Canad. Math. Bull. Vol. **54** (4), 2011 pp. 716–725
doi:10.4153/CMB-2011-033-6

Symplectic Lie–Rinehart–Jacobi Algebras and Contact Manifolds

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Abstract. We give a characterization of contact manifolds in terms of symplectic Lie–Rinehart–Jacobi algebras. We also give a sufficient condition for a Jacobi manifold to be a contact manifold.

1 Introduction

When E is a commutative graded algebra with unit 1_E over a commutative field K with characteristic zero, a K-linear map $\partial\colon E \to E$ is a differential operator of order ≤ 1 and degree r, $r \in \mathbb{Z}$, if ∂ is of degree r and satisfying

$$\partial(xy) = \partial(x)\cdot y + (-1)^{r\cdot|x|}x\cdot\partial(y) - (-1)^{r\cdot|x\cdot y|}x\cdot y\cdot\partial(1_E)$$

for homogeneous elements x, $y \in E$, where $|x|$ denotes the degree of x. If ∂ is a differential operator of order ≤ 1 and degree r, if ∂' is another differential operator of order ≤ 1 and degree r', then the bracket

$$[\partial, \partial'] = \partial\circ\partial' - (-1)^{r\cdot r'}\partial'\circ\partial\colon E \longrightarrow E$$

is a differential operator of order ≤ 1 and degree $r + r'$.

The pair (E, ∂) is a differential algebra if E is a commutative graded algebra with unit 1_E and $\partial\colon E \to E$ is a differential operator of order ≤ 1 and degree $+1$ such that $\partial^2 = 0$. The usual differential algebras correspond with the case $\partial(1_E) = 0$.

Let A be a commutative algebra with unit 1_A over a commutative field K with characteristic zero. We denote $\mathrm{Diff}_K(A)$, the A-module of differential operators of order ≤ 1 from A to A, *i.e.*, the A-module of K-linear endomorphisms $\varphi\colon A \to A$ such that for any $a, b \in A$, $\varphi(ab) = \varphi(a)\cdot b + a\cdot\varphi(b) - a\cdot b\cdot\varphi(1_A)$. A Lie–Rinehart algebra is a pair $(\mathcal{G}, \rho)$ where $\mathcal{G}$ is simultaneously an A-module and a K-Lie algebra, and $\rho\colon \mathcal{G} \longrightarrow \mathrm{Diff}_K(A)$ is simultaneously a morphism of A-modules and K-Lie algebras satisfying

$$[x, a\cdot y = [\rho(x)(a) - a\cdot\rho(x)(1_A)]\cdot y + a\cdot[x, y]$$

for any $a \in A$ and x , $y \in \mathcal{G}$ [3, 5].

Received by the editors October 7, 2008.
Published electronically March 8, 2011.
AMS subject classification: **13N05**, 53D05, 53D10.
Keywords: Lie–Rinehart algebras, differential operators, Jacobi manifolds, symplectic manifolds, contact manifolds.

Let $(\mathcal{G},\rho)$ be a Lie–Rinehart algebra and $\mathfrak{L}_{\text{sks}}(\mathcal{G},A)=\bigoplus_{p\in\mathbb{N}}\mathfrak{L}^p_{\text{sks}}(\mathcal{G},A)$, where $\mathfrak{L}^p_{\text{sks}}(\mathcal{G},A)$ is the module of skew-symmetric A-multilinear maps of degree p from $\mathcal{G}$ to A and finally $d_\rho\colon \mathfrak{L}_{\text{sks}}(\mathcal{G},A)\to\mathfrak{L}_{\text{sks}}(\mathcal{G},A)$ the cohomology operator associated with the representation ρ.

The pair $(\mathfrak{L}_{\text{sks}}(\mathcal{G},A),d_\rho)$ is a differential algebra [3, Proposition 18].

For any $x\in\mathcal{G}$, the map $i_x\colon \mathfrak{L}_{\text{sks}}(\mathcal{G},A)\to\mathfrak{L}_{\text{sks}}(\mathcal{G},A)$ defined by

$$(i_xf)(x_1,x_2,\dots,x_{p-1})=f(x,x_1,x_2,\dots,x_{p-1}),$$

for $x_1,x_2,\dots,x_{p-1}$ elements of $\mathcal{G}$ and for any $f\in\mathfrak{L}^p_{\text{sks}}(\mathcal{G},A)$ is a derivation of degree -1 [1]. The map

$$\theta_x=[i_x,d_\rho]\colon \mathfrak{L}_{\text{sks}}(\mathcal{G},A)\longrightarrow\mathfrak{L}_{\text{sks}}(\mathcal{G},A)$$

is a differential operator of order ≤ 1 and of degree zero satisfying, for any $y\in\mathcal{G}$, $a\in A$,

$$[\theta_x,i_y]=i_{[x,y]};\quad \theta_x\circ d_\rho=d_\rho\circ\theta_x;\quad \theta_xa=[\rho(x)](a).$$

For any $x\in\mathcal{G}$, the bracket that defines θ_x is the graded commutator.

A Lie–Rinehart–Jacobi algebra structure on a Lie–Rinehart algebra $(\mathcal{G},\rho)$ is defined by a skew-symmetric 2-form $\mu\colon\mathcal{G}\times\mathcal{G}\to A$ such that $d_\rho\mu=0$.

The triplet $(\mathcal{G},\rho,\mu)$ is a Lie–Rinehart–Jacobi algebra [3]. A Lie–Rinehart–Jacobi algebra $(\mathcal{G},\rho,\mu)$ is a Lie–Rinehart–Poisson algebra if $\rho(x)(1_A)=0$ for any $x\in\mathcal{G}$ [3].

A Lie–Rinehart–Jacobi algebra (a Lie–Rinehart–Poisson algebra, respectively) $(\mathcal{G},\rho,\mu)$ is said to be a symplectic Lie–Rinehart–Jacobi algebra (a symplectic Lie–Rinehart–Poisson algebra, respectively) if the skew-symmetric 2-form μ is nondegenerate [3], *i.e.*, the induced map $\mathcal{G}\to\mathcal{G}^*$, $x\mapsto i_x\mu$, is an isomorphism of A-modules where $\mathcal{G}^*$ is the A-module of linear forms on $\mathcal{G}$.

We recall that a commutative algebra A with unit 1_A over a commutative field K with characteristic zero is a Jacobi algebra if A is a K-Lie algebra, with bracket $\{\ ,\ \}$, such that for any $a\in A$, the inner derivation $\operatorname{ad}(a)\colon A\to A$, $b\mapsto\{a,b\}$, is a differential operator of order ≤ 1.

Let $(\mathcal{G},\rho,\mu)$ be a symplectic Lie–Rinehart–Jacobi algebra. For any $a\in A$, we denote x_a the unique element of $\mathcal{G}$ such that $i_{x_a}\mu=d_\rho a$.

We have $\theta_{x_a}\mu=0$.

For any $a\in A$ and for any $b\in A$, by letting $\{a,b\}=-\mu(x_a,x_b)$, we get

(i) $\{a,b\}=[\rho(x_a)](b)$,

(ii) $x_{\{a,b\}}=[x_a,x_b]$,

and the map $A\times A\to A$, $(a,b)\mapsto\{a,b\}=-\mu(x_a,x_b)$, defines a structure of Jacobi algebra on A [3].

When M is a smooth manifold and $C^\infty(M)$ the algebra of smooth functions on M, the manifold M is a Jacobi manifold if $C^\infty(M)$ is a Jacobi algebra.

If $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M, a locally conformal symplectic structure on M is a pair (ω,α) made up by a nondegenerate skew-symmetric 2-form $\omega\colon\mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$ and a closed 1-form

$$\alpha\colon\mathfrak{X}(M)\longrightarrow C^\infty(M)$$

https://doi.org/10.4153/CMB-2011-033-6 Published online by Cambridge University Press

such that $d\omega = -\alpha\Lambda\omega$, where d is the exterior differentiation operator.

When $\alpha = 0$, M is a symplectic manifold.

The characterization of locally conformal symplectic manifolds in terms of symplectic Lie–Rinehart–Jacobi algebras is the following one [3, 4]: a manifold M is a locally conformal symplectic manifold if and only if $\mathfrak{X}(M)$ admits a symplectic Lie–Rinehart–Jacobi algebra structure.

Let M be a Jacobi manifold and let

$$\omega_M\colon [C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M))] \times [C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M))] \longrightarrow C^\infty(M),$$

be the Jacobi 2-form of the Jacobi algebra $C^\infty(M)$ [3]. A sufficient condition for a Jacobi manifold M to be a locally conformal symplectic manifold is that the restriction

$$\omega_M|_{[\Omega_{\mathbb{R}}(C^\infty(M))]\times[\Omega_{\mathbb{R}}(C^\infty(M))]}\colon [\Omega_{\mathbb{R}}(C^\infty(M))] \times [\Omega_{\mathbb{R}}(C^\infty(M))] \longrightarrow C^\infty(M)$$

is nondegenerate [3].

A (exact) contact manifold is a manifold with dimension $2n + 1$ and with a differential 1-form α such that $\alpha\Lambda(d\alpha)^n$ is a volume form.

The main goal of this paper is to give a characterization of (exact) contact manifolds in terms of symplectic Lie–Rinehart–Jacobi algebras. We also give a sufficient condition for a Jacobi manifold to be a (exact) contact manifold.

In what follows, M denotes a smooth manifold, $\mathfrak{X}(M)$ the $C^\infty(M)$-module of vector fields on M, 1 the unit of $C^\infty(M)$ and $\mathcal{D}(M)$ the $C^\infty(M)$-module of differential operators of order ≤ 1 on $C^\infty(M)$. By "contact manifold" we will mean "exact contact manifold".

2 Contact Manifolds

Let $\mathrm{id}\colon \mathcal{D}(M) \to \mathcal{D}(M)$ be the identical map. We denote δ the cohomology operator associated with the representation id. The cohomology operator is restricted in this paper to the subspace of the usual Chevalley–Eilenberg complex that is made up of $C^\infty(M)$-multilinear skew-symmetric cochains.

We recall that if $\varphi = f + X$ and $\psi = g + Y$ are two differential operators of order ≤ 1, with f and g elements of $C^\infty(M)$, X and Y elements of $\mathfrak{X}(M)$, then the bracket on $\mathcal{D}(M)$ is given by $[\varphi, \psi] = X(g) - Y(f) + [X, Y]$.

When $\eta\colon \mathcal{D}(M)\times\mathcal{D}(M) \to C^\infty(M)$ is a skew-symmetric 2-form, then the 1-form

$$i_1\eta\colon \mathcal{D}(M) \longrightarrow C^\infty(M), \varphi \longmapsto \eta(1, \varphi),$$

is such that $i_1\eta|_{C^\infty(M)} = 0$.

Proposition 2.1 *If $\eta\colon \mathcal{D}(M) \times \mathcal{D}(M) \to C^\infty(M)$ is a skew-symmetric 2-form such that $\delta\eta = 0$, then $\eta = \delta(i_1\eta)$.*

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Proof For any $\varphi, \psi \in \mathcal{D}(M)$, we get

$$0 = (\delta\eta)(1, \varphi, \psi) = 1 \cdot \eta(\varphi, \psi) - \varphi[\eta(1, \psi)] + \psi[\eta(1, \varphi)] \\ - \eta([1, \varphi], \psi) + \eta([1, \psi], \varphi) - \eta([\varphi, \psi], 1).$$

As $[1, \varphi] = [1, \psi] = 0$, then

$$0 = (\delta\eta)(1, \varphi, \psi) = \eta(\varphi, \psi) - \varphi\left[(i_1\eta)(\psi)\right] + \psi\left[(i_1\eta)(\varphi)\right] + (i_1\eta)\left[\varphi, \psi\right].$$

Thus for any $\varphi, \psi \in \mathcal{D}(M)$

$$\eta(\varphi, \psi) = \varphi\left[(i_1\eta)(\psi)\right] - \psi\left[(i_1\eta)(\varphi)\right] - (i_1\eta)\left[\varphi, \psi\right] = \left[\delta(i_1\eta)\right](\varphi, \psi).$$

We deduce that $\eta = \delta(i_1\eta)$. ■

Thus if we denote $H_{\mathcal{D}}(M)$ the cohomology space of the differential complex $(\mathfrak{L}_{\text{sks}}(\mathcal{D}(M), C^\infty(M)), \delta)$, then $H^2_{\mathcal{D}}(M) = \{0\}$.

2.1 Symplectic Lie–Rinehart–Jacobi Algebra Structure on the Lie–Rinehart Algebra $(\mathcal{D}(M), \mathrm{id})$

In this part, we consider a symplectic Lie–Rinehart–Jacobi algebra structure $(\mathcal{D}(M), \mathrm{id}, \omega)$ on the Lie–Rinehart algebra $(\mathcal{D}(M), \mathrm{id})$.

In this case, $\omega\colon \mathcal{D}(M) \times \mathcal{D}(M) \to C^\infty(M)$ is a nondegenerate skew-symmetric 2-form such that $\delta\omega = 0$. For any $f \in C^\infty(M)$, we denote $\varphi_f \in \mathcal{D}(M)$ the unique differential operator of order ≤ 1 such that $i_{\varphi_f}\omega = \delta f$. For any $f \in C^\infty(M)$, we have $\theta_{\varphi_f}\omega = 0$, since $\theta_{\varphi_f}\omega = i_{\varphi_f}\delta\omega + \delta i_{\varphi_f}\omega = 0 + \delta(\delta f) = 0$. Let TM be the tangent vector bundle of M. The $C^\infty(M)$-module $\mathcal{D}(M)$ is the $C^\infty(M)$-module of sections of the vector bundle $\mathbb{R} \times TM \to M$. As ω is a nondegenerate skew-symmetric 2-form on $\mathcal{D}(M)$, then for any $p \in M$,

$$\omega(p)\colon (\mathbb{R} \times T_pM) \times (\mathbb{R} \times T_pM) \to \mathbb{R}$$

is a nondegenerate skew-symmetric 2-form on $(\mathbb{R} \times T_pM)$. Thus the dimension of M is odd.

Proposition 2.2 *For any $f, g \in C^\infty(M)$, the bracket $\{f, g\} = -\omega(\varphi_f, \varphi_g)$ is a Jacobi bracket on $C^\infty(M)$.*

Proof Since in this case the triplet $(\mathcal{D}(M), \mathrm{id}, \omega)$ is a symplectic Lie–Rinehart–Jacobi algebra. ■

Thus, M is a Jacobi manifold.

We note that for any $f, g \in C^\infty(M)$, we have $\{f, g\} = \varphi_f(g)$.

As $C^\infty(M)$ is a Jacobi algebra, we denote $\xi = \mathrm{ad}(-1)$ the fundamental vector field of the Jacobi manifold M. We deduce the following proposition.

Proposition 2.3 *We get*

(i) $\xi = -\varphi_1$;
(ii) $i_\xi \omega = -\delta 1$.

Proof For any $f \in C^\infty(M)$, we get $\xi(f) = [\mathrm{ad}(-1)](f) = \{-1, f\} = (-\varphi_1)(f)$. We deduce that $\xi = -\varphi_1$.

The second assertion is obvious. ■

Corollary 2.4 *We have* $(i_1\omega)(\xi) = 1$. *Moreover, for any* $\varphi \in \mathcal{D}(M)$, $(i_\xi\omega)(\varphi) = 0$ *if and only if* $\varphi \in \mathfrak{X}(M)$.

Proof As $i_\xi\omega = -\delta 1$, for any $\varphi \in \mathcal{D}(M)$ we get $(i_\xi\omega)(\varphi) = -\varphi(1)$. We deduce that $(i_\xi\omega)(-1) = 1$. Thus $(i_1\omega)(\xi) = 1$.

For any $\varphi \in \mathcal{D}(M)$, from the relation $(i_\xi\omega)(\varphi) = -\varphi(1)$, we deduce the second assertion. ■

Proposition 2.5 *We have* $\theta_\xi(i_1\omega) = 0$ *and* $(i_1\omega)[\xi, \varphi] = \xi[(i_1\omega)(\varphi)]$ *for any* $\varphi \in \mathcal{D}(M)$.

Proof As $\omega = \delta(i_1\omega)$ (see Proposition 2.1), for the first assertion, we obtain

$$\begin{aligned}\theta_\xi(i_1\omega) &= i_\xi\delta(i_1\omega) + \delta i_\xi(i_1\omega) = i_\xi\omega + \delta\left[(i_1\omega)(\xi)\right]\\ &= i_\xi\omega + \delta 1 = -\delta 1 + \delta 1 = 0.\end{aligned}$$

For the second assertion, for any $\varphi \in \mathcal{D}(M)$, we get

$$\begin{aligned}\omega(\xi,\varphi) &= \left[\delta(i_1\omega)\right](\xi,\varphi),\\ (i_\xi\omega)(\varphi) &= \xi\left[(i_1\omega)(\varphi)\right] - \varphi\left[(i_1\omega)(\xi)\right] - (i_1\omega)\left[\xi,\varphi\right],\\ -\varphi(1) &= \xi\left[(i_1\omega)(\varphi)\right] - \varphi(1) - (i_1\omega)\left[\xi,\varphi\right].\end{aligned}$$

Thus $(i_1\omega)\left[\xi,\varphi\right] = \xi\left[(i_1\omega)(\varphi)\right]$. ■

Corollary 2.6 *We have* $[\xi, \mathrm{Ker}(i_1\omega)] \subset \mathrm{Ker}(i_1\omega)$.

Let $\mathcal{D}(M)^*$ be the dual of the $C^\infty(M)$-module $\mathcal{D}(M)$. The sets

$$\mathcal{D}(M)^*_\xi = \{\eta \in \mathcal{D}(M)^* / \eta(\xi) = 0\}$$

and

$$\mathcal{D}(M)^*_{C^\infty(M),\xi} = \{\eta \in \mathcal{D}(M)^* / \eta|_{C^\infty(M)} = 0; \eta(\xi) = 0\}$$

are modules over $C^\infty(M)$.

For any $X \in \mathfrak{X}(M)$ ($X \in \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$, respectively), we verify that $i_X\omega \in \mathcal{D}(M)^*_\xi$ ($i_X\omega \in \mathcal{D}(M)^*_{C^\infty(M),\xi}$, respectively).

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Proposition 2.7 *We get* $\mathfrak{X}(M) = \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}] \oplus C^\infty(M) \cdot \xi$*. Moreover, the maps*

$$\mathfrak{X}(M) \longrightarrow \mathcal{D}(M)^*_\xi, X \longmapsto i_X\omega,$$

and

$$\mathrm{Ker}\left[i_1\omega|_{\mathfrak{X}(M)}\right] \longrightarrow \mathcal{D}(M)^*_{C^\infty(M),\xi}, X \longmapsto i_X\omega,$$

are isomorphisms of $C^\infty(M)$*-modules.*

Proof For any $X \in \mathfrak{X}(M)$, we write $X = [X - (i_1\omega)(X) \cdot \xi] + (i_1\omega)(X) \cdot \xi$. We verify that

$$[X - (i_1\omega)(X) \cdot \xi] \in \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$$

and

$$(\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]) \cap C^\infty(M) \cdot \xi = \{0\}\,.$$

Thus $\mathfrak{X}(M) = \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}] \oplus C^\infty(M) \cdot \xi$. Since the map

$$\mathcal{D}(M) \longrightarrow \mathcal{D}(M)^*, \varphi \longmapsto i_\varphi\omega,$$

is an isomorphism of $C^\infty(M)$-modules, then the maps

$$\mathfrak{X}(M) \longrightarrow \mathcal{D}(M)^*_\xi, X \longmapsto i_X\omega,$$

and

$$\mathrm{Ker}\left[i_1\omega|_{\mathfrak{X}(M)}\right] \longrightarrow \mathcal{D}(M)^*_{C^\infty(M),\xi}, X \longmapsto i_X\omega,$$

are injective.

Let $\eta \in \mathcal{D}(M)^*_\xi$ be a linear form on $\mathcal{D}(M)$ such that $\eta(\xi) = 0$ and let φ be the unique element of $\mathcal{D}(M)$ such that $i_\varphi\omega = \eta$. We get

$$0 = \eta(\xi) = (i_\varphi\omega)(\xi) = -(i_\xi\omega)(\varphi) = \varphi(1).$$

We deduce that $\varphi \in \mathfrak{X}(M)$. Thus the map $\mathfrak{X}(M) \to \mathcal{D}(M)^*_\xi, X \mapsto i_X\omega$, is also surjective.

Let $\sigma \in \mathcal{D}(M)^*_{C^\infty(M),\xi}$ be a linear form on $\mathcal{D}(M)$ such that $\sigma|_{C^\infty(M)} = 0$ and $\sigma(\xi) = 0$, and let φ be the unique element of $\mathcal{D}(M)$ such that $i_\varphi\omega = \sigma$. As $\sigma(\xi) = 0$, then $\varphi \in \mathfrak{X}(M)$.

Since $\sigma|_{C^\infty(M)} = 0$, we obtain $0 = \sigma(1) = (i_\varphi\omega)(1) = -(i_1\omega)(\varphi)$. We deduce that $\varphi \in \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$. Thus the map $\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}] \to \mathcal{D}(M)^*_{C^\infty(M),\xi}, X \mapsto i_X\omega$, is also surjective. ■

The $C^\infty(M)$-module $\mathcal{D}(M)^*_{C^\infty(M),\xi}$ is canonically isomorphic to the dual $\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]^*$ of the $C^\infty(M)$-module $\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$. As $\omega = \delta(i_1\omega)$ (see Proposition 2.1) if d is the de Rham differential, thus the restriction of ω to

$$\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}] \times \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$$

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is nondegenerate, *i.e.*, $d[i_1\omega|_{\mathfrak{X}(M)}]$ is a nondegenerate skew-symmetric 2-form on the $C^\infty(M)$-module $\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$.

For any $f \in C^\infty(M)$, we verify that the linear form

$$\xi(f)\cdot i_1\omega - \delta f + f\cdot\delta 1 \colon \mathcal{D}(M) \longrightarrow C^\infty(M)$$

belongs to $\mathcal{D}(M)^*_{C^\infty(M),\xi}$ and we denote X_f the unique element of $\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$ such that $i_{X_f}\omega = \xi(f)\cdot i_1\omega - \delta f + f\cdot\delta 1$.

Proposition 2.8 *For any $f \in C^\infty(M)$ and for any $g \in C^\infty(M)$,*

(i) $[\xi, X_f] = X_{\xi(f)}$;
(ii) $\varphi_f = \xi(f) - X_f - f\cdot\xi$;
(iii) $\{f, g\} = -\omega(X_f, X_g) - f\cdot\xi(g) + g\cdot\xi(f)$.

Proof (i) For any $f \in C^\infty(M)$, we get $i_{[\xi,X_f]}\omega = \theta_\xi[i_{X_f}\omega] - i_{X_f}[\theta_\xi\omega]$. As $\theta_\xi\omega = 0$, we obtain

$$\begin{aligned} i_{[\xi,X_f]}\omega &= \theta_\xi[i_{X_f}\omega] = \theta_\xi[\xi(f)\cdot i_1\omega - \delta f + f\cdot\delta 1] \\ &= \xi\left[\xi(f)\right]\cdot i_1\omega + \xi(f)\cdot\theta_\xi[i_1\omega] - \delta\xi(f) + \xi(f)\cdot\delta 1 + f\cdot\delta\xi(1). \end{aligned}$$

As $\theta_\xi(i_1\omega) = 0$, (see Proposition 2.5), we deduce that

$$i_{[\xi,X_f]}\omega = \xi\left[\xi(f)\right]\cdot i_1\omega - \delta\xi(f) + \xi(f)\cdot\delta 1 = i_{X_{\xi(f)}}\omega$$

and we conclude that $\left[\xi, X_f\right] = X_{\xi(f)}$.

(ii) For any $f \in C^\infty(M)$, we get

$$\begin{aligned} i_{\varphi_f - (\xi(f) - X_f - f\cdot\xi)}\omega &= i_{\varphi_f}\omega - \xi(f)\cdot i_1\omega + i_{X_f}\omega + f\cdot i_\xi\omega \\ &= \delta f - \xi(f)\cdot i_1\omega + \xi(f)\cdot i_1\omega - \delta f + f\cdot\delta 1 - f\cdot\delta 1 = 0. \end{aligned}$$

Thus $\varphi_f = \xi(f) - X_f - f\cdot\xi$.

(iii) We verify that $\{f, g\} = -\omega(X_f, X_g) - f\cdot\xi(g) + g\cdot\xi(f)$ for any $f, g \in C^\infty(M)$. The conclusion follows from the facts, $(i_1\omega)(\xi) = 1, (i_{X_f}\omega)(1) = 0$, and $(i_\xi\omega)(X) = 0$ for all vector fields on M. ∎

Theorem 2.9 *The* 1*-form $i_1\omega|_{\mathfrak{X}(M)} \colon \mathfrak{X}(M) \to C^\infty(M)$ is a contact form on M.*

Proof For any $x \in M$ we have $\xi_x \neq 0$ since $(i_1\omega)(\xi) = 1$. Thus the 1-form $i_1\omega|_{\mathfrak{X}(M)}$ is nonzero everywhere. Let $x \in M$ and let T_xM be the tangent vector space at x. As the dimension of M is odd, let $2n+1$ be the dimension of M. The set

$$(\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}])_x = \{X(x) \in T_xM / X \in \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]\}$$

is a vector space of dimension $2n$. Since $d[i_1\omega|_{\mathfrak{X}(M)}]$ is a nondegenerate skew-symmetric 2-form on the $C^\infty(M)$-module $\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$, then $(d[i_1\omega|_{\mathfrak{X}(M)}])(x)$ is a nondegenerate skew-symmetric 2-form on the vector space $(\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}])_x$ since for any $x \in M$ and for $X, Y \in \mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}]$

$$\left[(d[i_1\omega|_{\mathfrak{X}(M)}])(X,Y)\right](x) = \left[(d[i_1\omega|_{\mathfrak{X}(M)}])_x(X(x), Y(x))\right].$$

Thus $(\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}])_x$ is a symplectic vector space and $(d[i_1\omega|_{\mathfrak{X}(M)}])^n(x)$ is a volume form. We deduce that $(d[i_1\omega|_{\mathfrak{X}(M)}])^n(x) \neq 0$. Let $(v_1, v_2, \dots, v_{2n})$ be a basis of $(\mathrm{Ker}[i_1\omega|_{\mathfrak{X}(M)}])_x$. We have $(d[i_1\omega|_{\mathfrak{X}(M)}])^n(x)(v_1, v_2, \dots, v_{2n}) \neq 0$. We note that

$$\nu = [i_1\omega|_{\mathfrak{X}(M)}](x)\Lambda(d[i_1\omega|_{\mathfrak{X}(M)}])^n(x)$$

is nonzero since

$$\nu(\xi(x), v_1, v_2, \dots, v_{2n}) = (d[i_1\omega|_{\mathfrak{X}(M)}])^n(x)\,(v_1, v_2, \dots, v_{2n}) \neq 0.$$

As x is abitrary, we conclude that $[i_1\omega|_{\mathfrak{X}(M)}]\Lambda(d[i_1\omega|_{\mathfrak{X}(M)}])^n$ is a volume form on M. Thus $i_1\omega|_{\mathfrak{X}(M)}$ is a contact form on M. Therefore M is a contact manifold. ∎

We finally conclude that when the Lie–Rinehart algebra $(\mathcal{D}(M), \mathrm{id})$ admits a symplectic Lie–Rinehart–Jacobi algebra structure, then M is a contact manifold.

2.2 Characterization of Contact Manifolds

In this part, we give a characterization of contact manifolds in terms of symplectic Lie–Rinehart–Jacobi algebras.

Let $\alpha\colon \mathfrak{X}(M) \to C^\infty(M)$ be a differential 1-form on M. We denote $\widetilde{\alpha}\colon \mathcal{D}(M) \to C^\infty(M)$ the unique linear form on the $C^\infty(M)$-module $\mathcal{D}(M)$ such that $\widetilde{\alpha}|_{C^\infty(M)} = 0$ and $\widetilde{\alpha}|_{\mathfrak{X}(M)} = \alpha$.

We verify that $i_1(\delta\widetilde{\alpha}) = \widetilde{\alpha}$.

Theorem 2.10 ([4]) *If M is a contact manifold with contact form*

$$\alpha\colon \mathfrak{X}(M) \to C^\infty(M),$$

then the triplet $(\mathcal{D}(M), \mathrm{id}, \delta\widetilde{\alpha})$ is a symplectic Lie–Rinehart–Jacobi algebra.

Proof Let R be the Reeb vector field and let $\varphi = f + X_0 + h \cdot R$ be an element of $\mathcal{D}(M)$, with $f, h \in C^\infty(M)$, $X_0 \in \mathrm{Ker}\,\alpha$, such that $i_\varphi(\delta\widetilde{\alpha}) = 0$. We get

$$0 = [i_\varphi(\delta\widetilde{\alpha})](1) = -[i_1(\delta\widetilde{\alpha})](\varphi) = -\widetilde{\alpha}(\varphi) = -h \cdot \alpha(R) = -h.$$

Thus, $h = 0$.

On the other hand, we have

$$0 = [i_\varphi(\delta\widetilde{\alpha})](R) = (\delta\widetilde{\alpha})(\varphi, R) = \varphi(1) - \widetilde{\alpha}[f + X_0, R] = f - \alpha[X_0, R] = f,$$

since $[R, \mathrm{Ker}\,\alpha] \subset \mathrm{Ker}\,\alpha$ [2]. Thus $f = 0$.

As $f = h = 0$, we finally have $i_{X_0}(\delta\widetilde{\alpha}) = 0$. Thus for any $Y \in \mathrm{Ker}\,\alpha$, we obtain

$$0 = [i_{X_0}(\delta\widetilde{\alpha})](Y) = [i_{X_0}(d\alpha)](Y).$$

Since $d\alpha$ is a nondegenerate skew-symmetric 2-form on $\mathrm{Ker}\,\alpha$, therefore we get $X_0 = 0$. We conclude that $\varphi = 0$, *i.e.,* the map $\mathcal{D}(M) \to \mathcal{D}(M)^*, \varphi \mapsto i_\varphi(\delta\widetilde{\alpha})$, is injective.

https://doi.org/10.4153/CMB-2011-033-6 Published online by Cambridge University Press

The map $\mathcal{D}(M) \to \mathcal{D}(M)^*, \varphi \mapsto i_\varphi(\delta\widetilde{\alpha})$, is also surjective since if $\beta\colon \mathcal{D}(M) \to C^\infty(M)$ is a linear form on $\mathcal{D}(M)$ and if $X \in \operatorname{Ker}\alpha$ is the unique vector field such that $i_X d\alpha = \beta|_{\operatorname{Ker}\alpha}$, the differential operator $\varphi = \beta(R) + X - \beta(1)\cdot R$ is such that $i_\varphi(\delta\widetilde{\alpha}) = \beta$. Thus $\delta\widetilde{\alpha}\colon \mathcal{D}(M)\times\mathcal{D}(M) \to C^\infty(M)$ is a nondegenerate skew-symmetric 2-form.

As $\delta(\delta\widetilde{\alpha}) = 0$, we conclude that the triplet $(\mathcal{D}(M), \mathrm{id}, \delta\widetilde{\alpha})$ is a symplectic Lie–Rinehart–Jacobi algebra. ■

When (M,α) is a contact manifold, M is a Jacobi manifold. In this case, the Jacobi bracket of two smooth functions $f, g \in C^\infty(M)$ given by $\{f,g\} = -(\delta\widetilde{\alpha})(\varphi_f, \varphi_g)$ is the usual bracket for a contact manifold and the fundamental vector field $\xi = \mathrm{ad}(-1)$ is the Reeb vector field.

We state the following characterization.

Theorem 2.11 *A smooth manifold M is a contact manifold if and only if the Lie–Rinehart algebra $(\mathcal{D}(M), id)$ admits a symplectic Lie–Rinehart–Jacobi algebra structure.*

Let M be a Jacobi manifold and let

$$\omega_M\colon [C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M))] \times [C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M))] \longrightarrow C^\infty(M)$$

be the Jacobi 2-form of the Jacobi algebra $C^\infty(M)$ [3]. Following [3], we have $d_{\widetilde{\mathrm{ad}}}\omega_M = 0$ and if ω_M is nondegenerate, the map

$$\widetilde{\mathrm{ad}} : C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M)) \longrightarrow \mathcal{D}(M)$$

is an isomorphism of $C^\infty(M)$-modules.

A sufficient condition for a Jacobi manifold to be a contact manifold is the following one.

Proposition 2.12 *If M is a Jacobi manifold and if the Jacobi 2-form*

$$\omega_M\colon [C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M))] \times [C^\infty(M) \oplus \Omega_{\mathbb{R}}(C^\infty(M))] \longrightarrow C^\infty(M)$$

of the Jacobi algebra $C^\infty(M)$ is nondegenerate, then M is a contact manifold.

Proof In this case $\omega_M \circ ([\widetilde{\mathrm{ad}}]^{-1} \times [\widetilde{\mathrm{ad}}]^{-1})$ is a nondegenerate skew-symmetric 2-form on $\mathcal{D}(M)$ and

$$\delta\big(\omega_M \circ ([\widetilde{\mathrm{ad}}]^{-1} \times [\widetilde{\mathrm{ad}}]^{-1})\big) = (d_{\widetilde{\mathrm{ad}}}\omega_M) \circ ([\widetilde{\mathrm{ad}}]^{-1} \times [\widetilde{\mathrm{ad}}]^{-1} \times [\widetilde{\mathrm{ad}}]^{-1}) = 0.$$

Thus $(\mathcal{D}(M), \mathrm{id}, \omega_M \circ ([\widetilde{\mathrm{ad}}]^{-1} \times [\widetilde{\mathrm{ad}}]^{-1}))$ is a symplectic Lie–Rinehart–Jacobi algebra. Therefore M is a contact manifold. ■

In this paper we showed that a contact structure on a manifold M is due to the existence of a nondegenerate skew-symmetric 2-form $\omega\colon \mathcal{D}(M)\times\mathcal{D}(M) \longrightarrow C^\infty(M)$ such that $\delta\omega = 0$.

In this case, we will say that ω is the contact 2-form of the contact manifold M and $i_1\omega$ is the contact 1-form of the contact manifold M.

Thus the parallelism between symplectic manifolds and contact manifolds is obvious: a symplectic structure on a manifold M is due to the existence of a nondegenerate closed skew-symmetric 2-form on $\mathfrak{X}(M)$ whereas a contact structure on a manifold M is due to the existence of a nondegenerate closed skew-symmetric 2-form on $\mathcal{D}(M)$.

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