Glasgow Math. J. **47** (2005) 221–236.
doi:10.1017/S0017089505002430. Printed in the United Kingdom

ALGORITHMS FOR LABELING CONSTANT WEIGHT GRAY CODES

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(Received 13 August, 2001; accepted 28 October, 2004)

Abstract. Let n and r be positive integers with $1 < r < n$, and let $X_n = \{1, 2, \ldots, n\}$. An r-set A and a partition π of X_n are said to be orthogonal if every class of π meets A in exactly one element. We prove that if $A_1, A_2, \ldots, A_{\binom{n}{r}}$ is a list of the distinct r-sets of X_n with $|A_i \cap A_{i+1}| = r - 1$ for $i = 1, 2, \ldots, \binom{n}{r}$ taken modulo $\binom{n}{r}$, then there exists a list of distinct partitions $\pi_1, \pi_2, \ldots, \pi_{\binom{n}{r}}$ such that π_i is orthogonal to both A_i and A_{i+1}. This result states that any constant weight Gray code admits a labeling by distinct orthogonal partitions. Using an algorithm from the literature on Gray codes, we provide a surprisingly efficient algorithm that on input (n, r) outputs an orthogonally labeled constant weight Gray code. We also prove a two-fold Gray enumeration result, presenting an orthogonally labeled constant weight Gray code in which the partition labels form a cycle in the covering graph of the lattice of all partitions of X_n. This leads to a conjecture related to the Middle Levels Conjecture. Finally, we provide an application of our results to calculating minimal generating sets of idempotents for finite semigroups.

2000 *Mathematics Subject Classification.* 94B25, 05A18, 20M20.

1. Introduction and background. We prove a combinatorics result concerning constant weight Gray codes, with application to minimal generating sets of finite semigroups. The paper contributes techniques, algorithms, and problems aimed at understanding the combinatorics of functions on finite sets and their efficient listing.

Let $X_n = \{1, 2 \ldots, n\}$. In the 1950's Frank Gray [**9**] developed algorithms that for a positive integer n, list the 2^n subsets of X_n in such a way that successive subsets differ minimally, having a singleton symmetric difference (including the first and last set). The lists, now known as *Gray codes*, were used by Gray to minimize errors in certain analog computer operations.

For a given positive integer r with $1 \leq r < n$, refer to an r-element subset of X_n as an r-set. Gray [**9**] also constructed listings of the r-sets of X_n so that successive r-sets differ minimally, having two element symmetric difference (equivalently, successive r-sets intersect in an $(r-1)$-set). These r-set listings became known as *constant weight Gray codes*; we specify parameters by calling such a code a *constant weight* (n, r)-*Gray code*. A partition π of X_n is said to be a *weight-r* partition if π has r classes. In the present paper, we prove that every (n, r)-Gray code admits what we call an *orthogonal labeling by weight-r partitions*. An orthogonal labeling of a constant weight (n, r)-Gray code leads to a listing of a minimal generating set for the finite semigroup $K(n, r)$, consisting of all transformations on X_n with image sets of r or fewer elements. We describe connections between orthogonal labeling and semigroup theory in Section 5, where we also pose problems in combinatorics and semigroup theory.

1.1. Definitions and statements of theorems. Let n and r be positive integers such that $1 < r < n$. We consider the graph $G_{n,r} = (V_r, E_r)$ where the vertex set V_r is the set of the r-sets of X_n and the edges E_r connect precisely those pairs of sets in V_r that have $(r-1)$-element intersection. A *Hamiltonian cycle* of a graph G is a cycle that passes through each vertex of G exactly once; G is *Hamiltonian* if it has a Hamiltonian cycle. It is well-known that $G_{n,r}$ is Hamiltonian [**2**], [**3**], [**17**], [**21**]. Note that (n, r)-Gray codes and Hamiltonian cycles in $G_{n,r}$ are one and the same. In the sequel we use the more modern terminology, referring to an (n, r)-Gray code as a Hamiltonian cycle in $G_{n,r}$. In Section 3, we provide a construction for a certain Hamiltonian cycle in $G_{n,r}$ that arises in connection with the well-known *reflected Gray codes* [**17**], [**2**].

An r-set $A \subseteq X_n$ and a weight-r partition π are said to be *orthogonal* if A is a transversal of π; that is, if every class of π contains a single element of A. In what follows, for clarity we omit commas from subsets, from lists of subsets and from cycles in graphs.

DEFINITION 1.1. Let $\mathcal{C} = A_1A_2\ldots A_{\binom{n}{r}}$ be a list of the distinct r-sets of X_n. We say that $\mathcal{C}$ is *orthogonally labeled* if there exists a list of distinct weight-r partitions $\pi_1\pi_2\ldots\pi_{\binom{n}{r}}$ of X_n such that the partition π_i is orthogonal to both A_i and A_{i+1} for $1 \leq i \leq \binom{n}{r}$, modulo $\binom{n}{r}$. When $\mathcal{C}$ is a Hamiltonian cycle in $G_{n,r}$ it is said to be an *orthogonally labeled Hamiltonian cycle*. We say that π_i *labels* the edge A_iA_{i+1} and write $A_1\pi_1A_2\pi_2\ldots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ to describe the orthogonal labeling of $\mathcal{C}$.

EXAMPLE 1.2. An example of an orthogonal labeling of a Hamiltonian cycle $G_{4,2}$ follows. We omit commas between elements and italicize partitions.

$$\{12\}\mathit{134}\,|\,\mathit{2}\{23\}\mathit{124}\,|\,\mathit{3}\{13\}\mathit{14}\,|\,\mathit{23}\{34\}\mathit{123}\,|\,\mathit{4}\{24\}\mathit{12}\,|\,\mathit{34}\{14\}\mathit{234}\,|\,\mathit{1}$$

Our interest in orthogonally labeled lists of the r-sets of X_n stems from the J. M. Howie and R. B. McFadden paper [**12**], where the authors proved the following theorem.

THEOREM 1.3 [**12**]. *For positive integers n and r with $1 < r < n$ there exists an orthogonally labeled list of the r-sets of X_n.*

In general, the lists in the previous theorem are not Hamiltonian cycles in $G_{n,r}$. The authors of [**12**] were interested in finding minimal generating sets for the semigroup $K(n, r)$. In Section 5, we describe the connection between Theorem 1.4 below and the construction of minimal generating sets for $K(n, r)$.

The goal here is not to provide a new proof of Theorem 1.3; as our results show, [**12**] is a starting point for non-trivial combinatorial and semigroup theoretic investigations. We shall state our results and make some comments about the relations between them before presenting the proofs.

Our first theorem is combinatorial. It states that **every** Hamiltonian cycle in $G_{n,r}$ can be orthogonally labeled. Our proof is algorithmic: on input of a Hamiltonian cycle in $G_{n,r}$ we sequentially label its edges by orthogonal partitions.

THEOREM 1.4. *For positive integers n and r with $1 < r < n$, every Hamiltonian cycle in $G_{n,r}$ admits an orthogonal labeling by weight-r partitions.*

https://doi.org/10.1017/S0017089505002430 Published online by Cambridge University Press

Theorem 1.3 follows from Theorem 1.4. Our next result has a different flavor; instead of beginning with an input of a given Hamiltonian cycle, we input the pair of integers (n, r). With this input, we output an orthogonally labeled Hamiltonian cycle in $G_{n,r}$ in a surprisingly efficient manner. To do so, we use an algorithm from Theorem 1.3 in tandem with an algorithm from the literature on constant weight Gray codes [**2**].

THEOREM 1.5. *Let n and r be positive integers with $1 < r < n$. On input (n, r), we can output an orthogonally labeled Hamiltonian cycle in $G_{n,r}$ in $dn\binom{n}{r}$ time, where d is a constant independent of n and r.*

In [**2**] the authors provide an algorithm which produces a specific Hamiltonian cycle $H_{n,r}$ with sets $H_{n,r}(k)$. That algorithm calculates $H_{n,r}(k+1)$ from $H_{n,r}(k)$ in constant time independent of n and r (see Definition 3.1). Each r-set is stored as an n-vector of 0's and 1's. An algorithm so efficient is possible because the two positions that switch 0 and 1 have a high probability of occurring very near the positions that were switched at the previous calculation. We therefore pose the following:

PROBLEM 1. Is there an algorithm which on input (n, r) with $1 < r < n$ outputs an orthogonally labeled Hamiltonian cycle in such a way that the time needed for calculations is $c\binom{n}{r}$ where c is independent of n and r?

The term *combinatorial Gray code* came to mean a listing of a set of combinatorial objects so that successive objects differ in a prescribed minimal way [**17**]. The next result involves "two-fold" combinatorial Gray enumeration of interspersed r-sets and weight-r partitions of X_n. Recall that $Part(n)$ is the lattice of partitions of X_n. Associated with any finite lattice L is its *covering graph* $Cov(L)$, the graph whose vertices are elements of L, where two vertices are adjacent if one of the vertices covers the other in L. Let $Part_{n,r}$ be the graph whose vertices are weight-r partitions of X_n, with two partitions adjacent if the vertices are distance-two in $Cov(Part(n))$. That is, two vertices of $Part_{n,r}$ have an edge if they are as close as possible in the covering graph determined by $Part(n)$. It is not difficult to verify that two partitions of weight r are distance-two if and only if they agree on exactly $r - 2$ classes.

THEOREM 1.6. *Let n and r be positive integers with $1 < r < n$. There exists an orthogonally labeled Hamiltonian cycle $A_1\pi_1A_2\pi_2\dots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ in $G_{n,r}$ such that $\pi_1\pi_2\dots\pi_{\binom{n}{r}}$ is a cycle in $Part_{n,r}$.*

In the next subsection, we state a conjecture that, if true, significantly improves Theorem 1.6.

1.2. Extensions of the Middle Levels Conjecture. Observe that if $A_1A_2\dots A_{\binom{n}{r}}$ is a Hamiltonian cycle in $G_{n,r}$ then for $i = 1, \dots, \binom{n}{r}$, there exists a transposition β_i of X_n such that $A_i\beta_i = A_{i+1}$. In the conjecture below, for $n < 2r$, we ask if there exists an orthogonally labeled Hamiltonian cycle whose partitions can be obtained by successively applying transpositions.

The Middle Levels Conjecture that follows is attributed to Paul Erdos, and there is an extensive literature on the subject (see for example [**18**], [**5**], [**4**]). The Adjacent Levels Conjecture is a natural generalization of the Middle Levels Conjecture. It is known that if the Middle Levels Conjecture is true for all possible cases, then Adjacent Levels Conjecture is also true for all possible cases [**13**]. (In [**13**] it is shown that for

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any *given fixed partition type containing more than one non-singleton class* there exists a Hamiltonian cycle that can be labeled by partitions of this type.)

CONJECTURE 1. Adjacent Levels and Middle Levels Conjectures

The Adjacent Levels Conjecture states that for positive integers n and r with $1 < r < n$ *and* $n < 2r$, *there exists a sequence of distinct subsets* $A_1B_1A_2B_2 \dots A_{\binom{n}{r}}B_{\binom{n}{r}}$ *of* X_n *such that for* $i = 1, \dots, \binom{n}{r}$, $|A_i| = r$ *and* $|B_i| = r - 1$ *and* $A_i \cap A_{i+1} = B_i$, *where* $i + 1$ *is taken mod* $\binom{n}{r}$.

The Middle Levels Conjecture is the Adjacent Levels Conjecture for the special case $n = 2r - 1$.

For a Hamiltonian cycle $A_1A_2 \dots A_{\binom{n}{r}}$ in $G_{n,r}$ we can obtain A_{i+1} by applying a transposition to A_i, so it is natural to ask if for $1 < r < n$ there exist orthogonally labeled cycles such that successive partitions can be obtained by applying a transposition.

CONJECTURE 2. Transposition Listing Conjecture

For positive integers n and r with $1 < r < n$ *and* $n < 2r$, *there exists an orthogonally labeled Hamiltonian cycle* $A_1\pi_1A_2\pi_2 \dots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ *in* $G_{n,r}$ *such that for each* $i = 1, \dots, \binom{n}{r}$, *there is a transposition* α_i *of* X_n *with* $\pi_i\alpha_i = \pi_{i+1}$.

If π and θ are weight-r partitions of X_n such that $\theta = \pi\alpha$ for a transposition α, then π and θ are distance-two in *Part*(n). In particular, a positive answer to the Transposition Listing Conjecture for some n and r improves Theorem 1.6 for that n and r. We show that the Transposition Listing Conjecture is a logical consequence of the Middle Levels Conjecture.

Given a partition π of X_n that has the non-singleton classes $B_1, B_2, \dots, B_k$ and that also may have some singleton classes, we use the notation $\pi = B_1|B_2|\dots|B_k|$*singletons* to describe π.

DEFINITION 1.7. For a subset C of X_n let θ_C be the partition of X_n defined by $\theta_C := (X_n - C)|$ singletons.

Let $A_1B_1A_2B_2 \dots A_{\binom{n}{r}}B_{\binom{n}{r}}$ be a sequence of sets that satisfies the Adjacent Levels Conjecture for given n and r. Observe first that for $i = 1, \dots, \binom{n}{r}$, since $B_i = A_i \cap A_{i+1}$ and $|B_i| = r - 1$, it follows that $A_1A_2 \dots A_{\binom{n}{r}}$ is a Hamiltonian cycle in $G_{n,r}$. Moreover, observe that $B_i \cap B_{i+1}$ must be an $(r-2)$-set and that there exists a transposition β_i such that $B_i\beta_i = B_{i+1}$.

Observe that θ_{B_i} is orthogonal to both A_i and A_{i+1} and that if β_i is the transposition of X_n that satisfies $B_i\beta_i = B_{i+1}$, then $\theta_{B_i}\beta_i = \theta_{B_{i+1}}$. In particular, the orthogonally labeled Hamiltonian cycle $A_1\theta_{B_1} \dots A_{\binom{n}{r}}\theta_{B_{\binom{n}{r}}}$ satisfies the Transposition Listing Conjecture for n and r. Thus, if the Adjacent Levels Conjecture holds for n and r, the Transposition Listing Conjecture also holds for n and r. By **[13]**, it follows that if the Middle Levels Conjecture holds for all possible cases, then so does the Transposition Listing Conjecture.

With a weight-r partition π of X_n is associated a partition τ of n into a sum of r positive integers. Corresponding to the class sizes of π we write $\tau = a_1^{m_1} \dots a_k^{m_k}$ if π has m_i distinct classes of size a_i for $i = 1, \dots, k$. (So $\Sigma_{i=1}^{k} a_im_i = n$ and $\Sigma_{i=1}^{k} m_i = r$.) A partition π of X_n is said to be of *type* τ if the sizes of its classes determine the partition τ of n. If an orthogonally labeled Hamiltonian cycle $A_1\pi_1A_2\pi_2 \dots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ satisfies the

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condition described in the Transposition Listing Conjecture, then all partitions of the orthogonal labeling have the same fixed type.

EXAMPLE 1.8. Let $n = 4$ and $r = 2$. There are two possible partition types with which we have the potential to orthogonally label a Hamiltonian cycle in $G_{4,2}$, namely $\tau_1 = 3\,1$ and $\tau_2 = 2^2$. However, there are only four distinct partitions of type τ_1 and only three distinct partitions of type 2^2. Since $\binom{4}{2} = 6$, we can not extend the Transposition Listing Conjecture to the case $n = 4$ and $r = 2$.

In fact $n = 4$ and $r = 2$ is the only problem case in the following theorem.

THEOREM 1.9 [**14**]. *For positive integers n and r with* $1 < r < n$, *except for* $n = 4$ *and* $r = 2$, *there exists an orthogonally labeled Hamiltonian cycle* $A_1\pi_1A_2\pi_2 \ldots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ *in* $G_{n,r}$ *such that for each* $i = 1, \ldots, \binom{n}{r}$, *there exists a permutation* γ_i *of* X_n *with* $\pi_i\gamma_i = \pi_{i+1}$.

Theorem 1.9 and the lack of a refutation of the long-standing Middle Levels Conjecture point to a reasonable likelihood that the Transposition Listing Conjecture is true. If it is true, then the Transposition Listing Conjecture would enable us to use the word *transposition* rather than the word *permutation* in the statement of Theorem 1.9.

PROBLEM 2. Find an example of positive integers n and r with $1 < r < n$, other than $n = 4$ and $r = 2$, such that there exists no orthogonally labeled Hamiltonian cycle satisfying the condition of the Transposition Listing Conjecture.

2. Orthogonally labeled Hamiltonian cycles (Theorem 1.4). We begin by introducing a concept used in subsequent constructions. Let $\mathcal{C} = A_1A_2 \ldots A_{\binom{n}{r}}$ be a Hamiltonian cycle in $G_{n,r}$. For consecutive vertices A_i and A_{i+1} in $\mathcal{C}$, let $C_i = A_i \cap A_{i+1}$ be the *core* of the edge A_iA_{i+1}. Hence with $\mathcal{C}$ we may associate a *core sequence* $C_1C_2 \ldots C_{\binom{n}{r}}$. Note that core sequences depend on the selected orientation; we have chosen to start with A_1, move to A_2, and so on. We will use the core sequences to guide the orthogonal labeling of Hamiltonian cycles in $G_{n,r}$.

To each A_i with $1 \le i \le \binom{n}{r}$, we can associate at most two cores: $A_{i-1} \cap A_i$ and $A_i \cap A_{i+1}$. Some of the cores in the core sequence may appear more than once; we call such cores *repeated* cores. The *initial* occurrence of a core is the first time that it appears in the core sequence. An occurrence of a core other than its initial occurrence will be called a *repeated* occurrence. A core C is said to occur *consecutively* if C appears as $\ldots\ CC\ \ldots$ in the sequence $C_1C_2 \ldots C_{\binom{n}{r}}$, or if $C = C_{\binom{n}{r}} = C_1$.

LEMMA 2.1. *Let n and r be positive integers with* $1 < r < n$. *A given core C may be associated with at most* $n - r$ *edges of a Hamiltonian cycle. It can be associated with* $n - r$ *edges only if the occurrences of C are consecutive.*

Proof. Suppose C is a core in the Hamiltonian cycle $A_1A_2 \ldots A_{\binom{n}{r}}$. Since $|C| = r - 1$ there exist at most $n - (r - 1)$ sets representing vertices for which C is a core, therefore there are at most $n - r$ edges for which C is a core. If C occurs k times then C is associated with at least $k + 1$ vertices. The maximum happens precisely when those occurrences are consecutive. □

Recall that for $C \subseteq X_n$, the partition θ_C of X_n is defined by $\theta_C := (X - C)|$ *singletons*. If C is the core associated with the edge A_iA_{i+1}, then the partition θ_C is orthogonal to A_i and to A_{i+1}.

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DEFINITION 2.2. Let $1 < r < n-1$ and let $\mathcal{C} = A_1 A_2 \ldots A_{\binom{n}{r}}$ be a Hamiltonian cycle in $G_{n,r}$. Suppose that $A_j A_{j+1}$ is an edge in $\mathcal{C}$ and $C_j = A_j \cap A_{j+1}$ is a repeated core. If the previous occurrence of $C = C_j$ occurs at $A_i A_{i+1}$ (so $i < j$), let $A_i = C \cup \{u\}$. For each $x \in C$ let $\delta_{C,x,u}$ be the partition of X_n having precisely two non-singleton classes: $\{x, u\}$ and $X_n - (C \cup \{u\})$. Then each $\delta_{C,x,u}$ is said to be an *allowable partition* label for $A_j A_{j+1}$.

Note that there exist $r-1$ allowable partitions for $A_j A_{j+1}$. We list some observations regarding allowable partition labels.

OBSERVATION 2.3. 1. *$\delta_{C,x,u}$ is orthogonal to both A_j and A_{j+1}.*

2. *If a core C occurs three times or more, then (because of the way the element u was determined in $\delta_{C,x,u}$), the allowable partition labels for the different edges associated with repeated occurrences of C are distinct.*

Next we show that allowable partitions defined in terms of two (not necessarily distinct) edges are distinct.

LEMMA 2.4. *Let p, s be positive integers such that $1 \le p < s \le \binom{n}{r}$, and let C and D be repeated cores, say C is repeated at $A_p A_{p+1}$ and D at $A_s A_{s+1}$. Let $\delta_{C,x,u}$ be an allowable partition for $A_p A_{p+1}$ and let $\delta_{D,y,v}$ be an allowable partition for $A_s A_{s+1}$. Then $\delta_{C,x,u} \neq \delta_{D,y,v}$.*

Proof. Assume first that $n - r > 2$, and suppose that $\delta_{C,x,u} = \delta_{D,y,v}$. Then by Observation 2.3, $C \neq D$. Note that the singleton classes of $\delta_{C,x,u} = \delta_{D,y,v}$ consist of the elements of $C - \{x\} = D - \{y\}$. Because $C \neq D$, it follows that $x \neq y$. Because $|X_n - (C \cup \{u\})| = |X_n - (D \cup \{v\})| = n - r > 2$, each of $\delta_{C,x,u}$ and $\delta_{D,y,v}$ contains exactly one doubleton class, say $\{x, u\}$ and $\{y, v\}$ respectively, and so we have $\{x, u\} = \{y, v\}$. Thus $x = v$ and $y = u$. It follows that $D \cup \{v\} = D \cup \{x\} = C \cup D = C \cup \{y\} = C \cup \{u\}$.

From the definition of allowable partitions for $A_s A_{s+1}$, observe that $D \cup \{v\} = D \cup \{x\}$ is the left-most r-set of the last edge with core D prior to $A_s A_{s+1}$. Similarly, $C \cup \{u\} = C \cup \{y\}$ is also the left-most r-set of the last edge with core C prior to $A_p A_{p+1}$. But $C \cup \{u\} = D \cup \{v\}$, $C \neq D$ and each edge is assigned a single core (since we have chosen an orientation). This contradiction completes the proof of the Lemma for $n - r > 2$.

When $n - r = 2$, we encounter a difficulty because the allowable partitions each have two doubleton classes, so we must take up an additional case. We will assume, without loss of generality, that $C_{\binom{n}{r}} \neq C_1$. Because $n - r = 2$, if a core repeats then it repeats exactly twice, and it repeats consecutively (Lemma 1.1). Let C occur at $A_{p-1} A_p$, where $C \cup \{u\} = A_{p-1}$ and $A_p = C \cup \{a\}$, and also at $A_p A_{p+1}$ where $A_{p+1} = C \cup \{b\}$. Let D occur at $A_{s-1} A_s$ where $D \cup \{v\} = A_{s-1}$ and $A_s = D \cup \{c\}$ and also at $A_s A_{s+1}$, where $A_{s+1} = D \cup \{e\}$. We assume that $p + 1 \le s - 1$. Then $\delta_{C,x,u} = \{x, u\}|\{a, b\}|$ *singletons* and $\delta_{D,y,v} = \{y, v\}|\{c, e\}|$ *singletons*.

We assume $\{x, u\} \neq \{y, v\}$, since the proof of the first part of the Lemma can be applied when $\{x, u\} = \{y, v\}$. Since $\{x, u\} \neq \{y, v\}$, we have $\{c, e\} = \{x, u\}$ and $\{a, b\} = \{y, v\}$. Because $y \in D$ we have $x \neq y$. Since the sets of singleton classes of $\delta_{C,x,u}$ and $\delta_{D,y,v}$ are equal, being an $(r-2)$-set of both C and D, it follows that $D \cup \{x\} = C \cup \{y\}$. Now $D \cup \{x\} = A_s$ or A_{s+1}, while $C \cup \{y\} = A_p$ or A_{p+1}. But $p + 1 \le s - 1$, so $\{A_s, A_{s+1}\} \cap \{A_p, A_{p+1}\} = \emptyset$, contradicting the fact that $D \cup \{x\} = C \cup \{y\}$. □

In view of Lemma 2.4, to construct an algorithm that orthogonally labels the given Hamiltonian cycle, it suffices to provide an algorithm that specifies how the core element $x \in C$ is chosen in $\delta_{C,x,u}$.

Proof of Theorem 1.4. If $r = n - 1$ then clearly any list of all the $(n-1)$-sets of X_n is a Hamiltonian cycle in $G_{n,n-1}$. By Lemma 2.1 the cores of any such Hamiltonian cycle are distinct. Thus we can label each edge of the cycle with a partition of the form θ_C, where C is the core of the edge.

Now let $r < n - 1$. We use the core sequence of $\mathcal{C}$ to construct an orthogonal labeling. If the first occurrence of a core C occurs at the edge $A_i A_{i+1}$, label $A_i A_{i+1}$ with θ_C. If $C_j = A_j \cap A_{j+1}$ is a repeated core, label it with an allowable partition of the form $\delta_{C,x,u}$. Note that Lemma 2.4 guarantees the existence of an allowable label for $A_j A_{j+1}$ that has not been used earlier in the process. $\square$

Remark 2.5. Observe that we use only two partition types $(n - r + 1)\, 1^{r-1}$ and $(n - r)\, 2\, 1^{r-2}$ to label a given Hamiltonian cycle in $G_{n,r}$. The labeling procedure described in Theorem 1.4, which we call $\mathcal{A}$, is algorithmic once a method for choosing x in C is given. The most inefficient aspect of $\mathcal{A}$ involves a backtracking procedure: at each edge it must be determined if the associated core is an initial core or a repeated core. Depending on the Hamiltonian cycle that is given as input, there may be many backtracking searches each requiring on the order of $\binom{n}{r}$ checks. In particular, we can not claim that $\mathcal{A}$ can be implemented in less than $O\left(\binom{n}{r}^2\right)$ time.

In the next section, we describe a certain class of Hamiltonian cycles in $G_{n,r}$ for which we can determine in constant time whether a given core is repeated. With such Hamiltonian cycles, $\mathcal{A}$ will turn out to be remarkably efficient.

3. Efficient labeling algorithms (Theorem 1.5).

In order to prove Theorem 1.5 we require an algorithm with the following properties: for positive integers n and r with $1 < r < n$, on input (n, r) the algorithm quickly produces a Hamiltonian cycle in $G_{n,r}$ with properties that enable it to be orthogonally labeled in an efficient manner. We present the definition of *Hamiltonian cycles* $H_{n,r}$ and demonstrate that they have the required properties. The cycles $H_{n,r}$ have been widely studied ([**20**], [**16**], [**21**]), arising in the context of what are known as *reflected Gray codes* .

Definition 3.1. Let n, r be positive integers with $r \leq n$, and let $H_{n,r}$ be defined recursively as follows:

1. $H_{n,n} = X_n$.
2. $H_{n,1} = \{1\} \ldots \{n\}$.
3. For $1 < r < n$, given that $H_{n-1,r-1} = A_1 A_2 \ldots A_{\binom{n-1}{r-1}}$, let $H^{rev}_{n-1,r-1} \oplus n$ be the list

$$\left(A_{\binom{n-1}{r-1}} \cup \{n\}\right) \ldots (A_2 \cup \{n\})(A_1 \cup \{n\}),$$

that results by adjoining n to each set of $H_{n-1,r-1}$ and then reversing the order of the resulting listing.

4. For $1 < r < n$, let $H_{n,r} = H_{n-1,r}\ (H^{rev}_{n-1,r-1} \oplus n)$ be the list that results from concatenating $H_{n-1,r}$ and $H^{rev}_{n-1,r-1} \oplus n$.

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EXAMPLE 3.2. $H_{2,2} = \{12\}$.
$H_{2,1} = \{1\}\{2\}$.
$H_{3,2} = H_{2,2}(H^{rev}_{2,1} \oplus 3) = \{12\}\{23\}\{13\}$.
$H_{4,2} = H_{3,2}(H^{rev}_{3,1} \oplus 4) = \{12\}\{23\}\{13\}\{34\}\{24\}\{14\}$.
$H_{4,3} = \{123\}\{134\}\{234\}\{124\}$.
$H_{5,3} = H_{4,3}(H^{rev}_{4,2} \oplus 5) = \{123\}\{134\}\{234\}\{124\}\{145\}\{245\}\{345\}\{135\}\{235\}\{125\}$.

The next lemma is a collection of observations, each of which follows from the definition of $H_{n,r}$. For a positive integer k with $1 \leq k \leq \binom{n}{r}$, denote by $H_{n,r}(k)$ the k-th set in $H_{n,r}$.

LEMMA 3.3. *Let n and r be positive integers with $1 < r < n$. Then*
1. *$H_{n,r}$ is a Hamiltonian cycle.*
2. *$H_{n,r}(1) = \{1 \dots r\}$ and $H_{n,r}\left(\binom{n}{r}\right) = \{1 \dots (r-1)n\}$.*
3. *$H_{n,r}\left(\binom{n-1}{r}\right) = \{1 \dots (r-1)(n-1)\}$ and $H_{n,r}\left(\binom{n-1}{r}+1\right) = \{1 \dots (r-2)(n-1)n\}$.*

The next Lemma describes a crucial property of the cycles $H_{n,r}$ that will allow us to determine in constant time whether a core is repeated.

LEMMA 3.4. *Let n and r be positive integers with $1 \leq r \leq n$ and let a core C repeat in the core sequence of $H_{n,r}$. Then all occurrences of C in the core sequence are consecutive. Moreover, for $1 < r < n$ the set $\{1 \dots (r-1)\}$ occurs as a core exactly once, with $\{1 \dots (r-1)\} = C_{\binom{n}{r}}$.*

Proof. Note that the lemma holds vacuously for $n = r$ and for $r = 1$. We treat first the case when $n = r + 1$. Let $\mathcal{C} = A_1 A_2 \dots A_{r+1}$ be any Hamiltonian cycle in $H_{r+1,r}$ satisfying $A_1 = \{1 \dots r\}$ and $A_{r+1} = \{1 \dots (r-1)(r+1)\}$. By Lemma 2.1 for the case $r = n - 1$, the core sequence consists of distinct sets $C_1 \dots C_{r+1}$. Since $C_{r+1} = \{1 \dots (r-1)\}$, it follows that the lemma holds for $n = r + 1$.

Now assume that $n > r + 1$, and recall that $H_{n,r} = H_{n-1,r}(H^{rev}_{n-1,r-1} \oplus n)$, with the core sequence $C_1 C_2 \dots C_{\binom{n}{r}}$. By inductive hypothesis, repeated cores in $H_{n-1,r}$ are consecutive. So are repeated cores in $H_{n-1,r-1}$ and therefore so also are those in $H^{rev}_{n-1,r-1}$. Adjoining $\{n\}$ to each set in $H_{n-1,r}$ in the construction of $H^{rev}_{n-1,r-1} \oplus n$ preserves the fact that repeated cores here are consecutive and are distinct from cores in $H_{n-1,r}$.

To complete the proof of the lemma we only need to consider the cores $C_{\binom{n-1}{r}} = \{1 \dots (r-2)(n-1)\}$ and $C_{\binom{n}{r}} = \{1 \dots (r-1)\}$ (see Lemma 3.3), and show that all occurrences of $\{1 \dots (r-2)(n-1)\}$ in the core sequence of $H_{n,r}$ are consecutive, while the core $\{1 \dots (r-1)\}$ occurs only once.

Assume that the core $\{1 \dots (r-2)(n-1)\}$ repeats as the core $C_i \neq C_{\binom{n-1}{r}}$. Since n is an element of each core C_i with $\binom{n-1}{r} < i < \binom{n}{r}$, we restrict our attention to the cores C_i with $1 < i < \binom{n-1}{r}$. These cores are also cores of $H_{n-1,r} = H_{n-2,r}(H^{rev}_{n-2,r-1} \oplus (n-1))$. Since we are concerned with a core containing the element $n-1$, we may further restrict our attention to cores C_i with $\binom{n-2}{r} < i < \binom{n-1}{r}$.

If $r = 2$, then $C_{\binom{n-1}{2}} = \{n-1\}$, and the core sequence of $H^{rev}_{n-2,r-1}$ consists of empty sets. Therefore $C_i = \{n-1\}$ for all i with $\binom{n-2}{2} < i < \binom{n-1}{2}$, and so by the inductive assumption applied to $n-1$ and 2, we conclude that $\{n-1\}$ repeats consecutively in the core sequence of $H_{n,r}$.

If $r > 2$, we prove that $\{1 \dots (r-2)(n-1)\}$ occurs exactly once in the core sequence of $H_{n,r}$. Indeed if $\{1 \dots (r-2)(n-1)\} = C_i$ with $\binom{n-2}{r} < i < \binom{n-1}{r}$, then because of the recursive construction of $H_{n-1,r}$, the core $C_i = C \cup \{n-1\}$, where $C = \{1 \dots (r-2)\}$

https://doi.org/10.1017/S0017089505002430 Published online by Cambridge University Press

is an element of the core sequence of $H_{n-2,r-1}$. By the inductive assumption applied to $n-2$ and $r-1$, we have that C occurs uniquely in the core sequence of $H_{n-2,r-1}$ as a core of the edge $H_{n-2,r-1}(\binom{n-2}{r-1})H_{n-2,r-1}(1)$. However this edge is removed in the recursive construction of $H_{n-1,r-1}$. Hence the core $\{1\ldots(r-2)(n-1)\}$ occurs exactly once, as $C_{\binom{n-1}{r}}$ in the core sequence of $H_{n,r}$.

To show the uniqueness of the core $\{1\ldots(r-1)\}$, note that in the recursive construction of $H_{n,r}$ the edge $H_{n-1,r}(\binom{n-1}{r})H_{n-1,r}(1)$ is removed. By the inductive assumption, $H_{n-1,r}(\binom{n-1}{r}) \cap H_{n-1,r}(1) = \{1\ldots(r-1)\}$ and no other edge of $H_{n-1,r}$ has associated core equal to $\{1\ldots(r-1)\}$. In particular, for $i = 1,\ldots,\binom{n-1}{r}-1$, $C_i \neq \{1\ldots(r-1)\}$, and the core $C_{\binom{n-1}{r}} = \{1\ldots(r-2)(n-1)\} \neq \{1\ldots(r-1)\}$. Also for i such that $\binom{n-1}{r} < i < \binom{n}{r}$, we have $n \in C_i$. It now follows that $\{12\ldots(r-1)\}$ occurs uniquely in the core sequence of $H_{n,r}$ as the core C_1. □

Much attention has been given to developing algorithms which on input (n,r) efficiently output the cycles $H_{n,r}$ ([**20**], [**2**], [**21**]), algorithms that do not depend on prior knowledge of $H_{n-1,r}$ and $H_{n,r-1}$. The amount of computation time needed to transform $H_{n,r}(k)$ into $H_{n,r}(k+1)$ is constant, independent of n, r and k. The algorithm of [**2**] outputs an r-set at each of the $\binom{n}{r}$ stages. We use an n-tuple with entries in $\{0,1\}$ to encode an r-set, thereby introducing a factor n in the following upper bound for the time necessary to output $H_{n,r}$.

THEOREM 3.5 [**2**]. *Let n and r be positive integers with $1 < r < n$. There exists an algorithm $\mathcal{B}$ with an associated constant c, independent of n, r and k, such that the time needed to compute $H_{n,r}(k+1)$ from $H_{n,r}(k)$ is bounded above by c. In particular, on input (n,r), the algorithm $\mathcal{B}$ outputs the cycle $H_{n,r}$ in no more than $cn\binom{n}{r}$ time.*

The symmetric difference of two consecutive sets in $H_{n,r}$ has size 2. Therefore, there exists a transposition $\lambda = (u_k, v_k)$ such that $H_{n,r}(k)\lambda = H_{n,r}(k+1)$, where $u_k \in H_{n,r}(k)$ and $v_k \notin H_{n,r}(k)$, so that $H_{n,r}(k+1) = (H_{n,r}(k) - \{u_k\}) \cup \{v_k\}$. The algorithm $\mathcal{B}$ of Theorem 3.5 efficiently transforms $H_{n,r}(k)$ into $H_{n,r}(k+1)$ using the structure of the n-tuple that encodes $H_{n,r}(k)$ to locate in constant time the elements u_k, v_k.

Algorithm $\mathcal{B}$ is initialized with $\{1\ldots r\}$ in storage at its first stage. More generally, algorithm $\mathcal{B}$ requires n bits of temporary storage at each stage: it has the n-tuple $H_{n,r}(k)$ in storage as it begins to compute u_k, v_k. We determine a new algorithm which we call $\mathcal{B}^+$. We expand $\mathcal{B}$ to $\mathcal{B}^+$ by making use both of the output of $\mathcal{B}$ and also of its intermediate calculations u_k and v_k.

Our modification involves $6n$ bits of temporary storage, eventually of the form:

$$(H_{n,r}(k))(u_{k-1}, v_{k-1})(u_k, v_k)(c_k).$$

The set $H_{n,r}(k)$ is encoded as n-tuple, followed by two $2n$-tuples holding relevant information from $\mathcal{B}$, and an n-tuple in which a specially selected element c_k of C_k is recorded. The elements c_k are defined according to the following rules. The elements u_0 and v_0 are undefined. We let $c_1 = r$ and for $1 < k < \binom{n}{r} - 1$

$$v_{k-1} = u_k \Rightarrow c_k = c_{k-1}$$
$$v_{k-1} \neq u_k \Rightarrow c_k = v_{k-1}.$$

https://doi.org/10.1017/S0017089505002430 Published online by Cambridge University Press

We initialize the temporary memory of $\mathcal{B}^+$ with the following $6n$-tuple:

$$(\ )(\ ,\ )(\ ,\ )(\ ).$$

For $k = 1$, we call on $\mathcal{B}$ to provide $H_{n,r}(1)$, and to calculate u_1 and v_1. We write into the temporary memory of $\mathcal{B}^+$

$$(H_{n,r}(1))(\ ,\ )(u_1, v_1)(r).$$

For $k = 2, \ldots, \binom{n}{r} - 1$, we call on $\mathcal{B}$ to provide $H_{n,r}(k)$, and to calculate u_k and v_k. We write into the temporary memory of $\mathcal{B}^+$

$$(H_{n,r}(k))(u_{k-1}, v_{k-1})(u_k, v_k)(c_k).$$

At this point $\mathcal{B}^+$ has the information necessary to compute the corresponding orthogonal label by a procedure that we will describe in due course.

For $k = \binom{n}{r}$, when $\mathcal{B}$ has calculated $H_{n,r}\left(\binom{n}{r}\right)$, we write into the temporary memory of $\mathcal{B}^+$

$$\left(H_{n,r}\left(\binom{n}{n}\right)\right)\left(u_{\binom{n}{r}-1}, v_{\binom{n}{r}-1}\right)(\ ,\ )(\ ).$$

With algorithm $\mathcal{B}^+$ and its temporary memory at our disposal, we will be able to proceed with the proof of Theorem 1.5. First we examine c_k and C_k more closely, and provide an example.

OBSERVATION 3.6. *If* $1 < r < n$ *and* $k = 2, \ldots, \binom{n}{r} - 1$*, the cores* C_k *and* C_{k-1} *are equal if and only if* $u_k = v_{k-1}$*. By Lemma* 3.4 *we have an initial occurrence of* C_k *if and only if* $u_k \neq v_{k-1}$.

The next lemma validates our claim that $c_k \in C_k$.

LEMMA 3.7. *For* $1 \leq k \leq \binom{n}{r}$ *we have that* $c_k \in C_k$.

Proof. Assume first that $k = 1$, so that by the definition $c_1 = r$. Since $H_{n,r}(1) = \{1 \ldots r\}$ we only need to show that $r \in H_{n,r}(2)$. Recall that $H_{n,r} = H_{n-1,r}(H^{rev}_{n-1,r-1} \oplus n)$. If $n = r + 1$ then $H_{n,r}(2) = H_{r,r-1}\left(\binom{r}{r-1}\right) \cup \{r + 1\} = \{1 \ldots (r - 2)r(r + 1)\}$. If $n > r + 1$ then $H_{n,r}(2) = H_{n-1,r}(2)$ and the result follows inductively.

Now assume that $c_{k-1} \in C_{k-1}$ for some $k \geq 2$. If $v_{k-1} = u_k$, we have that $c_k = c_{k-1} \in C_{k-1} = C_k$, by Observation 3.6. If $v_{k-1} \neq u_k$ then $c_k = v_{k-1}$ and $H_{n,r}(k) = C_k \cup \{v_{k-1}\} = C_k \cup \{u_k\}$. Thus $c_k = v_{k-1} \in C_k$. □

Using the calculations from Example 3.2, we give a description of the temporary memory at each of the ten stages for $H_{5,3}$. In Figure 1, for $k = 1, \ldots, 9$, the k-th line gives the temporary memory at stage k after $\mathcal{B}$ has computed $H_{5,3}(k)$ and u_k, v_k where defined.

Note that $\mathcal{B}^+$ has the same output as $\mathcal{B}$ at this point. For $k = 1, \ldots, \binom{n}{r} - 1$, by Theorem 3.5 there exists a constant c, independent of n, r and k, such that $\mathcal{B}$ requires at most time c to go from the k-th stage to the $(k + 1)$-th stage. Once $\mathcal{B}$ determines u_k, v_k, the further steps involved in determining the temporary storage for the k-th stage involve copying (u_k, v_k) into memory, and both computing and copying the new value of c_k. This last operation only involves a comparison of two n-tuples and can be done in $O(n)$ time. Finally, before proceeding to the next stage, (u_k, v_k) is shifted

$H_{5,3}(k)$	$(u_{k-1}, v_{k-1})(u_k, v_k)$	(c_k)
123	(,) (2, 4)	(3)
134	(2, 4)(1, 2)	(4)
234	(1, 2)(3, 1)	(2)
124	(3, 1)(2, 5)	(1)
145	(2, 5)(1, 2)	(5)
245	(1, 2)(2, 3)	(5)
345	(2, 3)(4, 1)	(3)
135	(4, 1)(1, 2)	(3)
235	(1, 2)(3, 1)	(2)
125	(3, 1)(,)	( )

Figure 1. Temporary memory of $\mathcal{B}^+$ for $H_{5,3}$ at each stage

to the left and the information in the right-most $2n$-tuple is erased. The temporary memory storage is given as $\mathcal{B}^+$ adjusts the memory in preparation for the computation of $H_{5,3}(k+1)$ has the form:

$$(H_{5,3}(k))(u_k, v_k)(\ ,\ )(c_k).$$

We can accomplish all of these steps in time bounded by dn, where d is some constant independent of n, r and k. We summarize the discussion in this paragraph in the next proposition.

Proposition 3.8. *For positive integers n and r with* $1 < r < n$ *there is associated with* $\mathcal{B}^+$ *a constant d, such that for all* $k = 1, \ldots, \binom{n}{r} - 1$, *the time needed to proceed from the k-th stage to the* $(k+1)$*-st stage is bounded above by dn. Further, d is independent of n, r and k.*

The temporary memory that $\mathcal{B}^+$ provides will enable us to determine and output a weight-r partition π_k in constant time, independent of n, r and k, such that π_k is an orthogonal label for the edge $H_{n,r}(k)H_{n,r}(k+1)$, and $\pi_1, \ldots, \pi_{\binom{n}{r}-1}$ are distinct. We also define $\pi_{\binom{n}{r}}$ in such a way that it is an orthogonal label for the edge $H_{n,r}\left(\binom{n}{r}\right)H_{n,r}(1)$. This labeling procedure will coincide with that of algorithm $\mathcal{A}$ of Theorem 1.4, but without the backtracking. We now complete the proof of Theorem 1.5. Recall that the algorithm $\mathcal{A}$ used in the proof of Theorem 1.4 for labeling an edge A_kA_{k+1} of a Hamiltonian cycle involves determining whether the core $A_k \cap A_{k+1} = C$ is repeated or not. If C is not repeated, then the edge is labeled θ_C. If C is repeated, with its previous occurrence being $A_i \cap A_{i+1}$ (where $i < k$), then after determining $\{u\} = A_i - C$ and the least element c in C, we label the edge A_kA_{k+1} with $\delta_{C,c,u} = \{u, c\}|X_n - C \cup \{u\}|$*singletons*. To prove Theorem 1.5 we apply the labeling procedure of $\mathcal{A}$ to $H_{n-1,r}$ using $\mathcal{B}^+$ to produce $H_{n-1,r}$ together with the associated temporary memory.

Proof of Theorem 1.5. Recall that if k is a positive integer with $1 < k < \binom{n}{r}$ then at the k-th stage of $\mathcal{B}^+$, having computed $H_{n,r}(k)$ but before $H_{n,r}(k+1)$ is output, in the memory we have $H_{n,r}(k)(u_{k-1}, v_{k-1})(u_k, v_k)(c_k)$.

Using $\mathcal{A}$ for $1 < k < \binom{n}{r}$, we compute π_k, a weight-r partition of X_n which is orthogonal to $H_{n,r}(k)$ and $H_{n,r}(k+1)$. If $u_k \neq v_{k-1}$ then by Observation 3.6, we

$H_{5,3}(k)$	$(u_{k-1}, v_{k-1}), (u_k, v_k)$	(c_k)	$\pi(k)$	Output (= $H_{5,3}(k)\pi_k$)
123	(,) (2, 4)	(3)	θ_{13}	(11100(*01011*)
134	(2, 4)(1, 2)	(4)	θ_{34}	(10110)(*11001*)
234	(1, 2)(3, 1)	(2)	θ_{24}	(01110)(*10101*)
124	(3, 1)(2, 5)	(1)	θ_{14}	(11010)(*01101*)
145	(2, 5)(1, 2)	(5)	θ_{45}	(10011)(*11100*)
245	(1, 2)(2, 3)	(5)	$\delta_{45,5,1}$	(01011)(*10001*)(*01100*)
345	(2, 3)(4, 1)	(3)	θ_{35}	(00111)(*11010*)
135	(4, 1)(1, 2)	(3)	$\delta_{35,3,4}$	(10101)(*00110*)(*11000*)
235	(1, 2)(3, 1)	(2)	θ_{25}	(01101)(*10110*)
125	(3, 1)(,)	( )	θ_{12}	(11001)(*00111*)

Figure 2. Temporary memory of $\mathcal{B}^+$ for $H_{5,3}$ with $\mathcal{A}$-determined partition and output at each stage

have that $C_{k-1} \neq C_k$ and we have an initial occurrence of C_k. So we label the edge $H_{n,r}(k)H_{n,r}(k+1)$ with θ_{C_k}.

If $u_k = v_{k-1}$ then by Observation 3.6 the core C_k is repeated and equals C_{k-1}. Observe that $H_{n,r}(k-1) - H_{n,r}(k) = \{u_{k-1}\}$, which is contained in the memory of $\mathcal{B}^+$ at the k-th stage. By Lemma 3.7, we have that $c_k \in C_k$. We label the edge $H_{n,r}(k)H_{n,r}(k+1)$ with the partition $\delta_{C_k,c_k,u_{k-1}}$. When we come to the last set $H_{n,r}\left(\binom{n}{r}\right)$, algorithm $\mathcal{B}$ is programmed to output $H_{n,r}\left(\binom{n}{r}\right)$ and terminate. Algorithm $\mathcal{B}^+$ outputs the pair $H_{n,r}\left(\binom{n}{r}\right)\theta_{1\ldots(r-1)}$. It follows we have orthogonally labeled $H_{n,r}$.

The time involved in computing π_k is bounded above by cn, where c is independent of n, r and k. Because of the un-complicated nature of the two types of partition we use, we can calculate each of the labeling partitions of X_n in time independent of n, r and k, as we now describe. For $k = 1, \ldots, \binom{n}{r} - 1$, if $\pi_k = \theta_{C_k}$, we write, in an n-tuple, the unique non-singleton class (which consists of the complement C'_k of the core C_k). To do so, we consult $H_{n,r}(k)$ and u_k and switch appropriate 0's to 1's, and vice versa. We then output $H_{n,r}(k)C'_k$. On the other hand, if $\pi_k = \delta_{C_k,c_k,u_{k-1}}$, we output $\{c_k, u_{k-1}\}$ and $C'_k - \{u_{k-1}\}$ as n-tuples. At the last step, we output the pair $H_{n,r}\left(\binom{n}{r}\right)\theta_{1\ldots(r-1)}$. We note that the labeling of $H_{n,r}\left(\binom{n}{r}\right)H_{n,r}(1)$ coincides with the labeling of $\mathcal{A}$, since the core $H_{n,r}\left(\binom{n}{r}\right) \cap H_{n,r}(1) = \{1\ldots(r-1)\}$, and by Lemma 3.4, the core $\{1\ldots(r-1)\}$ occurs exactly once, as the core of the edge $H_{n,r}\left(\binom{n}{r}\right)H_{n,r}(1)$.

Since $\mathcal{B}$ (which only outputs $H_{n,r}$) can be implemented in $cn\binom{n}{r}$ time, for some constant c independent of n, it follows that the computation time required to output the orthogonally labeled cycle $H_{n,r}$ is $dn\binom{n}{r}$, where d is independent of n and r. □

Figure 2 indicates how the algorithm operates for the orthogonal labeling of $H_{5,3}$. The information to the left of the triple boundary lines is used to determine π_k, which is provided to the right of the boundary. The output at the k-th stage is indicated in the next box to the right, with the partition italicized.

REMARK 3.9. In Theorem 1.5, the computation of π_k is very quick, requiring only a constant number of comparisons of n-tuples. A weight r-partition π requires nr time to output since π consists of r distinct subsets of X_n. But each partition produced by $\mathcal{A}$ requires no more than $3n$ time. As a result, the output of the theorem is compact. The orthogonal labeling requires $dn\binom{n}{r}$ time. Also, it requires only space linear in n and r. In fact, it only requires tn space, where t is a constant independent of n and r.

https://doi.org/10.1017/S0017089505002430 Published online by Cambridge University Press

4. Distance two labeling partitions (Theorem 1.6). The next lemma is a list of easily proved and well known facts concerning the lattice $Part(n)$ of partitions of X_n.

LEMMA 4.1. *Let α and β be weight r partitions in $Part(n)$. Then the following are equivalent*:

1. *$\alpha\beta$ is an edge in $Part_{n,r}$*;
2. *$\alpha \vee \beta$ covers α* [1];
3. *α and β have exactly $r-2$ partition classes in common.*

We modify slightly the orthogonal labeling algorithm $\mathcal{A}$ of Theorem 1.4 to produce an orthogonal labeling of $H_{n,r}$ in which successive partitions are adjacent in $Part_{n,r}$.

Proof of Theorem 1.6. Let $C_1 \dots C_{\binom{n}{r}}$ be the core sequence associated with $H_{n,r}$. In algorithm $\mathcal{A}$, if we have an initial occurrence of C_k then we label the corresponding edge with θ_{C_k}. Otherwise C_k is a repeated core, so by Lemma 3.4 we have that $C_k = C_{k-1}$. Thus there exists a unique element $u_{k-1} \in H_{n,r}(k-1) - C_k$. For the first core $C \neq C_k$ that follows C_k in the core sequence, since $C \cap C_k$ is an $(r-2)$-set, there exists a unique element $w_k \in C_k - C$. We orthogonally label the corresponding edge with the partition $\delta_{C_k, w_k, u_{k-1}}$. Thus we have specified a sequence of partitions $\pi_1\pi_2 \dots \pi_{\binom{n}{r}}$ which, by the proof of Theorem 1.4, constitutes an orthogonal labeling of $H_{n,r}$.

Given any core $C_i = H_{n,r}(i)H_{n,r}(i+1)$, an element x of C_i and an element $w \notin H_{n,r}(i) \cup H_{n,r}(i+1)$, note that in the lattice $Part(n)$, the weight-$(r-1)$ partition $\theta_{C_i - \{x\}}$ covers both θ_{C_i} and $\delta_{C_i,x,w}$.

We will show that for any two successive partitions π_k and π_{k+1} in the sequence $\pi_1\pi_2 \dots \pi_{\binom{n}{r}}$, there exists a partition of the form $\theta_{C_k - \{x\}}$ which covers both π_k and π_{k+1} in $Part(n)$. As above, let w_k be the unique element of C_k that does not belong to the first successor C of C_k distinct from C_k in the core sequence. Then $\pi_k \theta_{C_k - \{w_k\}} \pi_{k+1}$ is the desired two-path. Indeed, either $\pi_k = \theta_{C_k}$ or $\pi_k = \delta_{C_k, w_k, u_{k-1}}$, so $\theta_{C_k - \{w_k\}}$ covers π_k. If $C_k \neq C_{k+1}$ then $\pi_k = \theta_{C_{k+1}}$ with $C_k - \{w_k\} \subset C_{k+1}$. If $C_k = C_{k+1}$ then by our construction above, $\pi_{k+1} = \delta_{C_{k+1}, w_{k+1}, u_k} = \delta_{C_k, w_k, u_k}$, so $\theta_{C_k - \{w_k\}}$ covers π_{k+1}. □

REMARK 4.2. Because a repeated core in $H_{n,r}$ can repeat no more than $n - r - 1$ times, using on the order of n^2 temporary memory space, standard arguments can be used to show that the time used by the above modification of $\mathcal{A}$ is $O\big(n^2\binom{n}{r}\big)$. Specifically, on input (n, r), we can output in time $O\big(n^2\binom{n}{r}\big)$ an orthogonally labeled Hamiltonian cycle in $Part_{n,r}$ satisfying the closeness-of-partitions condition described in Theorem 1.6.

5. An application to minimal generating sets for finite semigroups. In this section we elucidate connections between orthogonally labeled Hamiltonian cycles and generating sets of certain finite semigroups. For concepts of semigroup theory see, for example, [**11**]. A subset U of a semigroup S is a *generating set* for S if every element of S may be written as a finite product of elements of U; in this case we write $S = \langle U \rangle$. If $\langle U \rangle = S$ and no proper subset of U generates S, then U is called a *minimal generating set for S*. The *rank* of S is the size of a minimal generating set.

An element e of S is said to be idempotent if $e^2 = e$. If S has a generating set consisting of idempotents, then S is said to be *idempotent-generated*. The *idempotent*

[1] Note that because $Part(n)$ is upper semi-modular [**1**], the fact that $\alpha \vee \beta$ covers α is equivalent to α covers $\alpha \wedge \beta$.

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rank of an idempotent-generated semigroup is the size of a minimal generating set consisting of idempotents. Of course, the idempotent rank of S is at least as large as the rank of S. Idempotents are extremely important in the structure theory of semigroups, both finite and infinite. They help classify different types of semigroups, locate subgroups, locate (local) inverse elements, determine left or right ideals, and describe the general structure of a given semigroup. Our interest in idempotents stems from their role in sets of generators for a given semigroup.

Much of the work involving idempotent-generated semigroups arises in the context of the singular (non-invertible) endomorphisms of a structured set. J. A. Erdos [**6**] proved that the semigroup of singular endomorphisms of a finite dimensional vector space is idempotent-generated. J. B. Fountain and A. Lewin [**8**] proved that the semigroup of singular order-preserving endomorphisms of an independence algebra of finite rank is idempotent-generated, while A. Oliveira [**15**] proved a similar result for order-independence algebras. There are analogous results for other important classes of semigroups.

Our application is to the particular subsemigroup $K(n, r)$ of T_n. Here T_n is the full transformation semigroup on the set X_n, namely the set of all maps $\alpha : X_n \to X_n$ under the operation of composition. Each element $\alpha \in T_n$ has an image $im(\alpha) = \{x\alpha : x \in T_n\}$ and a kernel $ker(\alpha) = \{(x, y) \in X_n^2 : x\alpha = y\alpha\}$. We shall write $\alpha = [x_1x_2 \ldots x_n]$ to indicate that α maps 1 to x_1, 2 to x_2 ... and n to x_n. When $im(\alpha)$ is an r-set of X_n then $ker(\alpha)$ is a weight-r partition of X_n. For $1 \leq r \leq n$, we set $K(n, r) = \{\alpha \in T_n : |im(\alpha)| \leq r\}$. The singular part of T_n is $K(n, n-1)$. In [**7**], A. E. Evseev and N. E. Podran proved that $K(n, r)$ is generated by its idempotents whose images have r elements. In [**12**], John M. Howie and the second author proved the following theorem.

THEOREM 5.1. *The idempotent rank of $K(n, r)$ is $S(n, r)$, the Stirling number of the second kind.*

This last result is an existence theorem, proved by induction on n and r. To establish it, the authors showed the existence of disjoint sets U_1 and U_2 of idempotents in $K(n, r)$ such that $\langle U_1 \cup U_2 \rangle = K(n, r)$. Each idempotent ε in T_n is the identity on $im(\varepsilon)$, which is orthogonal to $ker(\varepsilon)$. Each pair consisting of an r-set A and a weight-r partition π orthogonal to A uniquely determines an idempotent $\varepsilon_{A\pi} : X_n \to X_n$ such that $im(\varepsilon_{A\pi}) = A$ and $ker(e_{A\pi}) = \pi$. The set U_1 was chosen as $\{\varepsilon_{A_1\pi_1}\varepsilon_{A_2\pi_2} \ldots \varepsilon_{A_{\binom{n}{r}}\pi_{\binom{n}{r}}}\}$ where $A_1\pi_1A_2\pi_2 \ldots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ satisfy the conditions of Theorem 1.3. The set U_2 need only be selected so that each of the remaining $S(n, r) - \binom{n}{r}$ partitions of X_n appears exactly once as the kernel of an element of U_2.

We will utilize our results from Sections 1, 2 and 3 to produce algorithmically a set U_1 as described above as part of a generating set for $K(n, r)$. A second set U_2 needed to complete the generating set may be chosen at will to represent the remaining weight-r partitions of X_n. Let $A_1\pi_1A_2\pi_2 \ldots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ be an orthogonally labeled Hamiltonian cycle in $G_{n,r}$ as described in Theorem 1.6. Then this Theorem implies that the associated sequence of idempotents $\varepsilon_{A_1\pi_1}\varepsilon_{A_2\pi_2} \ldots \varepsilon_{A_{\binom{n}{r}}\pi_{\binom{n}{r}}}$ differ in a prescribed minimal way: successive images differ minimally by intersecting in $(r-1)$-sets, and successive kernels differ minimally by being distance-two in $Part_{n,r}$.

Given an r-set A and a weight-r partition π orthogonal to A, the idempotent $\varepsilon_{A\pi} : X_n \to X_n$ is defined as follows: each class of π is mapped by $\varepsilon_{A\pi}$ is mapped to the unique element of A which belongs to that class.

$H_{5,3}(k)\pi_k$	$\varepsilon_{H_{5,3}(k)\pi_k}$
(11100(*01011*)	[12322]
(10110)(*11001*)	[11341]
(01110)(*10101*)	[32343]
(11010)(*01101*)	[12242]
(10011)(*11100*)	[11145]
(01011)(*10001*)(*01100*)	[52245]
(00111)(*11010*)	[44345]
(10101)(*00110*)(*11000*)	[11335]
(01101)(*10110*)	[32335]
(11001)(*00111*)	[12555]

Figure 3. The set U_1 of idempotents determined by $H_{5,3}(k)\pi_k$

One possible choice of the set U_2 that represents each of the remaining $S(5, 3) - \binom{5}{3} = 25 - 10 = 15$ partitions of X_5 exactly once is: {[11344], [11343], [12144], [12142], [12125], [12313], [12312], [12215], [12331], [12321], [12244], [12323], [12332]}. Then $\langle U_1 \cup U_2 \rangle = K(5, 3)$, a semigroup consisting of 1805 elements. (See Figure 3.)

The result [**12**] identifying the idempotent rank of $K(n, r)$ leads to the following definition and an open problem. Observe that $K(n, r)/K(n, r-1)$ is a completely 0-simple idempotent-generated semigroup with idempotent rank $S(n, r)$. Its maximal $\mathcal{D}$ class has $S(n, r)$ distinct $\mathcal{R}$-classes and $\binom{n}{r}$ distinct $\mathcal{L}$-classes. For a finite completely 0-simple semigroup S, let R_S and L_S denote respectively the number of $\mathcal{R}$ and $\mathcal{L}$-classes of the maximal $\mathcal{D}$-class of S. Observe that the rank and idempotent rank of S are both at least $max\{R_S, L_S\}$.

Definition 5.2. Let S be a finite idempotent-generated completely 0-simple semigroup. We say that S is *extremally-generated* if its idempotent rank is $max\{R_S, L_S\}$.

For example, since $R_{K(n,r)} = S(n, r) \geq \binom{n}{r} = L_{K(n,r)}$, Theorem 5.1 shows that $K(n, r)$ is extremally generated. Moreover, if τ is a weight-r partition type, and $S(\tau)$ is the subsemigroup of $K(n, r)$ generated by all the transformations α with $ker(\alpha)$ of type τ, it was shown in [**13**] that $S(\tau)$ is extremally generated. This leads to the following natural problem.

Problem 3. Characterize extremally-generated completely 0-simple semigroups.

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