A GENERALISATION OF DIVINSKY'S RADICAL

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1. Introduction. Let A be an associative ring. Given $a \in A$, an element $b \in A$ is called a *left identity* for a if

$$ba = a. \tag{1}$$

Given a subset S of A, an element $b \in A$ is called a left identity for S if (1) is satisfied for all $a \in S$. An element of A need not have a left identity; for example, if A is nilpotent then no non-zero element of A has a left identity. If a does have a left identity, the latter need not be unique; if every element of a subset S of A has a left identity, then it is not necessarily true that S has a left identity.

Divinsky [4] has shown that every ring A has a unique two-sided ideal $\Delta(A)$ with the following properties:

(i) *every element of* $\Delta(A)$ *has a left identity,*

(ii) $\Delta(A)$ *contains every left ideal I such that every element in I has a left identity,*

(iii) $\Delta(A/\Delta(A)) = 0$.

The ideal $\Delta(A)$ is a radical in the sense of Amitsur [1]; it is called the *left Divinsky radical* of A. The right Divinsky radical of A is defined similarly in terms of elements which possess right identities.

The ideal $\Delta(A)$ and the Jacobson-Perlis radical $\Gamma(A)$ have many properties in common, despite the fact that they are to a certain extent complementary (for a nilpotent ring $\Gamma(A) = A$ but $\Delta(A) = 0$). In some respects, $\Delta(A)$ is better behaved than $\Gamma(A)$. For example, as one of us has shown recently elsewhere [7], the Jacobson-Perlis radical of the ring $\mathscr{M}_\rho(A)$ of row-finite matrices over A is given by

$$\Gamma(\mathscr{M}_\rho(A)) = \mathscr{M}_\rho(\Gamma(A))$$

if and only if $\Gamma(A)$ is right-vanishing in the sense of Levitzki [6]. For $\Delta(A)$, however, we have

$$\Delta(\mathscr{M}_\rho(A)) = \mathscr{M}_\rho(\Delta(A))$$

in all cases; this is a consequence of Theorem 11 below.

In the present paper, we discuss a generalisation of the concept of the Divinsky radical. In (1), the important fact is not so much that b is an element of A, but that $x \to bx$ is an A-endomorphism which leaves a fixed. In our generalisation, the ring A is replaced by an A-module M and left identities for elements of A are replaced by A-endomorphisms of M, belonging to some fixed set F, which is a semi-group with respect to the circle operation. The Divinsky radical $\Delta(M, F)$ of the pair (M, F) is defined by considering the elements $m \in M$ for which there exists an element $f \in F$ such that $f(m) = m$.

In §§ 2 to 4 we establish the basic properties of $\Delta(M, F)$. The properties proved in § 3 can be compared with those given by Bourbaki [2] for the (Jacobson-Perlis) radical of a module. Those proved in § 4 are concerned with showing that, under finiteness assumptions of various kinds, there is an element of F which leaves fixed every element of $\Delta(M, F)$. This property corresponds in the special case of $\Delta(A)$ to the existence of a left identity for $\Delta(A)$. In §§ 5 and 6 we consider cases in which F is a ring of A-endomorphisms of M. In these cases, M can be regarded as an (F, A)-bimodule. In particular, we prove the theorem referred to above on row-finite matrices and an analogous result concerning polynomials.

2. Definition of the generalised Divinsky radical. Let A be a ring, let M be a right A-module and let F be a set of A-endomorphisms of M closed under the circle operation, so that

$$f \circ g = f+g-fg \tag{2}$$

belongs to F whenever f and g belong to F. An additive subgroup H of M will be called an *F-subgroup* if, for all $x \in H$ and $f \in F$, we have $f(x) \in H$. An F-subgroup H of M will be called *F-permissible* if, for each $x \in H$, there exists $g \in F$ such that $g(x) = x$.

Let $\mathscr{H}$ be the collection of F-permissible subgroups of M. Then $\mathscr{H}$ satisfies the following propositions.

PROPOSITION 1. *If $H \in \mathscr{H}$ and $a \in A$, then $Ha \in \mathscr{H}$.*

The proof of this is trivial, and so is omitted.

PROPOSITION 2. *If H_1, $H_2 \in \mathscr{H}$, then $H_1 + H_2 \in \mathscr{H}$.*

Proof. If $x \in H_1+H_2$, then $x = x_1+x_2$, where $x_1 \in H_1$ and $x_2 \in H_2$. Given $f \in F$, we have

$$f(x) = f(x_1+x_2) = f(x_1)+f(x_2) \in H_1 + H_2$$

and so H_1+H_2 is an F-subgroup. Since $H_1 \in \mathscr{H}$, there exists $g \in F$ such that $g(x_1) = x_1$; since $g(x_2)-x_2 \in H_2$ and $H_2 \in \mathscr{H}$, there exists $h \in F$ such that

$$h(g(x_2)-x_2) = g(x_2)-x_2.$$

Then

$$(h+g-hg)(x) = x$$

and $h+g-hg \in F$ because F is closed under the circle operation. It follows that H_1+H_2 is F-permissible. Thus Proposition 2 is proved.

The following proposition, which is used frequently in the sequel, can be proved by means of an elementary inductive argument based on the method used in the proof of Proposition 2.

PROPOSITION 3. *If $x_1, x_2, \ldots, x_n$ are elements of F-permissible subgroups of M, there exists an endomorphism $g \in F$ which leaves each of $x_1, x_2, \ldots, x_n$ fixed.*

Let $\Delta(M, F)$ denote the union of all F-permissible subgroups. From Propositions 1 and 2 it follows quickly that $\Delta(M, F)$ is a sub-module of M; moreover $\Delta(M, F)$ is itself an F-permissible subgroup. We shall call $\Delta(M, F)$ the *Divinsky radical* of the pair (M, F).

Given any A-endomorphism f of M, let f^* be the A-endomorphism of M defined by

$$f^* = e-f,$$

where $e\colon M \to M$ is the identity. Let F^* be the set of all endomorphisms f^* such that $f \in F$. Then F^* is a semi-group, since F is closed with respect to the circle operation. The definition given above of an F-permissible subgroup can be restated in terms of F^* as follows: a subgroup H is F-permissible if

(i) *for all* $x \in H$ *and all* $f^* \in F^*$, *we have* $f^*(x) \in H$,

(ii) *for each* $x \in H$, *there exists* $g^* \in F^*$ *such that* $g^*(x) = 0$.

This alternative is in some ways simpler, but is unsuitable for the discussions in §§ 5 and 6 concerning matrices and polynomials. In these cases we take F to be a ring (with respect to the usual operations) of endomorphisms, so that F is automatically closed with respect to the circle operation. To this ring there corresponds a matrix ring of endomorphisms, and in studying this situation we find that the first definition of F-permissibility fits in more naturally.

If F is taken to be a ring of A-endomorphisms of M, the latter can be regarded as a bimodule; as well as being a right A-module, it is a left F-module and the fact that the elements of F are endomorphisms of M ensures that M is a bimodule. In proving the basic properties of $\Delta(M, F)$, however, the only restriction that we make on F is that it be closed under the circle operation.

3. The basic properties of the Divinsky radical. In this section, we show that the submodule $\Delta(M, F)$ satisfies the standard properties of radicals (cf., for example, Bourbaki, [**2**]).

Let M, M' be right A-modules and let α: $M \to M'$ be an A-epimorphism. If F is a set of A-endomorphisms of M closed under the circle operation and the kernel of α is an F-subgroup, then the set F' of A-endomorphisms of M' induced by α is also closed under the circle operation. To each $f \in F$ there corresponds a unique $f' \in F'$ satisfying

$$\alpha f = f' \alpha \tag{3}$$

and to each $f' \in F'$ there corresponds at least one $f \in F$ satisfying (3). We shall write $\alpha(M)$, $\alpha(F)$ for M', F' respectively.

THEOREM 1. *Let* α *be an epimorphism such that the kernel of* α *is an F-subgroup. Then*

$$\alpha(\Delta(M, F)) \subset \Delta(\alpha(M), \alpha(F)).$$

Proof. Clearly $\alpha(\Delta(M, F))$ is an additive subgroup of $\alpha(M)$. If $x' \in \alpha(\Delta(M, F))$, then $x' = \alpha(x)$ for some $x \in \Delta(M, F)$ and, given $f' \in \alpha(F)$, we have

$$f'(x') = f'(\alpha(x)) = \alpha(f(x))$$

for some $f \in F$. Hence $f'(x') \in \alpha(\Delta(M, F))$, so that $\alpha(\Delta(M, F))$ is an F'-subgroup. Moreover, there exists an element $g \in F$ such that $g(x) = x$; hence there exists $g' \in F'$ such that

$$g'(x') = g'(\alpha(x)) = \alpha(g(x)) = \alpha(x) = x'.$$

Therefore $\alpha(\Delta(M, F))$ is F'-permissible.

THEOREM 2. *Let N be an F-permissible sub-module of M. Then*

$$\Delta(M, F)/N = \Delta(\eta(M), \eta(F)),$$

where η: $M \to M/N$ *is the natural homomorphism.*

Proof. By Theorem 1, we have

$$\Delta(M, F)/N = \eta(\Delta(M, F)) \subset \Delta(\eta(M), \eta(F)).$$

Let H be the complete inverse image of $\Delta(\eta(M), \eta(F))$ under η. Then, if $x \in H$ and $f \in F$, we have

$$\eta(f(x)) = f'(\eta(x))$$

for some $f' \in \eta(F)$ and so

$$\eta(f(x)) \in \Delta(\eta(M), \eta(F))$$

for all f; therefore $f(x) \in H$ for all $f \in F$, so that H is an F-subgroup.

Since $\eta(x) \in \Delta(\eta(M), \eta(F))$, there exists $g' \in \eta(F)$ such that $g'(\eta(x)) = \eta(x)$. Hence there exists $g \in F$ such that $\eta(g(x)) = \eta(x)$. Then $g(x) - x \in N$ and so $g(x) - x \in \Delta(M, F)$. It follows that there exists $h \in F$ such that

$$h(g(x) - x) = g(x) - x$$

and so

$$(h \circ g)(x) = x.$$

This proves that H is F-permissible and hence $H \subset \Delta(M, F)$. Therefore

$$\Delta(\eta(M), \eta(F)) \subset \Delta(M, F)/N$$

and so Theorem 2 is proved.

THEOREM 3. *If η: $M \to M/\Delta(M, F)$ is the natural homomorphism, then*

$$\Delta(\eta(M), \eta(F)) = 0.$$

Let L be a submodule of M which is also an F-subgroup. If $\Delta(\theta(M), \theta(F)) = 0$, where θ: $M \to M/L$ is the natural homomorphism, then

$$L \supset \Delta(M, F).$$

Proof. The first part follows at once from Theorem 2 if we take $N = \Delta(M, F)$. To prove the second part, we observe that

$$\theta(\Delta(M, F)) \subset \Delta(\theta(M), \theta(F))$$

by Theorem 1; whence $\Delta(M, F) \subset L$.

THEOREM 4. *If N is a sub-module of M which is also an F-subgroup, then*

$$\Delta(N, F) = \Delta(M, F) \cap N.$$

Proof. If $x \in \Delta(M, F) \cap N$, then $f(x) \in \Delta(M, F)$ because $\Delta(M, F)$ is an F-subgroup, and $f(x) \in N$ because N is an F-subgroup. Hence $\Delta(M, F) \cap N$ is an F-subgroup. Since $\Delta(M, F) \cap N$ is contained in $\Delta(M, F)$, it is F-permissible and so we have

$$\Delta(M, F) \cap N \subset \Delta(N, F).$$

The opposite inclusion is trivial; thus Theorem 4 is established.

Suppose now that I is any set and that $[M_i]_{i \in I}$ is an indexed family of right A-modules. Let F_i be a set of A-endomorphisms of M_i closed under the circle operation. Given any family $[f_i]_{i \in I}$, where $f_i \in F_i$, we define f by

$$f[m_i]_{i \in I} = [f_i(m_i)]_{i \in I}.$$

Then f is an A-endomorphism of the direct product

$$M = \prod_{i \in I} M_i.$$

Clearly if $g = [g_i]_{i \in I}$, then

$$f \circ g = [f_i \circ g_i]_{i \in I}$$

and so we have a set F of A-endomorphisms of M closed under the circle operation.

THEOREM 5. *For the direct product*

$$M = \prod_{i \in I} M_i,$$

we have

$$\Delta(M, F) = \prod_{i \in I} \Delta(M_i, F_i).$$

Proof. The projection p_i: $M \to M_i$ is an A-epimorphism whose kernel is an F-subgroup; hence, by Theorem 1,

$$p_i(\Delta(M, F)) \subset \Delta(p_i(M), p_i(F)).$$

Clearly $p_i(F) = F_i$; hence

$$p_i(\Delta(M, F)) \subset \Delta(M_i, F_i)$$

and so

$$\Delta(M, F) \subset \prod_{i \in I} \Delta(M_i, F_i).$$

If $x_i \in \Delta(M_i, F_i)$, then, for all $f_i \in F_i$, we have $f_i(x_i) \in \Delta(M_i, F_i)$. Hence, for all $x \in \prod_{i \in I} \Delta(M_i, F_i)$ and $f \in F$, we have $f(x) \in \prod_{i \in I} \Delta(M_i, F_i)$. It follows that $\prod_{i \in I} \Delta(M_i, F_i)$ is an F-subgroup.

For each $x_i \in \Delta(M_i, F_i)$, there exists $g_i \in F_i$ such that $g_i(x_i) = x_i$, and hence, if

$$x \in \prod_{i \in I} \Delta(M_i, F_i),$$

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there exists $g \in F$ such that $g(x) = x$. Therefore $\prod_{i \in I} \Delta(M_i, F_i)$ is F-permissible and so is contained in $\Delta(M, F)$. Thus Theorem 5 is proved.

We now examine a more general case in which the result established in the second part of the proof of Theorem 5 is extended. Suppose that I is any set and that $M_i (i \in I)$ is an A_i-module. With each indexed family $[x_i]_{i \in I}$, where $x_i \in M_i$, we associate an element x of an A-module M. With each family $[f_i]_{i \in I}$, where f_i is an A_i-endomorphism of M_i, we associate an A-endomorphism f of M such that $f(x)$ is the element corresponding to $[f_i(x_i)]_{i \in I}$. Thus, if we write

$$x = T[x_i], \quad f = T[f_i],$$

we have

$$(T[f_i])T[x_i] = T[f_i(x)] \tag{4}$$

(where, for convenience, we have abbreviated $[x_i]_{i \in I}$ to $[x_i]$).

It is easily seen from (4) that $T[f_i]$ is the zero homomorphism of M if $f_i = 0$ for all i and that $T[x_i]$ is the zero element of M if $x_i = 0$ for all i. As an example, let I consist of the integers 1 and 2, let M_1 be a right A_1-module, let M_2 be a left A_1-module and let M be the tensor product $M_1 \otimes M_2$; thus M is a Z-module, where Z is the ring of integers. Defining $T[x_1, x_2]$ to be $x_1 \otimes x_2$ and $T[f_1, f_2]$ to be $f_1 \otimes f_2$, we obtain a correspondence satisfying (4).

Let F_i be a set of A_i-endomorphisms of M_i closed with respect to the circle operation. Write F_i^* for the semi-group of A_i-endomorphisms of the form $e_i - f_i$, where $e_i\colon M_i \to M_i$ is the identity and $f_i \in F_i$. The set of A-endomorphisms of M of the form $f^* = T[f_i^*]$, where $f_i^* \in F_i^*$, is a semi-group which we denote by F^*. If F is the set of A-endomorphisms of M of the form $f = e - f^*$, where $f^* \in F^*$ and $e\colon M \to M$ is the identity, then F is closed with respect to the circle operation. (We cannot go directly from the F_i to F because we do not assume additive properties for T.)

THEOREM 6. *Let H be the additive subgroup of M generated by the elements of the form $T[x_i]$, where $x_i \in \Delta(M_i, F_i)$. Then*

$$H \subset \Delta(M, F).$$

Proof. If $x = T[x_i]$, where $x_i \in \Delta(M_i, F_i)$, then, for all $f \in F$, we have

$$f(x) = x - f^*(x) = x - T[f_i^*]T[x_i]$$

$$= x - T[f_i^*(x_i)].$$

Since $\Delta(M_i, F_i)$ is an F_i-subgroup, $f_i^*(x_i) \in \Delta(M_i, F_i)$. Hence $x - f(x) \in H$ and so $f(x) \in H$. This is true for each generator $T[x_i]$ of H and so H is an F-subgroup.

Since $x_i \in \Delta(M_i, F_i)$, there exists $g_i^* \in F_i^*$ such that $g_i^*(x) = 0$. Then

$$g(x) = x - g^*(x) = x - T[g_i^*(x_i)] = x.$$

Hence, for each generator $x = T[x_i]$ of H, there exists a $g \in F$ such that $g(x) = x$; but every element of H is a finite sum of elements of this form and so, by Proposition 3, for each $y \in H$ there exists $h \in F$ such that $h(y) = y$. Therefore H is F-permissible and we have

$$H \subset \Delta(M, F).$$

4. Endomorphisms which leave fixed each element of the Divinsky radical. In general, there is no element $f \in F$ such that $f(x) = x$ for all $x \in \Delta(M, F)$. However, as we show in the present section, the existence of such an element is implied by certain finiteness conditions: ascending or descending chain conditions, or compactness with respect to a suitable topology.

We begin by considering the ascending chain condition for A-submodules.

THEOREM 7. *If $\Delta(M, F)$ satisfies the ascending chain condition for A-submodules, then there is an element $g \in F$ which leaves fixed every element of $\Delta(M, F)$.*

Proof. The A-module $\Delta(M, F)$ is finitely generated; suppose that $x_1, x_2, \ldots, x_k$ is a system of generators. By Proposition 3, there exists an element $g \in F$ such that

$$g(x_i) = x_i \quad (i = 1, \ldots, k)$$

and so, for all $x \in \Delta(M, F)$, we have

$$g(x) = x.$$

In fact, we have assumed more than is necessary in the statement of Theorem 7, as the following argument shows. Given $f \in F$, define the set S_f by

$$S_f = \{x \mid x \in \Delta(M, F), f(x) = x\}.$$

Then S_f is an A-submodule of $\Delta(M, F)$. Suppose that, for each $f \in F$, we have $S_f \neq \Delta(M, F)$. Then, given $f \in F$, there exists $y \in \Delta(M, F)$ such that

$$f(y) \neq y.$$

But $f(y) - y \in \Delta(M, F)$ and so there exists $f' \in F$ such that

$$f'(f(y) - y) = f(y) - y.$$

Hence

$$(f' \circ f)(y) = y,$$

and it follows quickly that

$$S_f \subset S_g, \quad S_f \neq S_g,$$

where $g = f' \circ f$. By hypothesis $S_g \neq \Delta(M, F)$ and so we can repeat the process to obtain an A-submodule S_h for which

$$S_g \subset S_h, \quad S_g \neq S_h.$$

Continuing in this way, we can construct an infinite ascending chain of A-submodules of the form S_f, and this chain does not terminate. Thus, if $\Delta(M, F)$ satisfies the ascending chain condition for A-submodules of the form S_f, there exists an element $g \in F$ such that $g(x) = x$ for all $x \in \Delta(M, F)$.

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In the remaining theorems of this section, we consider cases in which finiteness conditions are imposed on the structure of the semi-group $(F, \circ)$ consisting of the set F together with the circle operation. Up to now, we have not taken the structure of $(F, \circ)$ into account, but it is easily seen that this structure is important. For example, if every element of F is quasi-regular (so that $0 \in F$ and, for each $f \in F$, there exists $f' \in F$ such that $f \circ f' = 0 = f' \circ f$), we have $\Delta(M, F) = 0$. In the general case the elements of F which are not quasi-regular form an ideal F_0 of $(F, \circ)$, and $\Delta(M, F) \subset \Delta(M, F_0)$.

THEOREM 8. *If $(F, \circ)$ satisfies the descending chain condition on left ideals, then there is an element $g \in F$ which leaves fixed every element of $\Delta(M, F)$.*

Proof. Let Φ be a subset of $\Delta(M, F)$. If

$$T_\Phi = \{f \mid f \in F,\ f(x) = x \text{ for all } x \in \Phi\},$$

then T_Φ is a left ideal of $(F, \circ)$. By Proposition 3, T_Φ is not empty when Φ is finite.

Suppose that there is no element $g \in F$ which leaves fixed every element of $\Delta(M, F)$. Let Φ_1 be a finite subset of $\Delta(M, F)$. If Φ_2 is any finite subset of $\Delta(M, F)$ containing Φ_1, then

$$T_{\Phi_1} \supset T_{\Phi_2}.$$

We can always choose Φ_2 to be such that

$$T_{\Phi_1} \neq T_{\Phi_2},$$

for otherwise there would be an element of F which leaves fixed every element of $\Delta(M, F)$. Repeating the process, we can find a finite subset Φ_3 such that

$$T_{\Phi_2} \supset T_{\Phi_3}$$

and

$$T_{\Phi_2} \neq T_{\Phi_3}.$$

Continuing in this way, we can construct a non-terminating descending chain of left ideals of $(F, \circ)$. It follows that $(F, \circ)$ does not satisfy the descending chain condition.

The result of Theorem 8 is not true if the descending chain condition on left ideals is replaced by the ascending chain condition on left ideals: neither is it true under the ascending chain condition on right (or two sided) ideals, as the following example shows.

Let Z be the set of positive integers, and let $[M_i]_{i \in Z}$ be an indexed family of Abelian groups, for which $M_i = M$ for all $i \in Z$. If $[x_i]_{i \in Z}$ denotes any element of the discrete direct sum $N = \sum_{i \in Z} M_i$, then, for each $r \in Z$, we define an endomorphism f_r of N by

$$f_r[x_i]_{i \in Z} = [y_i]_{i \in Z}, \quad \text{where} \quad \begin{cases} y_i = x_i & \text{for} \quad i \leqq r, \\ y_j = 0 & \text{for} \quad j > r. \end{cases}$$

It is easily verified that the set F of all such endomorphisms of N is closed under the circle composition. Evidently $\Delta(N, F) = N$, and there is no element of F which leaves fixed every element of N. However $(F, \circ)$ has the ascending chain condition on ideals.

We now consider the case in which F is a ring (with respect to the usual operations). The argument used in the proof of Theorem 8 can be adapted to show that the theorem still holds if the descending chain condition on the left ideals (in the ring sense) of F is satisfied. However, we give an alternative proof based on the structure theory of rings satisfying the descending chain condition either on left ideals or on right ideals.

THEOREM 9. *If F is a ring and satisfies the descending chain condition either on left ideals or on right ideals, then there is an element of F which leaves fixed every element of $\Delta(M, F)$.*

Proof. If F is nilpotent, then $\Delta(M, F) = 0$ and the theorem is trivial. If F is not nilpotent, there exists an idempotent e such that $f-fe$ is nilpotent for all $f \in F$. Suppose that $x \in \Delta(M, F)$ and that

$$y = x - e(x).$$

Then $y \in \Delta(M, F)$ and so there exists $f \in F$ such that

$$f(y) = y.$$

Therefore

$$(f-fe)(x) = y.$$

Since e is idempotent, we have $e(y) = 0$, and so

$$(f-fe)^2(x) = (f-fe)(y) = f(y) = y.$$

We can now prove by a simple inductive argument that

$$(f-fe)^n(x) = y,$$

for all positive integers n. But $f-fe$ is nilpotent; hence $y = 0$ and therefore $e(x) = x$. Thus e leaves fixed every element of $\Delta(M, F)$.

We conclude this section with a result, similar to the preceding propositions, which has topological connections. If $x \in \Delta(M, F)$, we write

$$T_x = \{f \mid f \in F;\ f(x) = x\}.$$

THEOREM 10. *If there exists a topology for F such that F is compact and the sets T_x are closed, then there is an element of F which leaves fixed every element of $\Delta(M, F)$.*

Proof. By Proposition 3, the intersection of any finite number of the T_x is non-empty; but F is compact and therefore, by the finite intersection property, the intersection of all the T_x is non-empty. The theorem follows at once.

By a similar argument, the following result is easily proved.

Let F be a compact topological ring and let M be a topological F-module. Then there is an element of F which leaves fixed every element of $\Delta(M, F)$.

COROLLARY. *If every element of a compact topological ring has a left identity, then the ring itself has a left identity.*

5. The Divinsky radical of a module of matrices. In this section and the next we assume that F is a ring. Let S be any set and let A be any ring. A matrix (see [3]) of type S over a ring A is a mapping α: $S \times S \to A$; the image of $(r, s) \in S \times S$ under α is denoted by α_{rs}. The matrix α is said to be *row-finite* if for each $r \in S$, α_{rs} is the zero element of A for all but a finite number of elements $s \in S$; *column-finite* matrices are defined similarly. Addition and multiplication of matrices are defined in the usual way. With respect to these operations, the set of all row-finite matrices over A is a ring $\mathcal{M}_\rho(A)$ and the set of all column-finite matrices is a ring $\mathcal{M}_\gamma(A)$.

Suppose now that M is a right A-module. A matrix of type S over M is a mapping μ: $S \times S \to M$. Defining *right-multiplication* of such a matrix by an element of $\mathcal{M}_\gamma(A)$ in the natural way, we obtain a right $\mathcal{M}_\gamma(A)$-module $\mathcal{M}(M)$; in this module the matrices over M are unrestricted. If we consider instead the set of row-finite matrices over M, then we obtain a right $\mathcal{M}_\rho(A)$-module, which we denote by $\mathcal{M}_\rho(M)$.

Let F be a ring of A-endomorphisms of M. If ϕ is any element of $\mathcal{M}_\rho(F)$ and μ is any matrix over M, then we define $\phi(\mu)$ to be the matrix obtained by applying the usual rule for matrix multiplication to ϕ and μ. Then ϕ becomes an $\mathcal{M}_\gamma(A)$-endomorphism of $\mathcal{M}(M)$, or an $\mathcal{M}_\rho(A)$-endomorphism of $\mathcal{M}_\rho(M)$. Thus we can define Divinsky radicals for the pairs $(\mathcal{M}(M), \mathcal{M}_\rho(F))$ and $(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$. Our main result on matrices is concerned with the second of these.

THEOREM 11. *The Divinsky radical of the pair $(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$ is given by*

$$\Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F)) = \mathcal{M}_\rho(\Delta(M, F)).$$

Proof. For fixed $p, q \in S$, let B_{pq} be the set of all elements $\beta_{pq} \in M$ such that β_{pq} is the (p, q)th element of a matrix $\beta \in \Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$. Then B_{pq} is an additive subgroup of M. If $f \in F$ and ϕ is the element of $\mathcal{M}_\rho(F)$ defined by

$$\phi_{pp} = f,$$
$$\phi_{rs} = 0 \quad \text{unless } r = p \text{ and } s = p,$$

then, since $\phi(\beta) \in \Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$ for all $\beta \in \Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$, we have $f(\beta_{pq}) \in B_{pq}$ for all $\beta_{pq} \in B_{pq}$. Moreover, given $\beta \in \Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$, there exists $\psi \in \mathcal{M}_\rho(F)$ such that $\psi(\beta) = \beta$; let ε be the element of $\mathcal{M}_\rho(F)$ defined by

$$\varepsilon_{ps} = \psi_{ps} \quad (s \in S),$$
$$\varepsilon_{rs} = 0 \quad (r \neq p).$$

Since $\varepsilon(\beta) \in \Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F))$, there exists $\theta \in \mathcal{M}_\rho(F)$ such that $\theta(\varepsilon(\beta)) = \varepsilon(\beta)$ and hence

$$\theta_{pp}(\beta_{pq}) = \beta_{pq}.$$

Since $\theta_{pp} \in F$, it follows that B_{pq} is an F-permissible subgroup of M; hence $\beta_{pq} \in \Delta(M, F)$ for all β_{pq}. Therefore

$$\Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F)) \subset \mathcal{M}_\rho(\Delta(M, F)).$$

It is clear that $\mathcal{M}_\rho(\Delta(M, F))$ is an additive subgroup of $\mathcal{M}_\rho(M)$ and that, for any element $\mu \in \mathcal{M}_\rho(F)$ and any element $\alpha \in \mathcal{M}_\rho(\Delta(M, F))$, we have $\mu(\alpha) \in \mathcal{M}_\rho(\Delta(M, F))$. Further, if β is any element of $\mathcal{M}_\rho(\Delta(M, F))$ and $r \in S$, then $\beta_{rs} = 0$ for all but a finite number of $s \in S$, since β is row-finite. Hence, by Proposition 3, there exists $f_r \in F$ such that

$$f_r(\beta_{rs}) = \beta_{rs} \quad (s \in S).$$

Let ϕ be the matrix in $\mathcal{M}_\rho(F)$ defined by

$$\phi_{rs} = 0 \quad (r \neq s),$$

$$\phi_{rr} = f_r\ ;$$

then $\phi(\beta) = \beta$. It follows that $\mathcal{M}_\rho(\Delta(M, F))$ is an $\mathcal{M}_\rho(F)$-permissible subgroup and so we have

$$\mathcal{M}_\rho(\Delta(M, F)) \subset \Delta(\mathcal{M}_\rho(M), \mathcal{M}_\rho(F)).$$

Theorem 11 follows at once.

THEOREM 12. *The Divinsky radical of* $(\mathcal{M}(M), \mathcal{M}_\rho(F))$ *satisfies*

$$\Delta(\mathcal{M}(M), \mathcal{M}_\rho(F)) \subset \mathcal{M}(\Delta(M, F)).$$

To prove this, we need only observe that the argument used in the proof of the first part of Theorem 11 does not depend on the row-finiteness of the elements of $\mathcal{M}_\rho(M)$.

Similar results to Theorems 11 and 12 can also be proved, by analogous methods, for other types of matrices.

Suppose, for example, that S is any totally ordered set. An upper triangular matrix of type S is a mapping α such that $\alpha_{rs} = 0$ if $r > s$. We denote the ring of all row-finite upper triangular matrices over A by $\mathcal{M}_\rho^t(A)$ and we denote the $\mathcal{M}_\rho^t(A)$-module of all row-finite upper triangular matrices over M by $\mathcal{M}_\rho^t(M)$. Then the result corresponding to Theorem 11 is

$$\Delta(\mathcal{M}_\rho^t(M), \mathcal{M}_\rho^t(F)) = \mathcal{M}_\rho^t(\Delta(M, F)).$$

6. The Divinsky radical of a polynomial module. Let A be a ring and let $\mathcal{P}(A)$ be the polynomial ring over A. If M is a right A-module, then we can define a $\mathcal{P}(A)$-module $\mathcal{P}(M)$ whose elements are polynomials over M. If F is a ring of A-endomorphisms of M, then the polynomial ring $\mathcal{P}(F)$ can be regarded as a ring of $\mathcal{P}(A)$-endomorphisms of $\mathcal{P}(M)$. The following is the result† for polynomials analogous to Theorem 11 for matrices.

THEOREM 13. *The Divinsky radical of* $(\mathcal{P}(M), \mathcal{P}(F))$, *is given by*

$$\Delta(\mathcal{P}(M), \mathcal{P}(F)) = \mathcal{P}(\Delta(M, F)).$$

Proof. If $\mu \in \mathcal{P}(F)$ and $\alpha \in \mathcal{P}(\Delta(M, F))$, then clearly $\mu(\alpha) \in \mathcal{P}(\Delta(M, F))$. Let α be any polynomial over $\Delta(M, F)$, with coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$. By Proposition 3, there exists an

† This theorem holds for polynomials in any number of indeterminates, but it is convenient here to give the proof in the case of one indeterminate.

element $f \in F$ such that $f(\alpha_i) = \alpha_i$ $(i = 0, 1, \ldots, n)$. Hence, if ϕ is the element of $\mathscr{P}(F)$ such that

$$\phi_0 = f, \quad \phi_i = 0 \quad (i \neq 0),$$

then $\phi(\alpha) = \alpha$. It follows that the subgroup $\mathscr{P}(\Delta(M, F))$ is $\mathscr{P}(F)$-permissible and hence

$$\mathscr{P}(\Delta(M, F)) \subset \Delta(\mathscr{P}(M), \mathscr{P}(F)).$$

Given any integer $i \geqq 0$, let B_i be the set of elements β_i such that β_i is the ith coefficient in a polynomial $\beta \in \Delta(\mathscr{P}(M), \mathscr{P}(F))$. Clearly B_i is an additive subgroup of M and, if $\beta_i \in B_i$, then $f(\beta_i) \in B_i$ for all $f \in F$. We now prove that B_i is an F-permissible subgroup of M by establishing the existence, for each $\beta_i \in B_i$, of an element $f_i \in F$ such that $f_i(\beta_i) = \beta_i$.

Suppose first that β_0 is an element of B_0. Then β_0 is the constant term in a polynomial $\beta \in \Delta(\mathscr{P}(M), \mathscr{P}(F))$. There exists an element $\phi \in \mathscr{P}(F)$ such that $\phi(\beta) = \beta$ and so, if ϕ_0 is the constant term of ϕ, then

$$\phi_0(\beta_0) = \beta_0.$$

Hence B_0 is an F-permissible subgroup of M.

Assume that B_i is F-permissible for $0 \leqq i \leqq k$. Let γ_{k+1} be an element of B_{k+1}. Then γ_{k+1} is the $(k+1)$th coefficient in a polynomial $\gamma \in \Delta(\mathscr{P}(M), \mathscr{P}(F))$. There exists an element $\psi \in \mathscr{P}(F)$ such that $\psi(\gamma) = \gamma$; hence

$$\sum_{r=0}^{k+1} \psi_r(\gamma_{k+1-r}) = \gamma_{k+1}$$

and so

$$\gamma_{k+1} - \psi_0(\gamma_{k+1}) = \sum_{r=1}^{k+1} \psi_r(\gamma_{k+1-r}).$$

Since $\gamma_0, \gamma_1, \ldots, \gamma_k$ belong to $B_0, B_1, \ldots, B_k$ respectively, then, by hypothesis,

$$\gamma_{k+1} - \psi_0(\gamma_{k+1}) \in \Delta(M, F).$$

Therefore there exists $g \in F$ such that

$$g(\gamma_{k+1} - \psi_0(\gamma_{k+1})) = \gamma_{k+1} - \psi_0(\gamma_{k+1})$$

and so

$$(\psi_0 + g - g\psi_0)(\gamma_{k+1}) = \gamma_{k+1}.$$

Hence there is an element $f = \psi_0 + g - g\psi_0$ of F such that $f(\gamma_{k+1}) = \gamma_{k+1}$, so that B_{k+1} is an F-permissible subgroup. Therefore, by induction, B_i is F-permissible for all $i \geqq 0$ and we have

$$\Delta(\mathscr{P}(M), \mathscr{P}(F)) \subset \mathscr{P}(\Delta(M, F)).$$

This completes the proof of Theorem 13.

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