since $d(u) \neq 0$ and R is of characteristic 0, we see that $d^n(u^{-1}) \neq 0$ for all positive integers n—that is, $[u^{-1},y]_n \neq 0$ for all positive integers n. This of course contradicts the fact that R was an E-ring.

Since rings without nilpotent elements are subdirect products of domains, Theorem 3 yields the following useful corollary.

COROLLARY 5. *If R is an EC-ring with no nonzero nilpotent elements, then R is commutative.*

Another corollary, extending the known results on E-rings, is

https://doi.org/10.4153/CMB-1991-048-x Published online by Cambridge University Press

THEOREM 6. *If R is an E-ring with no nonzero nil right ideals, then R is commutative.*

PROOF. We show that R has no nonzero nilpotent elements. Let $u^2 = 0$, and for $x \in R$ choose $k = k(u,x)$ such that $[u, ux]_k = 0$. Then $(ux)^k u = 0$, and it follows that the right ideal generated by u is nil. Therefore, $u = 0$.

From Corollary 5, it is immediate that any EC-ring R satisfying a condition which forces N to be an ideal must have $\mathcal{C}(R)$ nil. For example, an EC-ring with $N \subseteq Z$ must have nil commutator ideal. In fact, we can get a better result, reminiscent of Theorem 1 of [1].

THEOREM 7. *If R is an EC-ring in which N is commutative, then $\mathcal{C}(R)$ is nil.*

PROOF. We show that N is an ideal. It is immediate that N is an additive subgroup of R; and we proceed to show by induction on k that if $u^k = 0$, then $(xu)^k = (ux)^k = 0$ for all $x \in R$. We shall require the well-known fact that

$$[x, y]_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} y^i x y^{n-i} \tag{3}$$

for all $x, y \in R$ and all positive integers n.

Suppose that $u^2 = 0$. For $x \in R$, we get r and s such that $[u, xu]_r = [u, xu]_s$; and this equality reduces at once to $u(xu)^r = u(xu)^s$. It follows that $(ux)^{r+1} = (ux)^{s+1}$; hence, there exists an integer j such that $e = (ux)^j$ is idempotent. Since $xe - exe \in N$, we have $[u, xe - exe] = 0$—that is,

$$u\big(x(ux)^j - (ux)^j x(ux)^j\big) = \big(x(ux)^j - (ux)^j x(ux)^j\big)u. \tag{4}$$

Multiplying on the right by u shows that $(ux)^{j+2} = (xu)^{j+2} = 0$. We now know that ux and xu are in N, hence commute with u; therefore $(ux)^2 = (xu)^2 = 0$ as required.

Now suppose our result holds for all y with $y^m = 0$, $m < k$; and suppose $u^k = 0$. For $x \in R$, choose distinct r and s greater than $k - 2$ such that $[u, xu]_r = [u, xu]_s$. By (3) we see that

$$u(xu)^r - u(xu)^s = \sum w_q, \tag{5}$$

where each w_q is a product of u's and x's with at least k u's, including two adjacent u's. Since each u^i, $i = 2, \ldots, k-1$, has $(u^i)^t = 0$ for some $t < k$, our inductive hypothesis allows us to rewrite each w_q as a product having u^k as a factor; thus, each $w_q = 0$, and (5) yields $(ux)^{r+1} = (ux)^{s+1}$. Again there exists j such that $(ux)^j = e$ is idempotent. Looking at (4) again and right-multiplying by ux, we see that there exist $v, w \in R$ for which

$$(ux)^{j+2} = u^2 v + w u^2 x.$$

Since the right side of this equation is in N by the inductive hypothesis, we conclude that ux and xu are in N, hence $[u, ux] = [u, xu] = 0$ and $(xu)^k = (ux)^k = 0$.

4. **A further commutativity theorem.** Theorem 4 of [3] asserts that if R has the property that for each $y \in R$ there exists $n = n(y) > 1$ for which $[x, y] = [x, y]_n$ for all $x \in R$, then R is commutative. We can now prove an extension of this result.

THEOREM 8. *Let R be a ring such that for each $x, y \in R$ there exists $n = n(x, y) > 1$ for which $[x, y] = [x, y]_n$. Then R is commutative.*

PROOF. As in [3], we use results of Streb [9] to reduce the problem to showing commutativity in the absence of nil ideals.

Suppose, then, that R has no nonzero nil ideals, and write R as a subdirect product of prime rings R_α, each with no nonzero nil ideals. Suppose first that R_α has characteristic 0. Then for $x, y \in R_\alpha$ choose a single $n > 1$ for which $[x, y] = [x, y]_n$ and $[x, 2y] = [x, 2y]_n$. As usual we obtain $(2^n - 2)[x, y] = 0$, hence $[x, y] = 0$.

Now consider the case of R_α with prime characteristic p. For $x \in R$ and $u \in N$, there exists $n > 1$ such that $[x, u] = [x, u]_n$. Using Lemma 4(b) and the pigeonhole principle, we get $k \in \{2, 3, \ldots, n\}$ for which there exist arbitrarily large powers of p congruent to $k \pmod{n-1}$; and invoking Lemma 4(a), we see that $[x, u]_k = 0$, hence $[x, u] = 0$. Thus, $N \subseteq Z$, so that N is an ideal, necessarily trivial; and commutativity follows by Corollary 5.

REFERENCES

1. H. E. Bell, *Some commutativity results for periodic rings*, Acta Math. Acad. Sci. Hungar. **28**(1976), 279–283.
2. ———, *On commutativity of periodic rings and near-rings*, Acta Math. Acad. Sci. Hungar. **36**(1980), 293–302.
3. H. E. Bell and L. C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar. **53**(1989), 339–346.
4. J. Bergen and I. N. Herstein, *The algebraic hypercenter and some applications*, J. Algebra **85**(1983), 217–242.
5. R. Brandl, *Infinite soluble groups with Engel cycles; a finiteness condition*, Math. Z. **182**(1983), 259–264.
6. M. Chacron, *On a theorem of Herstein*, Canad. J. Math. **21**(1969), 1348–1353.
7. I. N. Herstein, *Sugli anelli soddisfacenti ad una condizione di Engel*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)**32**(1962), 177–180.
8. ———, *A remark on rings and algebras*, Michigan Math. J. **10**(1963), 269–272.
9. W. Streb, *Über einen Satz von Herstein und Nakayama*, Red. Sem. Mat. Univ. Padova **64**(1981), 159–171.

Mathematics Department
Brock University
St. Catharines, Ontario L2S 3A1

Raymond and Beverly Sackler Faculty of Exact Sciences
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv
69978 Tel-Aviv
Israel

https://doi.org/10.4153/CMB-1991-048-x Published online by Cambridge University Press