Chandrasekhar number to be zero, which physically corresponds to no impressed magnetic field. From Figure 1 it can be observed that the Rayleigh number-wave number dependence at marginal stability for the two sets of thermal boundary conditions differ significantly when \( Q = 0 \). In particular, when the constant flux boundary conditions apply, the convective regime will be established at a lower Rayleigh number for the larger cell sizes — i.e. at a lower wave number. The significance of these results, and the nature of the overall conclusions reached by Van der Borght (1974) regarding applications to supergranulation, indicate that an investigation into the effect of a magnetic field on the onset of cellular convection when the constant flux boundary conditions apply is clearly desirable. Irrespective of the geometry of the solar magnetic fields, we can expect some interaction between the convective motions associated with the supergranulation and the solar magnetic fields present.

The basic linear equations governing the onset of convective instability, in a horizontal layer of fluid heated from below and subject to a vertical magnetic field, are given by Chandrasekhar (1961). On eliminating the variable for the perturbed magnetic field from these equations we have the following differential system involving the vertical velocity \( W(z) \) and temperature fluctuation \( F(z) \) as functions of \( z \), with \( 0 < z < 1 \),

\[
(D^2 - a^2)W - QD^2W = Ra^2F,
\]

\[
(D^2 - a^2)F = -W,
\]

where \( D \equiv \frac{d}{dz} \), and \( R \) is the Rayleigh number, \( a \) the horizontal wave number and \( Q \) the Chandrasekhar number, \( Q \) being proportional to the square of the strength of the unperturbed magnetic field.

For astrophysical applications the free boundary conditions are appropriate:

\[
W = D^2W = 0 \text{ at } z = 0 \text{ and } 1.
\]

The temperature at any point within the fluid layer is given by

\[
T(x, y, z) = T_0(z) + F(z)f(x, y);
\]

here \( T_0(z) \) is the average (over the horizontal) temperature, \( F(z) \) the fluctuation from this average while \( f(x, y) \) establishes the planform of the convection pattern. The thermal boundary conditions applied, for no modulation of the heat flux, correspond to

\[
DF = 0, \text{ at } z = 0 \text{ and } 1,
\]

and it follows that

\[
DT = D T_0 + DF' = DT_0 = -N \text{ at } z = 0 \text{ and } 1,
\]

\( N \) being the Nusselt number. Further, the magnetic boundary conditions have no influence on the eigenvalues of this problem.

In this study two separate methods were employed to calculate the Rayleigh number, corresponding to a particular wave number and Chandrasekhar number, for the onset of stationary convection in a viscous fluid layer when the boundary conditions specified above apply. Good agreement was obtained between the two sets of results.

The variation of the Rayleigh number as a function of the horizontal wave number, when there is no modulation of the flux on the boundaries, is given in Figure 1 for the indicated values of \( Q \) when both surfaces are free (full curves). On comparing Chandrasekhar's (1961) results (broken curves) with those obtained here, it can be said that, overall, an impressed magnetic field inhibits the onset of cellular convection irrespective of the nature of the thermal boundary conditions. Nevertheless the contrasting features of these two sets of results are significant. If the magnetic field strength is such that \( Q \ll 10^{3.1} \), then minimum \( R = R_c \), occurs when \( a = a_c \approx 0 \) for \( DF = 0 \). A shallow minimum is only detected on the critical \( (R - a) \) curve when \( Q \) is large. This is in contrast to the constant temperature on the boundary results, where \( R \) is associated with a horizontal wave number \( a > \pi/\sqrt{2} \) and where \( R \rightarrow \infty \) as \( a \rightarrow 0 \) for any \( Q \geq 0 \).

These results have established that convection in the presence of a magnetic field will occur at a lower Rayleigh number if \( DF = 0 \) is the appropriate thermal boundary condition, rather than \( F = 0 \). However a better physical boundary condition is probably given by a combination of both these conditions, accepting that these two cases individually represent the two possible extremes. Further, the aspect ratio of the convective cells at the onset of instability, even in the presence of a magnetic field, will be large. Both these features have a bearing on the interpretation of convective theory as it applies to convection in stars and supergranulation.


A Semitheory for Semiconvection

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Semiconvection is the name given to a situation that arises in the evolution of large-mass main-sequence stars and lower-mass horizontal-branch stars, in which a layer forms where almost all of the heat flux is transported by radiation, but where slow convection is required to redistribute a stably stratified solute (e.g. helium). For a general discussion on semiconvection, see Spiegel (1969). The main problem is to determine the correct relationship between the solute distribution and the temperature gradient in the inhomogeneous layer. Ledoux (1947) and Schwarzschild and Harm (1958) have proposed two very different prescriptions. Since theory and experiment indicate that the Schwarzschild-Harm (SH) prescription is correct for the onset of convection (in the form of overstability), Spiegel (1969) has proposed that SH are correct. However, neither observation (oceanographic or laboratory) nor theory precludes a substantial deviation from the SH prescription as applied to finite amplitude semiconvection. The observational evidence is only readily applicable to stars if \( Pr > 1 \), where \( Pr \) is the Prandtl number of the fluid. In fact, \( Pr \ll 1 \) in stars. It is shown below that if one
could have stars in which $Pr \lesssim 1$, then the SH prescription would probably be wrong, and the Ledoux criterion might then be nearer to being correct. In the real situation ($Pr \ll 1$), observations or experiments are lacking and theory alone does not provide an unequivocal answer to the problem. This paper does not attempt a complete discussion of the problem, and a more detailed report will be submitted elsewhere. Some aspects of the problem have already been discussed in the context of the giant planets (Stevenson and Salpeter 1977).

Consider an infinite fluid in which the dimensionless measures of the destabilizing temperature gradient and the stabilizing solute gradient are respectively $\epsilon$ and $\chi$, defined by

$$
\epsilon = \frac{1}{\chi} \left( \frac{\partial \theta}{\partial T} \right) \frac{d}{d \theta} \left( \frac{\partial T}{\partial H_p} \right) \frac{d H_p}{d \theta} 
$$

and

$$
\chi = -\frac{1}{\chi} \left( \frac{\partial \chi}{\partial \theta} \right) \frac{d \chi}{d \theta} \frac{d H_p}{d \theta} \frac{d \theta}{d \chi}
$$

where $\theta$ is the fluid density, $T$ is the temperature, $P$ is the pressure, $x$ is the solute concentration, $r$ is a radial coordinate, $s$ is the entropy, and $H_p$ is the pressure scale height. The SH prescription for semiconvection is that the inequality $\epsilon > 0$ be marginally satisfied. The Ledoux prescription requires that $\epsilon > \chi$ be marginally satisfied. In both cases, the temperature gradient is chosen so that radiation can transport the required heat flux.

The Ledoux criterion is indeed correct for the onset of large scale monotonic flow. However, the different diffusivities of heat and solute allow small scale overstable modes as well. The criterion for the onset of overstability is (Walin 1964; Shirtcliffe 1967)

$$
(K + \nu)\epsilon > (D + \nu)\chi
$$

where $K$, $\nu$ and $D$ are respectively the thermal diffusivity, molecular viscosity and solute diffusivity. If $Pr \equiv \nu/K \ll 1$ and $D/K \ll 1$ then for $\chi \sim 1$, inequality (2) becomes essentially $\epsilon > 0$, the SH prescription. If $Pr \gtrsim 1$ then the Ledoux criterion would apply. Since $Pr \ll 1$ and $D \ll K$ in stars, it might appear that SH are correct.

However, this assumes infinitesimal perturbations. Experimental work at $Pr > 1$ (Turner and Stommel 1964; Turner 1968) indicate that the finite amplitude instability consists of mixed convective layers ($\epsilon, \chi \approx 0$) separated by thin purely diffusive layers ($\chi \gg \epsilon > 0$). In this situation, the solute flux $F_s$, and the convective thermal flux $F_{T, conv}$ (both in density units) are related by $F_s = (D/K)^{1/2} F_{T, conv}$ (Linden, 1974). The helium diffusion enters explicitly since the diffusive layers are highly stable and do not admit significant turbulent entrainment. Evolutionary models based on the SH prescription (for example, then 1955) indicate that typically $F_s \sim 10^{+4} F_{T, total}$ where $F_{T, total}$ is the total thermal flux. It follows that if $D \lesssim 10^{-8}$ then a substantial fraction of the thermal energy must be transported by convection or $F_s$ must be decreased. In either case, the SH prescription would be violated. Of course, this all assumes that $Pr \gtrsim 1$, a condition which is not satisfied in stars. Nevertheless, it does illustrate that, contrary to the assertions of Spiegel (1969), the oceanographic and laboratory evidence cannot be used to support the SH prescription. In a $15M_\odot$ star, $K \approx 3 \times 10^9$ cm$^2$ sec$^{-1}$ (Iben 1965), $D \approx \nu \approx 2 \times 10^3$ cm$^2$ sec$^{-1}$. (Chapman and Cowling 1970; Spitzer 1956) so that $D/K \sim 10^{-11}$ and the SH prescription would be only marginally satisfied at best.

Consider, now, the relevant case where $Pr \ll 1$. Oceanographic and laboratory experience do not provide guidance on the nature of the finite amplitude flow. Most likely it would consist of overstable motions or, perhaps, of uniformly mixed regions separated by overstable layers. It is plausible (but not obvious) that the solute diffusivity $D$ is now irrelevant, and the solute mixing is by turbulent motions ("entrainment"). Spiegel (1969) asserts that these motions could readily transport the required solute. However, entrainment can be inefficient in a very stably stratified system, and the rate of mixing can be orders of magnitude less than naive "eddy diffusivity" arguments would suggest (Turner 1973). Consider the following mechanistic argument for finite amplitude overstability: The fluid is assumed to be highly populated with uncorrelated "eddies", each of characteristic size $\lambda$. Assume that $(D/K)\chi < \epsilon \ll \chi$, which is the relevant domain if SH are approximately correct. The period of oscillation of an eddy is then $\tau \sim (H_s/\epsilon)^{1/3}$ where $g$ is the gravitational acceleration. The overstability is most efficiently driven when $\tau \sim \lambda^2/K$, the thermal diffusion time for the eddy. If the amplitude of the oscillation is $z$, then the energy production rate by the eddy is $a(z/\lambda) g z^2 \lambda^2/\tau$, where $a$ is a numerical coefficient somewhat less than unity. This is balanced by a dissipative term which might plausibly be of the form $C_D(\lambda/\chi)^2 z^2/\tau$, where $C_D$ is the drag coefficient (c.f. Landau and Lifshitz 1959) and $\nu \approx z/\tau$ is an average eddy velocity. This balance implies

$$
C_D \left( \frac{z}{\chi} \right)^2 \equiv a
$$

Since there exist small scale motions for which the Reynolds number is large but for which the stable stratification has little effect, one might expect $C_D \sim 0.05$ and $\epsilon \sim z/\lambda$. Such motions would not be very efficient in transporting solute upwards. If most of the available energy can be used to transport solute upward (rather than being lost ultimately by viscous dissipation), then $a(z/\lambda)^2 \approx F_s/F_{T, total} \sim 10^{-4}$ typically. These considerations suggest $\epsilon \approx 0.1$ and $z/\lambda \approx 0.1$ which would imply a significant deviation from the SH prescription.

It is clear that qualitative considerations alone are inadequate, and better quantitative calculations are needed. It may be, as Stothers and Chin (1976) suggest, that observations of massive stars can be used to determine which prescription for semiconvection is best. It is likely that both the Ledoux and the Schwarzschild-Harm prescriptions are oversimplifications of reality.

I acknowledge J. S. Turner for useful discussions.

Immiscibilities in cold, degenerate Stars

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The deep interiors of cold, degenerate stars consist of a mixture of elements, either because of primordial inhomogeneities or because of incomplete nuclear burning. However, most existing calculations for the cooling of such bodies (subsequent to any nuclear burning) assume that the only source of luminosity is the heat content of the star. An additional (and potentially much larger) energy source is available if the elements have limited mutual solubility below some temperature. The resulting differentiation and gravitational settling can dramatically decrease the rate of cooling, enhance the number of (potentially) observable low luminosity bodies, and may deplete the atmosphere of heavy elements (if fully pressure-ionized) tend to become super-saturated first, even if they are relatively underabundant. For example, 0.1% (by number) of iron becomes supersaturated in carbon at a much higher temperature than the temperature at which a 50-50 mixture of carbon and oxygen or carbon and helium would phase separate. The insolvency of Fe in C at $P \sim 10^{24}$ dyne-cm$^{-2}$ (the central pressure of a 1M$_{\odot}$ carbon white dwarf) occurs at a few million degrees, which is remarkably coincident with the freezing temperature of carbon (Lamb and Van Horn 1975).

The solubility in the solid phase is likely to be considerably less than in the fluid phase (see, for example, the calculations on solid H-He by Straus et al. 1977). The equilibrium at freezing may therefore be between a pure solid and an impure fluid. For definiteness, consider a cooling 1M$_{\odot}$ white dwarf that is pure carbon apart from a small mass fraction $\Delta$ of iron. If $\Delta \ll 1$, then the evolutionary calculations of Lamb and Van Horn (1975) can be adopted. They find that crystallization begins when $\log (L/L_\odot) \approx -3.6$. Instead of the simple outwardly growing solid core that they proposed, the following might happen in the presence of a small amount of iron: Crystals of almost pure carbon grow from the melt and rise (since they are less dense than the C-Fe fluid provided $\Delta \approx 10^{-2}$) to remelt higher up. The fluid at the centre becomes progressively enriched in iron, eventually reaching a eutectic composition at which almost pure solid iron grows from the melt to form a core. A small part of the gravitational energy released by this process is required for the increase in the zero temperature internal energy of the star but most of it must, by the virial theorem, be radiated away. The total radiated energy due to differentiation is therefore $a \Delta M g R$, where $g$ is the surface gravity, $R$ is the radius and $a$ is a numerical coefficient which, according to detailed calculations, is roughly unity. During the differentiation, the atmosphere and internal heat content changes little so the luminosity remains at roughly $10^{-4}$L$_\odot$. The time taken for differentiation is therefore

$$\tau_{\text{diff}} \approx 10^4 \Delta M g R/L_\odot \approx 3 \times 10^{19} \Delta \text{ years}$$

(2)

Even 0.1% of iron (comparable to the cosmic abundance)